Double Cycle Covers and the Petersen Graph

Paul A. Catlin
DEPARTMENT OF MATHEMATICS
WAYNE STATE UNIVERSITY
DETROIT, MICHIGAN

ABSTRACT

Let $O(G)$ denote the set of odd-degree vertices of a graph G. Let $t \in \mathbf{N}$ and let $\mathcal{S}_t$ denote the family of graphs G whose edge set has a partition

$$E(G) = E_1 \cup E_2 \cup \cdots \cup E_t,$$

such that $O(G) = O(G[E_i])\ (1 \leq i \leq t)$. This partition is associated with a double cycle cover of G. We show that if a graph G is at most 5 edges short of being 4-edge-connected, then exactly one of these holds: $G \in \mathcal{S}_3$, G has at least one cut-edge, or G is contractible to the Petersen graph. We also improve a sufficient condition of Jaeger for $G \in \mathcal{S}_{2p+1}(p \in \mathbf{N})$.

1. INTRODUCTION

A *double cycle cover* of a graph G is a collection of cycles of G (multiplicities allowed) such that each edge of G is in exactly two cycles of the collection. The family of graphs with double cycle covers is closed under contraction, and a graph with a double cycle cover posesses no cut edge. In [6] and [7] a general reduction method was presented that can be applied in certain situations to determine whether a graph belongs to a given family of graphs that is closed under contraction. We use that reduction method to show that certain graphs have a particular type of double cycle cover.

Szekeres [21] and Seymour [20] conjectured that every graph with no cut edge has a double cycle cover. This is trivial for planar graphs: the collection of facial cycles forms a double cover of a planar graph. Jaeger [14] has written a survey article on this problem, and he notes that it is sufficient to prove the conjecture for 3-regular graphs. Goddyn [10] has shown that if a counterexample exists, then a smallest 3-regular counterexample has girth at least seven. Tarsi [22] proved the conjecture for graphs with a hamilton path. Alspach and Zhang [1] have recently proved this conjecture for 2-connected 3-regular graphs containing no subdivision of the Petersen graph (compare with Conjecture 1 below).

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We use the terminology of Bondy and Murty [3], except that a *graph* is presumed to have no loops, and we regard K_1 as having infinite edge-connectivity. A *contraction* of G is a graph G' obtained from G by contracting a set (possibly empty) of edges and deleting all resulting loops. If H is a connected subgraph of G, then G/H denotes the graph obtained from G by contracting the edges of $E(G/[V(H)])$.

An *elementary homomorphism* of a graph G is a graph G' obtained from G by identifying two vertices in the same component and by deleting any resulting loops. Note: this is not the usual definition of homomorphisms. A *homomorphism* of G is a graph obtained from G by a sequence (possibly empty) of elementary homomorphisms.

Let $\mathcal{S}$ be a family of graphs (also called a *family*). We say that $\mathcal{S}$ is *closed under contraction* (respectively, *closed under homomorphisms*) if for any $G \in \mathcal{S}$, every contraction (respectively, homomorphism) of G is in $\mathcal{S}$. Let $\mathcal{S}^R$ denote the family of graphs having no nontrivial connected subgraph in $\mathcal{S}$. Any graph in $\mathcal{S}^R$ is called *$\mathcal{S}$-reduced*. For a family $\mathcal{S}$ and a graph G, the graph G_0 is called an *$\mathcal{S}$-reduction* of G if $G_0 \in \mathcal{S}^R$ and G_0 is obtained from G by a sequence of contractions of subgraphs in $\mathcal{S}$. For example, if $\mathcal{S}$ = {2-edge-connected graphs} then $\mathcal{S}^R$ = {forests}. Reductions are used, with (2) of Theorem 2, to determine which graphs lie in a given family.

Theorem 1 [7]. If a family $\mathcal{T}$ is closed under homomorphisms, then every graph G has a unique $\mathcal{T}$-reduction.

If a family $\mathcal{T}$ is closed under homomorphisms and if G is a graph, then let $G/\mathcal{T}$ denote the unique $\mathcal{T}$-reduction of G.

