Discrete Mathematics 76 (1989) 95-116
North-Holland

SPANNING EULERIAN SUBGRAPHS AND MATCHINGS

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Received 25 March 1986
Revised 1 October 1987

Let G be a 2-edge-connected graph of order n. For a matching M_3 consisting of three independent edges of $E(G)$, let $\sum(M_3)$ denote the sum of the degrees of the six vertices incident with M_3. We show that if $\sum(M_3) \geq 2n+2$ for all 3-matchings M_3 of G, then either G has a spanning eulerian subgraph, or there is a connected subgraph H of G such that the contraction G/H is $K_{2,t}$ for some odd t. We describe the nature of this contraction. The inequality is best-possible. We obtain several previous results as special cases.

We shall follow the notation of Bondy and Murty [4].

For $xy \in E(G)$, an *elementary contraction* of G is the graph G/xy obtained from G by deleting $\{x, y\}$ and inserting a new vertex v and edges joining v to each $w \in V(G-\{x, y\})$ with exactly as many edges as join $\{x, y\}$ to w in G. Thus, an elementary contraction can create multiple edges where none existed in G. A *contraction* of G is a graph G/H obtained from G by a sequence of elementary contractions which contract a connected subgraph H of G to a vertex.

The degree of a vertex is the number of incident edges. The degree of v in G is denoted $d(v)$, and the degree of v in G_1 is denoted $d_1(v)$.

A matching $M_k = \{u_1v_1, u_2v_2, \ldots, u_kv_k\}$ of k edges will be called a *k-matching*. Define $\sum(M_k)$ by

$$\sum(M_k) = \sum_{i=1}^{k} d(u_i) + d(v_i)$$

when M_k is a k-matching of G, and define

$$\sum_1(M_k) = \sum_{i=1}^{k} d_1(u_i) + d_1(v_i),$$

when M_k is a k-matching in G_1. The vertex set of M_k is denoted $V(M_k)$ or $V_1(M_k)$, respectively, according as M_k is regarded as being in G or in G_1.

By the definition of contractions, if G_1 is a contraction of G and if M_k is a k-matching in G_1, then there is a corresponding k-matching in G, which will also be called M_k.

Theorem 1. *Let the graph G_1 be a contraction of G, where G is a simple graph of* *order n and n_1 denotes the order of G_1. If*

$$\sum (M_3) \geq 2n + 2 \tag{1}$$

for every 3-matching M_3 in G, then

$$\sum_1 (M_3) \geq 2n_1 + 2 \tag{2}$$

for every 3-matching M_3 in G_1.

Proof. Let M_3 be a 3-matching in G_1. Then M_3 is also a 3-matching in G. Let W denote the vertices of $V(G) - V(M_3)$ that are identified with a vertex of $V(M_3)$ by the contraction-mapping $\Theta: G \to G_1$. Choose a subset $E_1 \subseteq E(G)$ so that $G[E_1]$ is a forest whose six components span the six connected subgraphs $G[\theta^{-1}(v)]$, where v runs over the six members of $V(M_3)$. Then Θ may be considered to contract each edge of E_1. By definition, $|E_1| = |W|$. Hence,

$$\sum_1 (M_3) \geq \sum (M_3) - |E_1| \geq (2n + 2) - |W|$$
$$\geq 2(n - |W|) + 2 \geq 2n_1 + 2. \quad \square$$

We define a graph G to be *collapsible* if for every even set $S \subseteq V(G)$, there is a subgraph Γ in G such that

(i) $G - E(\Gamma)$ is connected; and
(ii) The set S is the set of vertices of odd degree in Γ.

This concept was defined in [8], as a tool for determining the existence of spanning eulerian subgraphs. In [8] it was observed that a collapsible graph has a spanning eulerian subgraph.

We define a graph G to be *reduced* if no nontrivial subgraph of G is collapsible. The only graph that is both reduced and collapsible is K_1. By definition, any subgraph of a reduced graph is reduced.

The following two lemmas are proved in [8] (Corollary of Theorem 3 and Theorem 7):

Lemma 1. *Let H be a subgraph of G. If H is collapsible, then G is collapsible iff G/H is collapsible.*

Lemma 2. *If $|E(G)| \geq 2n - 3$, then G is reduced if and only if $G = K_1$ or $G = K_2$.*

In fact, as we observed in Theorem 1 of [8], if G has two edge-disjoint spanning trees, then G is collapsible. It is easy to show that the cycles C_2 and C_3 are collapsible, whereas C_n is not collapsible if $n \geq 4$.

Lemma 3. *The graph obtained from $K_{3,3}$ by deleting one edge is collapsible.*

Proof. By inspection. □

Theorem 2. *Let $E \subseteq E(G)$ be a minimum edge set such that every component of $G-E$ is collapsible, and let G_1 denote the reduced graph obtained from G by contracting each component of $G-E$ to a single vertex. Then G is collapsible if and only if $G_1 = K_1$; and G has a spanning eulerian subgraph if and only if G_1 has a spanning eulerian subgraph.*

This result is straightforward. The first part is trivial, and will be used in the next proof. We omit the details, since we will not need the latter part here. This result is contained in [8].

Theorems 1 and 2 reduce the problem of whether G, satisfying (1), is collapsible to the special case where G is reduced. Before we present the main result, we state and prove Theorem 3:

Theorem 3. *Let G be a reduced graph of order n. If every 3-matching M_3 of satisfies*

$$\sum (M_3) \geq 2n + 2, \tag{4}$$

then exactly one of the following holds:

(a) *G is collapsible (i.e. $G = K_1$);*
(b) *$G = K_{2,n-2}$ $(n \geq 4)$;*
(c) *$\kappa'(G) \geq 2$ and for some edge $e \in E(G)$, $G/e = K_{2,n-3}(n \geq 5)$;*
(d) *$G = G_d$ of Fig. 1;*
(e) *G is disconnected or G has a cut-edge.*

The hypothesis (4) may hold vacuously. In Theorem 4, we drop the hypothesis

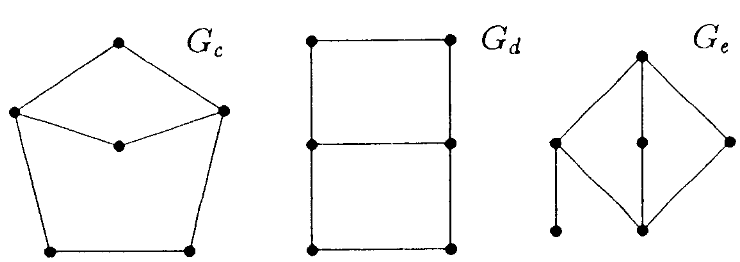

Fig. 1.

that G is reduced, and we thus generalize Theorem 3. The only conclusion that changes as Theorem 3 is generalized to Theorem 4 is (b).

We now show that the inequality (4) of Theorem 3 is sharp. These examples also show that (18) of Theorem 4 is best-possible.