For a family $\mathcal{S}$, we define the *kernel* of $\mathcal{S}$ to be the family of connected graphs

$$\mathcal{S}^O = \{H\text{: For every supergraph } G \text{ of } H,\ G \in \mathcal{S} \Leftrightarrow G/H \in \mathcal{S}\}. \qquad (1)$$

2. PRIOR RESULTS ON KERNELS

The kernel of a family is often just $\{K_1\}$, an uninteresting case in which the reduction method says nothing. However, in this paper we will consider certain families with large kernels. The basic reduction theorem is this:

Theorem 2 [7]. Let $\mathcal{S}$ and $\mathcal{T}$ be families such that $\mathcal{S}$ is closed under contraction and

$$\mathcal{T} \subseteq \mathcal{S}^O.$$

Let G be a graph and let G' be a $\mathcal{T}$-reduction of G. Then

$$G' \in \mathcal{S} \Leftrightarrow G \in \mathcal{S}. \qquad (2)$$

If $\mathcal{T}$ is also closed under homomorphisms, then the $\mathcal{T}$-reduction of G is unique.

Of course, (2) is a straightforward consequence of the definition of the kernel $\mathcal{S}^O$ and the fact that $\mathcal{T} \subseteq \mathcal{S}^O$. Only the last part of Theorem 2, which follows from Theorem 1, is nontrivial. The point of Theorem 2 is to simplify problems of characterizing graphs in $\mathcal{S}$ by restricting the problem to graphs in $\mathcal{T}^R$.

Define a family $\mathscr{C}$ of connected graphs to be *complete* if $\mathscr{C}$ satisfies these three axioms:

(C1) $K_1 \in \mathscr{C}$;
(C2) $\mathscr{C}$ is closed under contraction; and
(C3) $H \subseteq G, H \in \mathscr{C}, G/H \in \mathscr{C} \Rightarrow G \in \mathscr{C}$.

Proofs of the following theorems may be found in the indicated sources. Theorems 4 and 5 are quite easy.

Theorem 3 [6]. For any family $\mathcal{S}$ of graphs, closed under contraction, these are equivalent:

(a) $\mathcal{S}$ is a complete family,
(b) $\mathcal{S} = \mathcal{S}^O$, and
(c) $\mathcal{S}$ is the kernel of some family closed under contraction. ∎

Theorem 4 [6]. For any family $\mathcal{S}$, $\mathcal{S}^O \subseteq \mathcal{S} \Leftrightarrow K_1 \in \mathcal{S}$. ∎

Theorem 5 [6]. For any family $\mathcal{S}$, $\mathcal{S} \cap \mathcal{S}^R \subseteq \{K_1\}$. ∎

Theorem 6 [7]. Any complete family is closed under homomorphisms. ∎

3. PRIOR RESULTS ON EDGE-CONNECTIVITY

Let $\tau(G)$ be the maximum number of edge-disjoint spanning trees in G. The following theorem improves a result of Kundu [15] and Gusfield [11]. For $k = 2$, Zhan [24] proves the "$\Rightarrow$" case of this result:

Theorem 7 [6]. Let G be a nontrivial graph, let k be an integer at most $|E(G)|$, and let $\mathscr{E}_k$ be the collection of k-element subsets of $E(G)$. Then

$$\kappa'(G) \geq 2k \Leftrightarrow \forall E' \in \mathscr{E}_k, \tau(G - E') \geq k. \quad ∎$$

Let uv and vw be two edges of a graph G with $u \neq w$. Define the graph $G(uv, vw)$ to be obtained from $G - \{uv, vw\}$ by adding a new edge uw. We say that $\{uv, vw\}$ has been *lifted* to change G into $G(uv, vw)$.

Let uv and vw be two edges of a graph G, and suppose that $d(v) = 2$. If v and the two incident edges $\{uv, vw\}$ are deleted from G and replaced by the new edge uw, then we say that v has been *dissolved*.

For the vertices $x, y \in V(G)$, let $\kappa'_G(x, y)$ be the minimum cardinality of a set $E \subseteq E(G)$ such that x and y are in different components of $G - E$.