Consider the star $H = K_{1,3}$, with center w and ends x_1, x_2, x_3. For nonnegative integers, s_{12}, s_{13}, s_{23}, define the graph $G(s_{12}, s_{13}, s_{23})$ to be the graph of order $4 + s_{12} + s_{13} + s_{23}$ obtained from H by adding:

s_{12} vertices with neighborhood $\{x_1, x_2\}$;
s_{13} vertices with neighborhood $\{x_1, x_3\}$; and
s_{23} vertices with neighborhood $\{x_2, x_3\}$.

For example, $G(1, 1, 1) = Q_3 - v$, a cube minus a vertex, and $G(1, 1, 0)$ is the graph G_d of Fig. 1.

Let M_e be a 3-matching in $G(s_{12}, s_{13}, s_{23})$. If w is not incident with an edge of M_3, then

$$\sum (M_3) = \sum_{i=1}^{3} (d(x_i) + 2) = 2(s_{12} + s_{13} + s_{23}) + 9 = 2n + 1.$$

If w is incident with an edge of M_3, then

$$\sum (M_3) = 2n + 2.$$

Now, if $s_{12} \geq s_{13} \geq 1$ and $s_{23} = 0$, then w is necessarily incident with an edge of M_3. Otherwise, if $s_{12}s_{13}s_{23} \geq 1$, then there are some 3-matchings M_3 not covering w, and for them,

$$\sum (M_3) = 2n + 1.$$

Therefore, the graphs $G(s_{12}, s_{13}, s_{23})$, with $s_{12}s_{13}s_{23} \geq 1$, show that (4) is best-possible.

Another graph showing (4) to be best-possible is obtained by adding to $K_{2,3}$ a path of length 3, whose ends are distinct divalent vertices of the $K_{2,3}$. This graph has order 7.

Proof of Theorem 3. Let G be a graph of order n with no nontrivial collapsible subgraph. Suppose, inductively, that G is a smallest counterexample. As a basis for induction, note that the theorem holds if $n \leq 3$. If any subgraph H of G has

$$|E(H)| \geq 2\,|V(H)| - 3,$$

then $H = K_1$ or $H = K_2$, by Lemma 2, since a subgraph of a reduced graph is reduced. Thus, for any nontrivial subgraph H of G,

$$|E(H)| \leq 2\,|V(H)| - 4 \quad \text{or} \quad H = K_2, \tag{5}$$

and hence G is simple. Also, since K_3 is collapsible and G has no nontrivial collapsible subgraph,

$$G \text{ is } K_3\text{-free}. \tag{6}$$

If $|E(G)| \geq 2n-3$, then by Lemma 2, G satisfies a conclusion of Theorem 3. Hence, we suppose

$$|E(G)| \leq 2n-4. \tag{7}$$

Let M be a maximum matching of G.

Case 1. Suppose $|M| \geq 4$, and set

$$M_4 = \{u_1v_1,\ u_2v_2,\ u_3v_3,\ u_4v_4\} \subseteq M,$$

where

$$d(u_4) + d(v_4) \geq \max_{1 \leq i \leq 3} (d(u_i) + d(v_i)). \tag{8}$$

Set $M_3 = M_4 - u_4v_4$. By (8) and (4),

$$\sum (M_4) \geq \tfrac{4}{3} \sum (M_3) \geq \tfrac{4}{3}(2n+2). \tag{9}$$

Let $E' \subseteq E(G)$ denote the edges with both ends in $\bigcup_{i=1}^{4} \{u_i, v_i\}$. By (5),

$$|E'| \leq 2(8) - 4 = 12. \tag{10}$$

By (9), (7), and (10),

$$\begin{aligned} \tfrac{4}{3}(2n+2) &\leq \sum_{i=1}^{4} d(u_i) + d(v_i) \leq |E(G)| + |E'| \\ &\leq (2n-4) + 12 = 2n+8. \end{aligned} \tag{11}$$

Therefore, $n \leq 8$, and $M_4 \subseteq E(G)$ implies $n = 8$. Equality holds in (11) and so $|E(G)| = 12$. In the remainder of Case 1, we show that this graph G of order 8 satisfies Theorem 3.

If (e) of Theorem 3 holds for G, then we are done. Suppose otherwise. Then $\delta(G) \geq 2$ and G is not collapsible. Suppose $\delta(G) = 2$, and let $v \in V(G)$ have degree 2 in G. Then

$$|V(G-v)| = 7; \qquad |E(G-v)| = 10.$$

By way of contradiction, suppose that $G - v$ has a cut-edge, say e. Since each component of $(G-v)-e$ satisfies (5), we have

$$10 = |E(G-v)| = |E((G-v)-e)| + 1 \leq 2\,|V((G-v)-e)| - 6 + 1 = 9,$$

a contradiction. By this and since G is reduced, $G - v$ does not satisfy (a) or (e), and since (4) holds for $G - v$, the induction hypothesis implies that $G - v$ satisfies

(b) or (c). If $G-v$ satisfies (b), then $|M| \leqslant 3$, a contradiction. If $G-v$ satisfies (c), then $|E(G)|=11$, also a contradiction.

Hence $\delta(G) \geqslant 3$, and so G must be 3-regular. We claim that each edge of G lies in a C_4. Suppose the edge wx is an exception, and set

$$N(w)=\{u, v, x\}, \qquad N(x)=\{w, y, z\}.$$

Since G is reduced, $|N(w) \cup N(x)|=6$, and since wx is in no C_4, $\{u, v, y, z\}$ is an independent set. Hence, $E(G)$ consists of five edges incident with $\{x, w\}$ and at most 6 edges incident with the two remaining vertices of G, for a total of at most 11 edges, contrary to $|E(G)|=12$. Hence, each edge of G is in a C_4.

Let H_1 be a C_4 in G, and let $H_2=G-V(H_1)$. Since G is reduced and 3-regular, with $|E(G)|=12$, four edges of G are in H_1, four edges join H_1 and H_2, four edges are in H_2, and so H_2 is a C_4, since G is reduced. Also, the four edges joining H_1 and H_2 in G must be a matching, since G is 3-regular. Hence, either G is a cube Q_3, or there are nonadjacent edges uv, $wx \in E(G)$ such that $G-\{uv, wx\}+\{ux, vw\}$ is a cube Q_3. In either case, G is collapsible, a contradiction.