Theorem 8 (Mader's Lifting Theorem [17]). Let G be a graph and suppose $v \in V(G)$ has $d(v) \geq 4$, is not a cutvertex, and has at least two neighbors. Then there are edges e', e'' incident with v such that $G' = G(e', e'')$ satisfies

$$\kappa'_{G'}(x, y) = \kappa'_G(x, y) \quad \forall\, x, y \in V(G) - v . \quad \blacksquare$$

4. EXAMPLES

The following examples will be used throughout the paper. For a family $\mathcal{S}$ and an integer $k \in \mathbf{N} \cup \{0\}$, we say that a graph G is *at most k edges short* of being in $\mathcal{S}$ if $\mathcal{S}$ has a graph G' such that G can be obtained from G' by deleting at most k edges.

Let $t \in \mathbf{N}$. Denote

$$\mathcal{S}_{t,0} = \{G\colon \tau(G) \geq t\}$$

and

$$\mathcal{S}(t, 0) = \{G\colon \kappa'(G) \geq t\}.$$

For $k, t \in \mathbf{N}$, denote

$$\mathcal{S}_{t,k} = \{G\colon G \text{ is at most } k \text{ edges short of being in } \mathcal{S}_{t,0}\};$$
$$\mathcal{S}(t, k) = \{G\colon G \text{ is at most } k \text{ edges short of being in } \mathcal{S}(t, 0)\}.$$

We proved [6] that $\mathcal{S}_{t,0}$ equals the kernel $\mathcal{S}^O_{t,k}$ of $\mathcal{S}_{t,k}$, for any k and t, and that $\mathcal{S}^O(t, k) = \mathcal{S}(t, 0)$. Hong-Jian Lai [17] proved for any complete family $\mathcal{C}$, if $\mathcal{C}(k)$ is the family of graphs k edges short of being in $\mathcal{C}$, then $\mathcal{C}^O(k) = \mathcal{C}$.

For any graph H, let $O(H)$ denote the set of odd-degree vertices of H. Let G be a graph and let $V \subseteq V(G)$. We define a V-*join* to be a subset $E \subseteq E(G)$ such that

$$O(G[E]) = V.$$

Define $\mathcal{SL}$ to be the family of graphs having a spanning closed trail. We call $\mathcal{SL}$ the family of *supereulerian graphs*. We regard $K_1 \in \mathcal{SL}$. Define $\mathcal{CL}$ to be the family of graphs G such that for any even subset $V \subseteq V(G)$, G has a V-join E such that $G - E$ is connected. In [4] we proved that $\mathcal{CL}$ is contained in $\mathcal{SL}^O$, the kernel of $\mathcal{SL}$, and we proved

Lemma 1. The family $\mathcal{CL}$ is complete. $\blacksquare$

Theorem 9 [4]. If H is a nontrivial subgraph of a $\mathscr{CL}$-reduced graph G, then

$$|E(H)| \leq 2|V(H)| - 3\,, \tag{3}$$

with equality if and only if $H = K_2$. Also, $\mathscr{S}_{2,0} \subseteq \mathscr{CL}$. ∎

Theorem 9 is part (iv) of Theorem 8 of [4].

For $t \in \mathbf{N}$, let $\mathscr{S}_t$ denote the family of graphs G whose edge set has a partition

$$E(G) = E_1 \cup E_2 \cup \cdots \cup E_t\,,$$

such that each E_i $(1 \leq i \leq t)$ is an $O(G)$-join of G. By Theorems 7, 9, and 4, and by a result in [8] that $\mathscr{SL} \subseteq \mathscr{S}_3$, we have

$$\mathscr{S}(4,0) \subseteq \mathscr{S}_{2,0} \subseteq \mathscr{CL} \subseteq \mathscr{SL}^o \subseteq \mathscr{SL} \subseteq \mathscr{S}_3\,. \tag{4}$$

Lemma 2. The families $\mathscr{S}_t$, $\mathscr{S}(t,k)$ and $\mathscr{S}_{t,k}$ are closed under contraction.

Proof. This follows routinely from the respective definitions. ∎

To obtain our main result, we shall improve the following result:

Theorem 10 (Jaeger [11]). For all $p \in \mathbf{N}$, $\mathscr{S}_{2p,0} \subseteq \mathscr{S}_{2p+1}$, i.e., if a graph G has $2p$ edge-disjoint spanning trees, then $G \in \mathscr{S}_{2p+1}$.

For the graph H, if $O(H) = \emptyset$, then H is called an *even* graph. For $t \in \mathbf{N}$, let $E_1 \cup E_2 \cup \cdots \cup E_t$ be a partition of $E(G)$ into O(G)-joins, and define $E_{t+1} = E_1$. Then for $i \in \{1, 2, \ldots, t\}$, $G[E_i \cup E_{i+1}]$ is an even subgraph of G. Thus, if $t = 3$ (i.e., if $G \in \mathscr{S}_3$) then three $O(G)$-joins $\{E_1, E_2, E_3\}$ induce a double cycle cover $G[E_1 \cup E_2], G[E_1 \cup E_3], G[E_2 \cup E_3]$ of G, consisting of three even subgraphs. Conversely, a double cycle cover of G that consists of three even subgraphs induces a partition of $E(G)$ into three $O(G)$-joins. Our main result is a condition sufficient for $G \in \mathscr{S}_3$.

Tutte [23] conjectured that if a 3-regular graph G is not in $\mathscr{S}_3$ (i.e., if $\chi'(G) = 4$ and G is 3-regular), then G has a subgraph contractible to the Petersen graph. Matthews [18] extended this conjecture by dropping the hypothesis of regularity:

Conjecture 1 [18,23]. If $G \notin \mathscr{S}_3$ then G has a cut-edge or G has a subgraph contractible to the Petersen graph.

Theorem 14 confirms Conjecture 1 whenever $G \in \mathscr{S}(4,5)$.

Celmins [9] and Preismann [19] conjectured that any graph G with no cut-edge contains a collection of at most five even subgraphs (multiplicities allowed) such that each edge of G is in exactly two of the even subgraphs. Since an even graph is an edge-disjoint union of cycles (Euler's Theorem), this is a refinement of the double cycle cover conjecture.

5. THE KERNEL OF $\mathcal{S}_{2p+1}$

Before proving the main result, we need to find a large subfamily of the kernel $\mathcal{S}_3^O$ of $\mathcal{S}_3$. In the process, we shall also improve Theorem 10, due to Jaeger [13], which gives a sufficient condition for $G \in \mathcal{S}_{2p+1}$, where $p \in \mathbf{N}$.

Let $k \in \mathbf{N}$. Denote by $\mathcal{C}_{k+1}$ the family of graphs H with the property that for any k even subsets $V_1, V_2, \ldots, V_k \subseteq V(H)$, there are disjoint sets $E_1, E_2, \ldots, E_k \subseteq E(H)$ such that $O(H[E_i]) = V_i (i = 1, 2, \ldots, k)$, i.e., such that E_i is a V_i-join in H. The family of connected graphs is just $\mathcal{C}_2$.

Theorem 11. For any $k \in \mathbf{N}$, the family $\mathcal{C}_{k+1}$ is complete.

Proof. Let $k \in \mathbf{N}$. It is trivial that $K_1 \in \mathcal{C}_{k+1}$, and it is routine to show that $\mathcal{C}_{k+1}$ is closed under contraction. Thus, to prove that $\mathcal{C}_{k+1}$ is complete, it remains to prove axiom (C3) for $\mathcal{C}_{k+1}$.

Let $H \in \mathcal{C}_{k+1}$. Suppose G is a supergraph of H such that $G/H \in \mathcal{C}_{k+1}$.