This concludes Case 1, and so

$$|M| \leqslant 3.$$

Case 2. Suppose that a maximum matching of G has 3 edges. For any maximum matching

$$M=\{u_1v_1, u_2v_2, u_3v_3\} \subseteq E(G),$$

denote the six incident vertices

$$X=X(M)=\{u_1, u_2, u_3, v_1, v_2, v_3\},$$

and set $G'=G[X]$ and $E'=E(G')$. Also, define

$$Y=Y(M)=V(G)-X(M).$$

By the maximality of M, each edge of G is incident with X, and so

$$E(G[Y])=\emptyset \tag{12}$$

and edges of G' are those that are twice incident with X. Hence, if $|E'| \leqslant 5$, then (4) and (7) give

$$2n+2 \leqslant \sum(M)=|E(G)|+|E'| \leqslant |E(G)|+5 \leqslant 2n+1,$$

a contradiction. Therefore,

$$6 \leqslant |E'|. \tag{13}$$

Lemma 4. *If G' is a reduced graph of order 6 with at least 7 edges and a perfect matching, then G' is one of the graphs G_c, G_d, or G_e of Fig. 1.*

Proof. A reduced graph is simple and K_3-free. By inspection, the only simple K_3-free graphs of order 6 with 7 edges which contain a 3-matching and are not collapsible, are G_c, G_d, and G_e of Fig. 1. □

Lemma 5. *If $y \in Y$, if $|E(G)| = 2n - 4$, and if $d(y) = 2$, then a conclusion of Theorem 3 holds.*

Proof of Lemma 5. Suppose $|E(G)| = 2n - 4$ and let y be a vertex of Y with $d(y) = 2$. Since G is reduced, so is $G - y$.

By the induction hypothesis, $G - y$ satisfies one of (b), (c), (d), or (e) of Theorem 3. Since $|E(G)| = 2n - 4$, $G - y$ cannot satisfy (c) or (d). Since $G - y$ has a 3-matching (by the definition of Y), $G - y$ cannot satisfy (b). Hence, $\kappa'(G - y) \leq 1$. If $G - y$ is disconnected, then G satisfies (e) of Theorem 3. Hence, we can assume that $G - y$ has a cut-edge e, where $(G - y) - e$ has components G_1 and G_2. We have

$$|E(G_1)| + |E(G_2)| + 3 = |E(G)| = 2n - 4 = 2(n_1 + n_2 - 1),$$

where $n_i = |V(G_i)|$ $(i = 1, 2)$. Hence,

$$|E(G_1)| + |E(G_2)| = (2n_1 - 2) + (2n_2 - 2) - 1.$$

Without loss of generality, suppose

$$2n_1 - |E(G_1)| \geq 2n_2 - |E(G_2)|.$$

Since G is reduced, (5) implies

$$|E(G_i)| \leq 2n_i - 2 \qquad (1 \leq i \leq 2),$$

and hence

$$|E(G_1)| = 2n_1 - 3, \qquad |E(G_2)| = 2n_2 - 2,$$

and since G is reduced, Lemma 2 implies $G_1 = K_2$ and $G_2 = K_1$. Hence, $G = K_{2,2}$ or G has a cut-edge, and so either (b) or (e) of Theorem 3 holds. □

Proof of Theorem 3, continued. Either (e) of Theorem 3 holds, or $d(y) \geq 2$ for each $y \in Y$. We consider two subcases.

2*A*. Suppose that each $y \in Y$ has $d(y) \geq 3$. Set $k = |Y|$ and

$$r = \sum_{y \in Y} (d(y) - 3).$$

By (12), we have $|E(G)| = |E'| + \sum_{y \in Y} d(y)$, and hence

$$2(k + 6) + 2 = 2n + 2 \leq \sum (M_3) \leq |E(G)| + |E'| = 3k + r + 2\,|E'|.$$

Hence,

$$14 \leqslant k + r + 2\,|E'|\,. \tag{14}$$

Since G is reduced, (7), (12), and (13) give

$$2n - 4 \geqslant |E(G)| = |E'| + \sum_{y \in Y} d(y) \geqslant 6 + 3k + r$$
$$= (12 + 2k - 4) + (k + r - 2) = (2n - 4) + (k + r - 2).$$

Thus,

$$k + r \leqslant 2.$$

By the definition of r, if $r > 0$ then $k > 0$. Since k and r are nonnegative integers,

$$(k, r) \in \{(0, 0), (1, 0), (2, 0), (1, 1)\}.$$

Suppose $k = 0$. Then $r = 0$, and so (14) gives $|E'| \geqslant 7$. By Lemma 4, $G \in \{G_c, G_d, G_e\}$. Hence, Theorem 3 holds for G.

Suppose $k = 1$ and $r = 1$. Let y be the unique vertex of Y. Then $d(y) = 4$ and $N(y)$ contains both ends of some edge of M, thus forming a K_3. This contradicts the assumption that G is reduced.

Suppose $k = 1$ and $r = 0$. By (14),

$$|E'| \geqslant 7.$$

By Lemma 5, $G' = G - y$ is one of G_c, G_d, or G_e. By inspection, in any case, the graph G has a nontrivial collapsible subgraph, a contradiction.

Suppose $k = 2$. Then $r = 0$. Hence, $n = 6 + k = 8$ and

$$|E(G)| = |E'| + 6 \geqslant 12.$$

Hence, by (7),

$$|E(G)| = 12.$$

Since a maximum matching of G has only 3 edges, Tutte's Matching Theorem [12] (in combination with a parity argument) implies that there is a set $S \subseteq V(G)$ with $|S| = 3$ such that $\omega(G - S) \geqslant 5$. Since $n = 8$, $G - S$ consists of 5 isolated vertices. If (e) holds, we are done, and so we suppose that $\kappa'(G) \geqslant 2$. Therefore, for all $w \in V(G) - S$, $N(w) \subseteq S$ and $d(w) \geqslant 2$. If two vertices, say $w_1, w_2 \in V(G) - S$, both have degree 3, then for any $w_3 \in V(G) - (S \cup \{w_1, w_2\})$, we have $d(w_3) \geqslant 2$ and by Lemma 3, $G[S \cup \{w_1, w_2, w_3\}]$ is a collapsible subgraph of G. But G has no nontrivial collapsible subgraph, and so at most one vertex of $V(G) - S$ has degree 3. Suppose that just one vertex $w \in V(G) - S$ has degree 3. Since $|E(G)| = 12$ and since $V(G) - S$ is incident with 11 edges, there is an edge in $G[S] = G[N(w)]$, and so G has a K_3, contrary to (6). Hence, each vertex of $V(G) - S$ has degree 2, and since $|E(G)| = 12$, $G[S]$ has 2 edges. By (6), each $w \in V(G) - S$ must be adjacent to the pair of nonadjacent vertices of $G[S]$.

Therefore, $G = K_{2,6}$. But then G has no 3-matching, contrary to the assumption of Case 2.

2B. Suppose that some $y \in Y$ has $d(y) = 2$.

Since $\sum (M_3) \geq 2n + 2$ for any 3-matching M_3 of G, (4) holds for $G - y$, too. Also, $G - y$ is not collapsible, since G is reduced. By the induction hypothesis, $G - y$ satisfies a conclusion of Theorem 3, other than (a).

Suppose $G - y$ satisfies (b) of Theorem 3. By (6) and Lemma 3, $G = K_{2,n-2}$, since G is reduced. If $G - y$ satisfies (c) of Theorem 3, then $G/e = K_{2,n-3}$, for some edge e, since G is reduced and (4) holds. Suppose $G - y$ satisfies (d) of Theorem 3. Then for some 3-matching M_3 of G, $\sum (M_3) = 2n + 1$, a contradiction.