Now let $X_1, X_2, \ldots, X_k$ be k even subsets of $V(G)$. Denote the vertex of G/H corresponding to H by v_H and define, for $i \in \{1, 2, \ldots, k\}$,

$$X_i/H = \begin{cases} X_i - V(H) & \text{if } |V(H) \cap X_i| \text{ is even;} \\ X_i \cup \{v_H\} - V(H) & \text{if } |V(H) \cap X_i| \text{ is odd.} \end{cases} \tag{5}$$

Since $G/H \in \mathcal{C}_{k+1}$, there are k disjoint sets $F_1, F_2, \ldots, F_k \subseteq E(G) - E(H)$ such that

$$O((G/H)[F_i]) = X_i/H \quad (1 \le i \le k)\,. \tag{6}$$

Since $H \in \mathcal{C}_{k+1}$ there are disjoint sets $E_1, E_2, \ldots, E_k \subseteq E(H)$ such that

$$O(H[E_i]) = (O(G[F_i]) \cap V(H)) \,\Delta (V(H) \cap X_i)\,, \tag{7}$$

where Δ denotes the symmetric difference. By (7),

$$\begin{aligned} O(G[E_i \cup F_i]) \cap V(H) &= (O(G[E_i]) \,\Delta O(G[F_i])) \cap V(H) \\ &= (O(H[E_i]) \,\Delta O(G[F_i])) \cap V(H) \\ &= V(H) \cap X_i\,. \end{aligned} \tag{8}$$

By $O(G[E_i]) \subseteq V(H)$, by (6), and by (5),

$$\begin{aligned} O(G[E_i \cup F_i]) - V(H) &= O(G[E_i])\,\Delta O(G[F_i]) - V(H) \\ &= O(G[F_i]) - V(H) \\ &= O((G/H)[F_i]) - v_H \\ &= X_i/H - v_H = X_i - V(H)\,. \end{aligned} \tag{9}$$

By (8) and (9),

$$O(G[E_i \cup F_i]) = X_i \quad (1 \leq i \leq k)\,,$$

and since $X_1, X_2, \ldots, X_k$ are arbitrary even subsets of $V(G)$, $G \in \mathscr{C}_{k+1}$. Therefore, $\mathscr{C}_{k+1}$ is complete. ∎

Theorem 12. For any $p \in \mathbf{N}$, $\mathscr{C}_{2p+1} \subseteq \mathscr{S}^{O}_{2p+1}$.

Proof. Let $p \in \mathbf{N}$. Let $H \in \mathscr{C}_{2p+1}$ and let G be a supergraph of H. It is routine to prove that if $G \in \mathscr{S}_{2p+1}$ then $G/H \in \mathscr{S}_{2p+1}$. Thus, to prove $H \in \mathscr{S}^{O}_{2p+1}$, we must assume that $G/H \in \mathscr{S}_{2p+1}$ and prove that $G \in \mathscr{S}_{2p+1}$.

Suppose

$$G/H \in \mathscr{S}_{2p+1}\,.$$

Then we have a partition

$$E(G/H) = F_1 \cup F_2 \cup \cdots \cup F_{2p+1}$$

such that

$$O((G/H)[F_i]) = O(G/H) \quad (1 \leq i \leq 2p+1)\,.$$

Define

$$W_i = O(G[F_i]); \quad (1 \leq i \leq 2p+1)\,, \tag{10}$$

and set

$$V_i = W_i \Delta O(G) \quad (1 \leq i \leq 2p+1)\,. \tag{11}$$

Then

$$V_i \subseteq V(\mathrm{H})\,,$$

since any vertex of $V(G) - V(H)$ is in $W_i = O(G[F_i])$ if and only if it is in $O((G/H)[F_i]) = O(G/H)$.

Since $H \in \mathscr{C}_{2p+1}$, there is a partition

$$E(H) = E_1 \cup E_2 \cup \cdots \cup E_{2p+1}$$

such that E_i is a V_i-join in H $(1 \le i \le 2p)$. Hence by (10) and (11),

$$O(G[E_i \cup F_i]) = O(G[E_i])\,\Delta O(G[F_i]) = V_i\,\Delta W_i = O(G)\,,$$

for $1 \le i \le 2p$, and it follows that $O(G[E_{2p+1} \cup F_{2p+1}]) = O(G)$.