Hence, $\kappa'(G - y) \leq 1$. We may assume that (e) fails for G. Let e be a cut-edge of $G - y$, and denote by G_1 and G_2 the two components of $(G - y) - e$. We can choose a 3-matching M_3 of $G - y$ such that either $e \in M_3$ or e separates edges of M_3 in $G - y$. Hence, for the subgraph G' induced by $V(M_3)$,

$$\kappa'(G') \leq 1. \tag{15}$$

If $|E'| \geq 8$, then (15) implies that G' has a K_3, contrary to (6). This and (13) imply

$$6 \leq |E'| \leq 7.$$

By (15) and Lemma 4, either $|E'| = 6$ or $G' = G_e$.

First, suppose $G' = G_e$ and let $xz \in E'$, where z has degree 1 in G'. We shall reduce this to the case $|E'| = 6$. Let $y \in Y$. Since $M_3 \subseteq E'$ is a maximum matching of G, $N(y) \subseteq V(G')$. If all $y \in Y$ have $z \notin N(y)$, then (e) of Theorem 3 holds. Hence, some $y_0 \in Y$ is adjacent to z. Let M'' be a 3-matching containing y_0z and two edges of $G' - x$. Then the subgraph G'' of G, induced by $V(M'')$, has an edge set E'' with $|E''| = 6$, since G is reduced. If no vertex of $G - V(M'')$ has degree 2, then Case 2A applies with $M = M''$. Hence, it suffices to consider the case $|E'| = 6$.

Let $s = \sum_{y \in Y} (d(y) - 2)$.

Then

$$\sum_{y \in Y} d(y) = s + 2\,|Y|\,,$$

and so by (7)

$$2\,|Y| + 8 = 2n - 4 \geq |E(G)| = |E'| + \sum_{y \in Y} d(y) = 6 + s + 2\,|Y|\,,$$

which gives

$$2 \geq s.$$

By (4),

$$2\,|Y| + 14 = 2n + 2 \leq \sum (M_3) \leq |E(G)| + |E'| = (6 + s + 2\,|Y|) + 6.$$

Hence,

$$2 \leq s.$$

Therefore, $2 = s$, and since equalities hold everywhere,

$$|E(G)| = 2n - 4.$$

By Lemma 5, a conclusion of Theorem 3 holds. This concludes Case 2.

Case 3. Suppose

$$|M| = 2.$$

Let $M = \{uv, wx\}$, and set $X = \{u, v, w, x\}$. Define

$$Y = V(G) - X.$$

We may assume

$$\delta(G) \geq 2, \tag{16}$$

for otherwise (e) of Theorem 3 holds. If $Y = \emptyset$, then (6), $G = G[X]$ and (16) imply $G = K_{2,2}$, and (b) of Theorem 3 holds. Suppose, instead that

$$Y \neq \emptyset.$$

By the maximality of M, $G[Y]$ is edgeless. Hence, by (6), for any $y \in Y$, $N(y)$ is one of $\{u, w\}$, $\{u, x\}$, $\{v, w\}$, or $\{v, x\}$.

Let $y_1 \in Y$. Without loss of generality, suppose

$$N(y_1) = \{u, w\}.$$

By the maximality of M, $N(v) \cap Y = \emptyset$, for if instead

$$y_2 \in N(v) \cap Y,$$

then $\{vy_2, uy_1, wx\}$ is a 3-matching. Likewise, $N(x) \cap Y = \emptyset$. Hence, by (6) and (16), either

$$N(v) = \{u, w\}, \qquad N(x) = \{u, w\}$$

or

$$N(v) = \{u, x\}, \qquad N(x) = \{v, w\}.$$

In the latter case, $G[X \cup \{y_1\}] = C_5$, and if $G[X \cup \{y_1\}]$ is a proper subgraph of the connected graph G, then M is not a maximum matching, a contradiction. In the former case,

$$G[X \cup \{y_1\}] = K_{2,3},$$

and

$$d(v) = d(x) = d(y_1) = 2, \tag{17}$$

for otherwise M is not a maximum matching. If $G[X \cup \{y_1\}] = G$, then (b) or (c) of Theorem 3 holds; if $G[X \cup \{y_1\}]$ is a proper subgraph of G, then by (17), any $y_2 \in Y - y_1$ has $N(y_2) = \{u, w\}$, since $d(y_2) = 2$. Then $G = K_{2,n-2}$, and so (b) holds.

Case 4. Finally, suppose

$$|M| = 1.$$

By (6), $G \neq K_3$. Hence, by the maximality of M, either G is disconnected or $G = K_{1,n-1}$. In either case, (e) holds.

This completes the proof of Theorem 3. □

Theorem 4. *Let G be a 2-edge-connected simple graph of order n. If for every 3-matching M_3 of G.*

$$\sum (M_3) \geq 2n + 2, \tag{18}$$

then exactly one of the following holds:

(a) *G is collapsible;*

(b) *For some integer $t \geq 2$ and for some collapsible subgraph H of G,*

$$G/H = K_{2,t},$$

and the contraction-mapping $G \to G/H$ maps H to a vertex of degree t in $K_{2,t}$;

(c) *For some edge $e \in E(G)$, $G/e = K_{2,n-3}$ $(n \geq 5)$;*

(d) *$G = G_d$ of Fig. 1.*

Proof. The conclusions are mutually exclusive. Let $E \subseteq E(G)$ be a minimal set such that each component $H_1, H_2, \ldots, H_c$ of $G - E$ is collapsible, and arrange these components so that

$$|V(H_1)| \geq |V(H_2)| \geq \cdots \geq |V(H_c)|. \tag{19}$$

Let G_1 denote the graph obtained from G by contracting each component of $G - E$ to a single vertex. Let

$$V(G_1) = \{v_1, v_2, \ldots, v_c\}$$

be arranged such that v_i is the image of H_i under the contraction-mapping $G \to G_1$ $(1 \leq i \leq c)$. We call G_1 the *reduction* of G.

By the minimality of E, no nontrivial subgraph of G_1 is collapsible. Hence, G_1 is reduced. If $G_1 = K_1$ then (a) holds. Hence assume $G_1 \neq K_1$. As in the proof of

Theorem 3,

$$|E| \leqslant 2c - 3; \tag{20}$$

$$G_1 \text{ has no } C_3; \quad \text{and} \tag{21}$$

$$G_1 \text{ has no } C_2 \ (G_1 \text{ is simple}). \tag{22}$$

Properties (21) and (22) imply that for any three distinct components H_i, H_j, H_k of $G - E$, at most two edges of E join them.

For a given 3-matching M_3 of G, let $i(M_3, E)$ denote the number of incidences of $V(M_3)$ and E.

Since $\kappa'(G) \geqslant 2$, we have $\kappa'(G_1) \geqslant 2$. Hence, $c \geqslant 3$.

Case 1. Suppose $|V(H_3)| \geqslant 2$.

Since H_3 is collapsible, $\kappa'(H_3) \geqslant 2$. This and (19) imply

$$|V(H_1)| \geqslant |V(H_2)| \geqslant |V(H_3)| \geqslant 3.$$

Choose $e_i \in E(H_i)$ for $1 \leqslant i \leqslant 3$, and set

$$M_3 = \{e_1, e_2, e_3\}.$$

By (21) and (22), the subgraph $G' = G_1[\{v_1, v_2, v_3\}]$ has at most two edges. The edges of $E(G') \subseteq E$ are the only edges of E with both ends incident in G with $V(H_1) \cup V(H_2) \cup V(H_3)$. By this and (20),

$$i(M_3, E) \leqslant |E| + 2 \leqslant 2c - 1. \tag{23}$$

By (18),

$$2n + 2 \leqslant \sum (M_3) \leqslant \sum_{i=1}^{3} 2(|V(H_i)| - 1) + i(M_3, E). \tag{24}$$

We subtract $\sum_1^3 2\,|V(H_i)|$ from each side of (24) and we use (23) to get

$$2(c-3) + 2 \leqslant 2\left(\sum_4^c |V(H_i)|\right) + 2 \leqslant -6 + 2c - 1,$$

a contradiction. Therefore, Case 1 is impossible.

Case 2. Suppose $|V(H_2)| \geqslant 2$, $|V(H_3)| = 1$, and that $G - (V(H_1) \cup V(H_2))$ has an edge e_3.

Since H_2 is collapsible, $\kappa'(H_2) \geqslant 2$. Together with (19) we have

$$|V(H_1)| \geqslant |V(H_2)| \geqslant 3.$$

Hence, we can choose $e_1 \in E(H_1)$ and $e_2 \in E(H_2)$ so that their ends are not joined by E to either end of e_3, because (21) and (22) imply that e_3 is joined by E to at most one vertex of H_i $(i \leqslant c)$. Let $M_3 = \{e_1, e_2, e_3\}$. By our choice of e_1 and e_2, only e_3 and at most one other edge v_1v_2 of E, if it exists, have two incidences

Case 4. Suppose $|V(H_1)| \geqslant 2$ and $|V(H_2)| = 1$.

Let s denote the order of H_1. Since H_1 is collapsible and nontrivial,

$$s = |V(H_1)| \geqslant 3. \tag{28}$$

By Lemma 1, G/H_1 is not collapsible, and so Theorem 3 implies that either $G/H_1 = K_{2,t}$ for some $t \geqslant 2$, or $(G/H_1)/e = K_{2,t}$ for some $t \geqslant 2$ and some e, or $G/H_1 = G_d$ of Fig. 1. In the first of these three possibilities, H_1 is mapped to a vertex of degree t in $K_{2,t}$, for otherwise (18) would be violated. Hence in this case, (b) of Theorem 4 holds. It suffices to reduce the latter two cases to (c) and (d) of Theorem 4. This we do next.

4A. Suppose that $G/H_1 = G_d$.

We can choose a 3-matching $M_3 \subseteq E(G) - E(H_1)$ such that

$$\sum (M_3) \leqslant 4 + 6 + 4 + (|V(H_1)| - 1) = |V(H_1)| + 13,$$

Hence, by (18),

$$2(|V(H_1)| + 5) + 2 = 2n + 2 \leqslant \sum (M_3) \leqslant |V(H_1)| + 13,$$

and so $|V(H_1)| \leqslant 1$, a contradiction.

4B. Suppose that G/H_1 is the subdivision of $K_{2,t}$ of order $t+3$, where $t \geqslant 2$. If $t = 2$, then a contradiction with (18) is easily obtained.

From (18) we deduce that, under the contraction-mapping $G \to G_1$, H_1 is mapped to a vertex of degree t. Hence there is a matching

$$M_3 = \{e_1, e_2, e_3\}$$

in $G - E(H_1)$ such that both ends of e_1 have degree 2, one end of e_2 has degree t and the other end has degree 2, and exactly one end of e_3 lies in $V(H_1)$, and thus has degree at most $t + s - 1$, while the other end of e_3 has degree 2. Hence,

$$\sum (M_3) \leqslant (2+2) + (t+2) + (t+s-1+2) = 2t + s + 7. \tag{29}$$

Since $n = s + t + 2$, (18) and (29) give

$$2(s + t + 2) + 2 = 2n + 2 \leqslant \sum (M_3) \leqslant 2t + s + 7,$$

and so $s \leqslant 1$, a contradiction.

4C. Suppose that G/H_1 has an edge $e = xy$ whose ends both have degree at least 3, such that $(G/H_1)/e = K_{2,t}$ where $t \geqslant 2$, and the vertex of $K_{2,t}$ formed by the contraction of $e \in E(G/H_1)$ has degree t.

with $V(M_3)$. This and (20) imply

$$i(M_3, E) \leqslant |E| + 2 \leqslant 2c - 1. \tag{25}$$

By (18),

$$2 + 2n \leqslant \sum (M_3) \leqslant 2(|V(H_1)| - 1) + 2(|V(H_2)| - 1) + i(M_3, E),$$

and so by (25), we can subtract $2(|V(H_1)| + |V(H_2)|)$ on each side to get

$$2 + 2(c-2) = 2 + 2\sum_{i=3}^{c} |V(H_i)| \leqslant -4 + 2c - 1,$$

a contradiction.

Case 3. Suppose $|V(H_2)| \geqslant 2$, $|V(H_3)| = 1$, and suppose $G - (V(H_1) \cup V(H_2))$ is edgeless.

Thus, in G_1, all edges are incident with $\{v_1, v_2\}$.

Let $y \in V(H_i)$, for some $i \geqslant 3$. Since $\kappa'(G) \geqslant 2$, and since $G - (V(H_1) \cup V(H_2))$ is edgeless, (22) implies that $N(y)$ overlaps both $V(H_1)$ and $V(H_2)$. Hence, by (21), no edge of E joins $V(H_1)$ and $V(H_2)$. Hence,

$$G_1 = K_{2,t}$$

for some $t \geqslant 2$. Also, by (19) and $|V(H_3)| = 1$,

$$d(y) = 2. \tag{26}$$

Since H_2 is collapsible and $|V(H_2)| \geqslant 2$, we have $\kappa'(H_2) \geqslant 2$. This and (19) imply

$$|V(H_1)| \geqslant |V(H_2)| \geqslant 3.$$

Choose $e \in E(H_2)$ so that its ends are incident with the fewest possible number of edges of E. Then we can choose $e_1' = x_1 y_1 \in E$ and $e_2' = x_2 y_2 \in E$ such that $x_1 \in V(H_1)$, $x_2 \in V(H_2)$ and $\{e, e_1', e_2'\}$ is a matching, which we denote M_3. Note that y_1 and y_2 satisfy (26).