Hence, $G \in \mathscr{S}_{2p+1}$. Since G is an arbitrary supergraph of H such that $G/H \in \mathscr{S}_{2p+1}$, we have $H \in \mathscr{S}^O_{2p+1}$, and Theorem 12 is proved. ∎

Corollary 12A. For all $p \in \mathbf{N}$,

$$\mathscr{S}_{2p,0} \subseteq \mathscr{C}_{2p+1} \subseteq \mathscr{S}^O_{2p+1} \subseteq \mathscr{S}_{2p+1}\,.$$

Proof. Let $G \in \mathscr{S}_{2p,0}$. Then G has $2p$ edge-disjoint spanning trees, say $T_1, T_2, \ldots, T_{2p}$. Let $V_1, V_2, \ldots, V_{2p}$ be $2p$ even subsets of $V(G)$. Each tree T_i contains a V_i-join $(1 \le i \le 2p)$, and since these V_i-joins are edge-disjoint, $G \in \mathscr{C}_{2p+1}$. Hence $\mathscr{S}_{2p,0} \subseteq \mathscr{C}_{2p+1}$, and by Theorems 12 and 4,

$$\mathscr{C}_{2p+1} \subseteq \mathscr{S}^O_{2p+1} \subseteq \mathscr{S}_{2p+1}\,. \quad ∎$$

Jaeger [12] proved that $\mathscr{S}(4,0) \subset \mathscr{S}_{2,0} \subset \mathscr{SL}$, and in [4], [6], and [8] (combined), we improved that by proving (4). Corollary 12A is a similar improvement of an analogous result of Jaeger (Theorem 10).

Theorem 13. The kernel of $\mathscr{S}_3$ contains the four-cycle.

Proof. By Lemma 2, $\mathscr{S}_3$ is closed under contraction, and so it suffices to show that for a four-cycle H and for any supergraph G of H,

$$G/H \in \mathscr{S}_3 \Rightarrow G \in \mathscr{S}_3.$$

Let H be the four-cycle $wxyzw$, and suppose

$$G/H \in \mathscr{S}_3\,.$$

Then there is a partition

$$E(G/H) = F_1 \cup F_2 \cup F_3\,,$$

such that $O((G/H)[F_i]) = O(G/H)\,(1 \le i \le 3)$. Hence,

$$G/H - F_i \text{ is an even graph } (1 \le i \le 3).$$

Define

$$U_i = O(G - E(H) - F_i) \quad (1 \le i \le 3).$$

Hence, $U_i \subseteq V(H)$. We have

$$O(G[F_i]) = U_i \Delta O(G) \quad (1 \le i \le 3), \tag{12}$$

and since $|O(G - E(H) - F_i)|$ is even,

$$|U_i| \text{ is even} \quad (1 \le i \le 3). \tag{13}$$

Case 1. Suppose that H contains a U_1-join E_1 and a U_2-join E_2 such that $E_1 \cap E_2 = \emptyset$. For $i \in \{1,2\}$, (12) gives

$$O(G[E_i \cup F_i]) = O(G[E_i]) \Delta O(G[F_i]) = U_i \Delta U_i \Delta O(G) = O(G).$$

It follows that $O(G[E(G) - (E_1 \cup F_1) - (E_2 \cup F_2)]) = O(G)$, and so $G \in \mathcal{S}_3$.

Case 2. Suppose Case 1 does not apply. Check that neither U_1 nor U_2 can be empty; check that no U_i ($i = 1,2$) consists of two adjacent vertices of H; and check that if U_1 and U_2 are equal even sets, then Case 1 applies. Hence by (13) and by symmetry on the four-cycle $H = wxyzw$, we are left with two possibilities:

(a) $U_1 = V(H)$ and $U_2 = \{w, y\}$;
(b) $U_1 = \{x, z\}$ and $U_2 = \{w, y\}$.

Since $G - E(H) - F_3 = G[F_1 \cup F_2]$, (12) gives

$$\begin{aligned} U_3 &= O(G - E(H) - F_3) \\ &= O(G[F_1 \cup F_2]) = O(G[F_1]) \Delta O(G[F_2]) \\ &= U_1 \Delta U_2 . \end{aligned} \tag{14}$$

If (b) holds, then (14) would imply $U_3 = V(H)$, and by a change of subscripts, this is equivalent to (a). Thus, without loss of generality, we assume that (a) holds and hence by (14) that

$$U_3 = \{x, z\}.$$

By (12) and (a), $O(G - F_1) = U_1 = V(H)$. Since each component of $G - E(H) - F_1$ has evenly many odd-degree vertices, all in $V(H)$, the component G_w (say) of $G - E(H) - F_1$ containing w contains one or three members of $\{x, y, z\}$.

Subcase 2A. Suppose $y \in V(G_w)$. Then there is a (w, y)-path P (say) in $G_w \subseteq G - E(H) - F_1$. Thus, by (12),

$$\begin{aligned} O(G[F_1 \Delta\{wx, yz\}]) &= O(G[F_1]) \Delta\{w, x, y, z\} \\ &= U_1 \Delta O(G) \Delta V(H) = O(G); \\ O(G[F_2 \Delta E(P)]) &= O(G[F_2]) \Delta\{w, y\} \\ &= U_2 \Delta O(G) \Delta\{w, y\} = O(G); \\ O(G[F_3 \Delta E(P) \Delta\{wz, xy\}]) &= O(G[F_3]) \Delta\{w, y\} \Delta\{w, x, y, z\}) \\ &= U_3 \Delta O(G) \Delta\{x, z\} = O(G). \end{aligned}$$

Also,

$$(F_1 \Delta\{wx, yz\}) \cup (F_2 \Delta E(P)) \cup (F_3 \Delta E(P) \Delta\{wz, xy\})$$

is a partition of $E(G)$, and hence it is an $O(G)$-join of G.

Subcase 2B. Suppose $x \in V(G_w)$ (by symmetry, this is similar to the case $z \in V(G_w)$). Then there is a (w, x)-path P (say) in $G_w \subseteq G - E(H) - F_1$. By (12), as in Subcase 2A,

$$\begin{aligned} O(G[F_1 \Delta\{wx, yz\}]) &= O(G); \\ O(G[F_2 \Delta E(P) \Delta\{xy\}]) &= O(G[F_2]) \Delta\{w, y\} = O(G); \\ O(G[F_3 \Delta E(P) \Delta\{wz\}]) &= O(G[F_3]) \Delta\{x, z\} = O(G). \end{aligned}$$

As before, we have an $O(G)$-join in G, formed by the partition

$$(F_1 \Delta\{wx, yz\}) \cup (F_2 \Delta E(P) \Delta\{xy\}) \cup (F_3 \Delta E(P) \Delta\{wz\}).$$

In both subcases, we have $G \in \mathcal{S}_3$, as desired.
Therefore, $H \in \mathcal{S}_3^O$. ▮

Corollary 13A. Both $\mathcal{CL} \subseteq \mathcal{C}_3$ and

$$\mathcal{C}_3 \cup \{C_4\} \subseteq \mathcal{S}_3^O \subseteq \mathcal{S}_3.$$

Proof. Suppose that $G \in \mathcal{CL}$ and let V_1 and V_2 be even subsets of $V(G)$. Since $G \in \mathcal{CL}$, there is a V_1-join E_1 in G such that $G - E_1$ is connected. But $G - E_1$ thus has a V_2-join, and so $G \in \mathcal{C}_3$. Therefore, $\mathcal{CL} \subseteq \mathcal{C}_3$. The latter part of Corollary 13A comes from Theorems 12 and 13. ▮

It can be shown that each containment in Corollary 13A is strict. For example, $Q_3 - v$ (the cube minus a vertex) is in $\mathcal{C}_3 - \mathcal{CL}$.

6. THE MAIN RESULT

Theorem 14. Let G be a graph. If G is at most 5 edges short of being 4-edge-connected, then exactly one of the following holds:

(a) $G \in \mathcal{S}_3$;
(b) G has at least one cut-edge;
(c) The $(\mathcal{CL} \cup \{C_4\})$-reduction of G is the Petersen graph (G is contractible to the Petersen graph).

Proof. It is easy to check that the conclusions are mutually exclusive, since the Petersen graph is not in $\mathcal{S}_3$.