The only edges of E that could have both ends incident with $V(M_3)$ are those edges incident with $\{y_1, y_2\}$. By (26), there are at most four such edges, and so (20) gives

$$i(M_3, E) \leqslant |E| + 4 \leqslant 2c + 1. \tag{27}$$

We combine (27) with (18) and (19) to get

$$\begin{aligned} 2n + 2 \leqslant \sum (M_3) &\leqslant 2(|V(H_2)| - 1) + (|V(H_1)| - 1) + (|V(H_2)| - 1) + i(M_3, E) \\ &\leqslant 2\,|V(H_2)| + 2\,|V(H_1)| + 2(c-2) + 1 \\ &\leqslant 2\Big(\sum_{i=1}^{c} |V(H_i)|\Big) + 1, \end{aligned}$$

a contradiction. Therefore, Case 3 fails.

Case 4. Suppose $|V(H_1)| \geqslant 2$ and $|V(H_2)| = 1$.

Let s denote the order of H_1. Since H_1 is collapsible and nontrivial,

$$s = |V(H_1)| \geqslant 3. \tag{28}$$

By Lemma 1, G/H_1 is not collapsible, and so Theorem 3 implies that either $G/H_1 = K_{2,t}$ for some $t \geqslant 2$, or $(G/H_1)/e = K_{2,t}$ for some $t \geqslant 2$ and some e, or $G/H_1 = G_d$ of Fig. 1. In the first of these three possibilities, H_1 is mapped to a vertex of degree t in $K_{2,t}$, for otherwise (18) would be violated. Hence in this case, (b) of Theorem 4 holds. It suffices to reduce the latter two cases to (c) and (d) of Theorem 4. This we do next.

4A. Suppose that $G/H_1 = G_d$.

We can choose a 3-matching $M_3 \subseteq E(G) - E(H_1)$ such that

$$\sum (M_3) \leqslant 4 + 6 + 4 + (|V(H_1)| - 1) = |V(H_1)| + 13,$$

Hence, by (18),

$$2(|V(H_1)| + 5) + 2 = 2n + 2 \leqslant \sum (M_3) \leqslant |V(H_1)| + 13,$$

and so $|V(H_1)| \leqslant 1$, a contradiction.

4B. Suppose that G/H_1 is the subdivision of $K_{2,t}$ of order $t+3$, where $t \geqslant 2$. If $t = 2$, then a contradiction with (18) is easily obtained.

From (18) we deduce that, under the contraction-mapping $G \to G_1$, H_1 is mapped to a vertex of degree t. Hence there is a matching

$$M_3 = \{e_1, e_2, e_3\}$$

in $G - E(H_1)$ such that both ends of e_1 have degree 2, one end of e_2 has degree t and the other end has degree 2, and exactly one end of e_3 lies in $V(H_1)$, and thus has degree at most $t + s - 1$, while the other end of e_3 has degree 2. Hence,

$$\sum (M_3) \leqslant (2+2) + (t+2) + (t+s-1+2) = 2t + s + 7. \tag{29}$$

Since $n = s + t + 2$, (18) and (29) give

$$2(s+t+2) + 2 = 2n + 2 \leqslant \sum (M_3) \leqslant 2t + s + 7,$$

and so $s \leqslant 1$, a contradiction.

4C. Suppose that G/H_1 has an edge $e = xy$ whose ends both have degree at least 3, such that $(G/H_1)/e = K_{2,t}$ where $t \geqslant 2$, and the vertex of $K_{2,t}$ formed by the contraction of $e \in E(G/H_1)$ has degree t.

Since both ends of e have degree at least 3, we must have

$$t \geqslant 4.$$

There are integers t_1, t_2 satisfying

$$t_1 + t_2 = t, \tag{30}$$

such that in G/H_1, we have $d(x) = t_1 + 1$ and $d(y) = t_2 + 1$. It follows from (18) that $V(H_1) \cap \{x, y\} \neq \emptyset$. Without loss of generality, suppose $x \in V(H_1)$, $y \notin V(H_1)$. Then $n = s + t + 2$, and we can choose a matching

$$M_3 = \{e_1, e_2, e_3\}$$

in $E(G) - E(H_1)$, such that e_1 has ends of degree 2 and t, e_2 is incident with y and has ends of degree 2 and $t_2 + 1$, and e_3 is incident with x and has ends of degree 2 and at most $s + t_1$ in G. Then by (18) and (30),

$$\begin{aligned} 2s + 2t + 6 = 2n + 2 \leqslant \textstyle\sum (M_3) \\ &\leqslant (2 + t) + (2 + t_2 + 1) + (2 + s + t_1) \\ &= 2t + s + 7, \end{aligned}$$

and so $s \leqslant 1$, a contradiction. Therefore, 4C and Case 4 are complete.

Case 5. If $|V(H_1)| = 1$, then Theorem 3 applies directly. This proves Theorem 4. □

If (b) holds in Theorem 4, then (18) forces certain other restrictions that are not stated explicitly in (b).

The following result is implied by Theorem 4. Its proof is straightforward and hence omitted.

Corollary 1. *Let G be a simple graph on n vertices. If*

$$d(u) + d(v) \geqslant \tfrac{2}{3}(n + 1) \tag{31}$$

whenever $uv \in E(G)$, then exactly one of the following holds:

(a) *G is collapsible*;

(b) $G = K_{2,n-2}$ $(n \geqslant 4)$;

(c) *$G = G(k)$ for some $k \geqslant 2$, where $G(k)$ is the graph of Fig.* 2;

(d) *G is disconnected or G has a cut-edge.*

Corollary 2 (Catlin [7]). *If the hypothesis of Corollary* 1 *holds, then exactly one of the following holds*:

(a) *G has a spanning eulerian subgraph*;

(b) *$G = K_{2,n-2}$ and n is odd,* $(n \geqslant 5)$;

(c) *G is disconnected or has a cut-edge.*

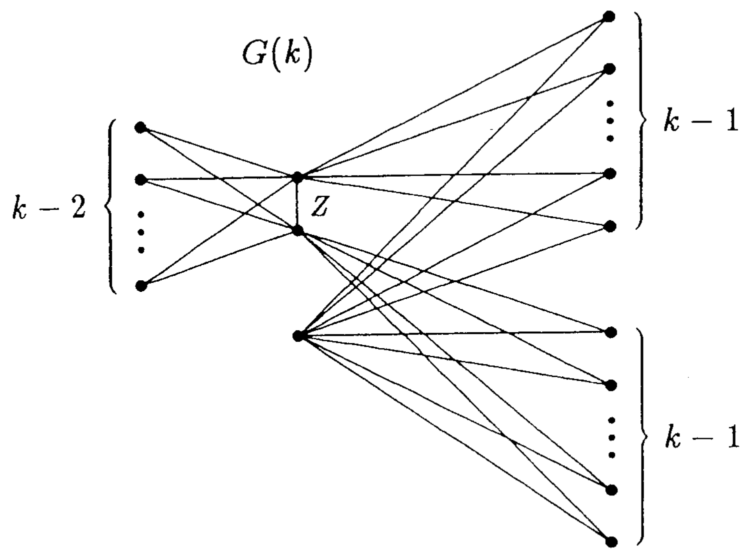

Fig. 2.

Proof. Parts (a) and (b) of Corollary 2 follow from (a) and (b) of Corollary 1 and by the fact that a collapsible graph has a spanning eulerian subgraph. The graph $G(k)$ of Fig. 2 has a spanning eulerian subgraph. □

Corollary 2 improves upon previous results due to Brualdi and Shanny [5], Catlin [6], Clark [10], and Veldman ([14], Theorem 5). A closely related result on hamiltonian line graphs was obtained independently by Catlin [7] and by Benhocine, Clark, Köhler, and Veldman [3]:

Theorem 5. *Let G be a simple graph of order n. If*

$$d(u) + d(v) \geq \tfrac{1}{3}(2n + 1) \tag{32}$$

whenever $uv \in E(G)$, then exactly one of the following holds:

(a) *$L(G)$, the line graph of G, is hamiltonian;*

(b) *G is not cyclically 2-edge-connected.*

Examples showing Corollary 2 to be best-possible are found among the examples presented earlier that show that Theorems 3 and 4 are best-possible.

Theorem 6. *Let G be a 2-edge-connected simple graph of order n. If*

$$d(u) + d(v) + d(w) \geq n + 1 \tag{33}$$

for every independent subset $\{u, v, w\}$ of $V(G)$, then exactly one of the following holds:

(a) *G is collapsible;*

(b) *$G \in \{C_4, C_5, K_{2,3}, G_d\}$ (see Fig. 1).*

Proof. As in the proof of Theorem 4, we let E be a minimal subset of $E(G)$ such that every component of $G-E$ is collapsible. Let $H_1, H_2, \ldots, H_{n_1}$ denote the components of $G-E$, where $n_1 = |V(G_1)|$ and G_1 is the reduction of G. Thus, G_1 is obtained from G by contracting the respective subgraphs $H_1, H_2, \ldots, H_{n_1}$ to the respective vertices $x_1, x_2, \ldots, x_{n_1}$ of $V(G_1)$.

Case 1. It is easily checked that, if $\{x_i, x_j, x_k\}$ is an independent set of three vertices in G_1, then

$$n_1 + 1 \leqslant d_1(x_i) + d_1(x_j) + d_1(x_k). \tag{34}$$

If G_1 satisfies the hypothesis of Theorem 3, then it is straightforward to reach a conclusion of Theorem 6.

Case 2. Hence, suppose that some 3-matching M_3 of G_1 does not satisfy

$$\sum_1 (M_3) \geqslant 2n_1 + 2.$$

Let X be the set of six vertices of G_1 incident with M_3, and define

$$Y = V(G_1) - X.$$

If $\chi(G_1[X]) = 2$, then the existence of M_3 implies that both color classes of $G_1[X]$ have three members. Hence, (34) holds for both color classes of $G_1[X]$, contrary to the condition of Case 2.

Therefore, $\chi(G_1[X]) \geqslant 3$, and so $G_1[X]$ must have an odd cycle. Since G_1 is reduced, $G_1[X]$ has no 3-cycle, and since $|X| = 6$, any odd cycle in $G_1[X]$ has length 5. Since $M_3 \subseteq E(G_1[X])$ and since $G_1[X]$ has a 5-cycle but no 3-cycle, we must have

$$C_5 \cup K_1 \subset G_1[X] \subseteq G_c, \qquad C_5 \cup K_1 \neq G_1[X], \tag{35}$$

where G_c appears in Fig. 1. Denote

$$X = \{u_1, u_2, u_3, v_1, v_2, v_3\},$$

where $u_1 v_1 u_2 v_3 u_3 u_1$ is a 5-cycle of $G_1[X]$, and where $N(v_2) = \{u_1, u_2\}$ if $G_1[X] = G_c$ and $N(v_2) = \{u_2\}$ otherwise. Define

$$m = \max(d_1(u_3), d_1(v_3)) \tag{36}$$

and

$$Y' = Y - (N(u_1) \cup N(u_2)).$$

First, suppose

$$|Y'| \leqslant m - t, \tag{37}$$

where

$$t = \begin{cases} 1 & \text{if } G_1[X] = G_c; \\ 2 & \text{if } G_1[X] \neq G_c. \end{cases}$$

By the definition of Y', by $n_1 = |Y| + 6$, and by (37), we have

$$d_1(u_1) + d_1(u_2) \geqslant |Y \cap (N(u_1) \cup N(u_2))| + (|X \cap N(u_1)| + |X \cap N(u_2)|)$$
$$= (|Y| - |Y'|) + (7 - 7)$$
$$\geqslant n_1 + 1 - m. \tag{38}$$

Since $\{v_1, v_2, v_3\}$ and $\{v_1, v_2, u_3\}$ are independent vertex sets in G_1, (34) gives

$$d_1(v_1) + d_1(v_2) + d_1(v_3) \geqslant n_1 + 1; \tag{39}$$
$$d_1(v_1) + d_1(v_2) + d_1(u_3) \geqslant n_1 + 1. \tag{40}$$

By (36), (38) holds for some $m \in \{d(u_3), d(v_3)\}$, and so (38) and one of (39) or (40) can be added to give

$$\sum_1 (M_3) \geqslant 2n_1 + 2,$$

contrary to the condition of Case 2. Hence, (37) is false.

Since (37) is false,

$$|Y'| \geqslant m + 1 - t, \tag{41}$$

and (36) gives

$$m \geqslant d_1(u_3); \qquad m \geqslant d_1(v_3).$$

Since u_3 and v_3 are each adjacent in G_1 to two vertices of X, this implies

$$m - 2 \geqslant |N(u_3) \cap Y'|; \qquad m - 2 \geqslant |N(v_3) \cap Y'|. \tag{42}$$

By (41) and $t \in \{1, 2\}$,

$$|Y'| \geqslant m + 1 - t > m - 2,$$

and so by (42) there are vertices

$$u_4 \in Y' - N(u_3), \qquad v_4 \in Y' - N(v_3), \tag{43}$$

and if $t = 1$, then such vertices u_4 and v_4 can be chosen to be distinct. If possible, choose u_4 and v_4 satisfying (43) to be distinct.

2A. Suppose u_4 and v_4 are distinct. Define

$$S = \{u_1, u_2, u_3, u_4, v_3, v_4\}.$$

By (43) and the definition of Y', $\{u_1, v_3, v_4\}$ and $\{u_2, u_3, u_4\}$ are two independent vertex sets in G_1, and so by (34),

$$d_1(u_1) + d_1(v_3) + d_1(v_4) \geqslant n_1 + 1$$
$$d_1(u_2) + d_1(u_3) + d_1(u_4) \geqslant n_1 + 1.$$

Hence, the number of incidences of edges of G_1 with vertices of S is at least $2n_1 + 2$. We distinguish two subcases:

– u_4 (or v_4) is adjacent to neither of the vertices v_1 and v_2. Then $|E(G_1[\{u_2, u_3, u_4, v_1, v_2, v_3\}])| = 5$ and application of (34) to $\{u_2, u_3, u_4\}$ and $\{v_1, v_2, v_3\}$ gives the desired contradiction.

– Both u_4 and v_4 are adjacent to a vertex in $\{v_1, v_2\}$. Since G_1 is K_3-free, both u_4 and v_4 have exactly one neighbour in $\{v_1, v_2\}$. Suppose, e.g. $u_4v_1 \in E(G_1)$ and $v_4v_2 \in E(G_1)$. (The remaining case is similar). Then $u_1v_2 \notin E(G_1)$, for otherwise $G_1[\{u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4\}]$ would be collapsible. Now $|E(G_1[\{u_1, u_3, v_1, v_2, v_3, v_4\}])| = 5$ and (34) can be applied to $\{u_3, v_1, v_2\}$ and $\{u_1, v_3, v_4\}$ to obtain a contradiction.

2B. Suppose $u_4 = v_4$. By $t \in \{1, 2\}$ and by the choice of u_4 and v_4 we have $t = 2$, and so $G_1[X] \neq G_c$ and $v_2u_1 \notin E(G_1)$. Define

$$S = \{u_1, u_2, u_3, u_4, v_2, v_3\}.$$

By (43) and the definition of Y', $\{u_1, v_2, v_3\}$ and $\{u_2, u_3, u_4\}$ are independent sets, and so (34) gives

$$d_1(u_1) + d_1(v_2) + d_1(v_3) \geq n_1 + 1,$$
$$d_1(u_2) + d_1(u_3) + d_1(u_4) \geq n_1 + 1,$$

and so there are at least $2n_1 + 2$ incidences of edges of G_1 with vertices of S. By (43) and the definition of Y', at most five edges incident with S have been counted twice: u_2v_2, u_2v_3, u_1u_3, u_3v_3, and possibly v_2u_4. Thus,

$$|E(G_1)| \geq 2n_1 + 2 - 5 = 2n_1 - 3,$$

and so by Lemma 2, $G_1 \in \{K_1, K_2\}$, a contradiction. This completes Case 2, and Theorem 6 is proved. □

Let G be a connected graph of order n obtained from K_{n-3} by adding a path P of length 4, such that the ends of P, but not the internal vertices of P, are in the K_{n-3}. For any independent set $\{u, v, w\} \subseteq V(G)$,

$$d(u) + d(v) + d(w) = n,$$

and none of the conclusions of Theorem 6 holds. Hence, (33) is best-possible in Theorem 6. The graphs $K_{2,4}$ and G_c (see Fig. 1) also show that (33) is best-possible.

Corollary 3 (Benhocine, Clark, Köhler, and Veldman [3]). *Let G be a 2-edge-connected simple graph of order n. If*

$$d(u) + d(v) \geq \tfrac{1}{3}(2n + 3) \tag{44}$$

whenever $uv \notin E(G)$, then G has a spanning eulerian subgraph.

Proof. Since (44) implies (33), and since a collapsible graph has a spanning eulerian subgraph, Corollary 3 follows directly from Theorem 6. □

As Benhocine, Clark, Köhler, and Veldman state, Corollary 3 implies a result of Lesniak-Foster and Williamson (the case $p=2$ of Theorem 9).

A result of Veldman ([14], Theorem 3) is analogous to Theorem 6:

Theorem 7 (Veldman [14]). *Let G be a connected simple graph of order n. If*

$$d(u)+d(v)+d(w)\geqslant n-1$$

for every independent set $\{u, v, w\}\subseteq V(G)$, then G has a spanning trail (possibly open).

Define, for any edge $xy\in E(G)$,

$$d(xy)=|N(x)\cup N(y)|\,. \tag{45}$$

Corollary 4. *Let G be a simple graph of order n. If*

$$d(e_1)+d(e_2)+d(e_3)\geqslant 2n+2 \tag{46}$$

for every matching $\{e_1, e_2, e_3\}\subseteq E(G)$, then G satisfies a conclusion of Theorem 3.

Proof. Write $e_i=x_iy_i$, for $1\leqslant i\leqslant 3$. By (45),

$$d(e_i)\leqslant d(x_i)+d(y_i), \tag{47}$$

and so if $M_3=\{e_1, e_2, e_3\}$, then (47) and (46) give

$$\sum(M_3)\geqslant\sum_{i=1}^{3} d(e_i)\geqslant 2n+2,$$

and so either (e) of Theorem 3 holds, or the hypothesis of Theorem 4 holds. It is easy to show that (b) of Theorem 4 and (46) together imply (b) of Theorem 3. The corollary follows. □

Examples showing that Theorem 3 is best possible also show that (46) is best-possible.

Veldman [13, 14] has used hypotheses somewhat similar to (46) as sufficient conditions for G to have a cycle or trail that contains a vertex of every edge of G. (His definition of $d(xy)$ is slightly different than (45).) We shall state the result of his that is most analogous to Corollary 4. Two edges uv and wx are *remote* if the distance in G between $\{x, w\}$ and $\{u, v\}$ is at least 2.

Theorem 8 (Veldman [13], Corollary 3.2). *Let G be a simple 2-connected graph* *of order n. If*

$$d(e_1) + d(e_2) + d(e_3) \geq n + 5 \tag{48}$$

for every three mutually remote edges e_1, e_2, e_3, then G has a cycle that passes through at least one end of each edge of G.

We have obtained the following generalization of Corollary 3, to appear separately [9]:

Theorem 9. *Let G be a simple connected graph of order n, and let $p \geq 2$ be an integer. If*

$$d(u) + d(v) > \frac{2n}{p} - 2, \tag{49}$$

whenever $uv \notin E(G)$, and if n is sufficiently large compared to p, then exactly one of the following holds:

(a) *G has a spanning eulerian subgraph;*

(b) *G is contractible to a graph G_1 of order less than p, such that G_1 has no spanning eulerian subgraph;*

(c) *$p = 2$, and $G - x = K_{n-1}$ for some $x \in V(G)$ with $d(x) = 1$.*

The case $p = 2$ of Theorem 9 is a theorem of Lesniak-Foster and Williamson [11]. The case $p = 3$ is similar to Corollary 3. The case $p = 5$ was conjectured by Benhocine, Clark, Köhler, and Veldman [3]. In [8], we proved an analogous result with the hypothesis $\delta(G) \geq \frac{1}{5}n - 1$ in place of (49), thereby proving a conjecture of Bauer [1, 2]. The inequality (49) is best-possible.

Acknowledgement

The author is grateful for the referee's suggestions.

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