By Corollary 13A, $\mathcal{CL} \cup \{C_4\} \subset \mathcal{S}_3^O$. Hence, we can define

$$\mathcal{T} = \mathcal{CL} \cup \{C_4\},$$

knowing that $\mathcal{T} \subseteq \mathcal{S}_3^O$ in Theorem 2. By Lemma 1, $\mathcal{CL}$ is complete, and so by Theorem 6, $\mathcal{CL}$ is closed under homomorphisms. Since every elementary homomorphism of C_4 is in $\mathcal{CL}$, $\mathcal{T}$ is thus closed under homomorphisms. By Lemma 2, $\mathcal{S}_3$ is closed under contractions, and so $\mathcal{T}$ and $\mathcal{S}_3$ satisfy the hypothesis of Theorem 2. Hence, the $\mathcal{T}$-reduction of any graph G is unique, and by (2),

$$G \in \mathcal{S}_3 \Leftrightarrow G/\mathcal{T} \in \mathcal{S}_3 .$$

By way of contradiction, suppose that Theorem 14 is false, and let G be a counterexample with the fewest edges possible. Thus, we suppose that

$$G \in \mathcal{S}(4,5)$$

and that (a), (b), and (c) of Theorem 14 are false. By Lemma 2, $\mathcal{S}(4,5)$ and $\mathcal{S}_3$ are closed under contraction, and so if G is a counterexample, then so is $G/\mathcal{T}$. By the minimality of G, therefore,

$$G \text{ is } \mathcal{T}\text{-reduced}. \tag{15}$$

This means that

$$G \text{ has no subgraph in } \mathcal{CL} \cup \{C_4\}. \tag{16}$$

Also, (a), (b), and (c) of Theorem 14 are false, i.e.,

$$G \notin \mathcal{S}_3 \quad \text{and} \quad G \notin \{\text{Petersen graph}\} \tag{17}$$

and $\kappa'(G) \neq 1$. If $\kappa'(G) \geq 4$ then $G \in \mathcal{S}(4,0) \subseteq \mathcal{S}_{2,0} \subseteq \mathcal{CL}$, by (4). But since (16) says $G \in \mathcal{CL}^R$, Theorem 5 gives $G = K_1$, and so $G \in \mathcal{S}_3$, contrary to (17). If G is disconnected, then each component is in $\mathcal{S}(4,5)$ and thus satisfies a conclusion of Theorem 14, by the minimality of G. But then G satisfies Theorem 14, too. Hence,

$$\kappa'(G) \in \{2,3\}. \tag{18}$$

Lemma 3. Let G be a graph, let $v \in V(G)$, and let e' and e'' be distinct edges incident with v. If $G(e',e'') \in \mathcal{S}_3$, then $G \in \mathcal{S}_3$. ∎

This lemma is routine and its proof is omitted. (Recall that $G(e',e'')$ is the graph obtained from G when e' and e'' are lifted.)

Lemma 4. If G is a $\mathcal{T}$-reduced graph with $\delta(G) \geq 3$, then the order of G is at least $1 + 3\Delta(G)$. If also $\Delta(G) = 3$ and if G has order at most 10, then G is the Petersen graph.

Proof. Let G be a graph with $\delta(G) \geq 3$ and let v be a vertex with $d(v) = \Delta(G)$. Let $X_t = X_t(v)$ denote the set of vertices of G at distance t from v, where $t \geq 0$. Since G is $\mathcal{T}$-reduced, G has girth at least 5, by (16). Hence, each vertex of X_t is connected to v by a unique path of length t, when $t \in \{1,2\}$. Therefore, $|X_1| = d(v) = \Delta(G)$, and $\delta(G) \geq 3$ implies $|X_2| \geq 2|X_1|$. Hence,

$$|V(G)| \geq |X_0| + |X_1| + |X_2| \geq 1 + 3|X_1| = 1 + 3\Delta(G).$$

Now suppose that $\Delta(G) = 3$ and that G has order at most 10. By the inequality above, G has order exactly 10, and since G has no cycle of length less than 5, it is routine to show that equality can only hold when $G[X_2]$ is a 6-cycle such that G is the Petersen graph. ∎

Proof of Theorem 14, continued. By the hypothesis $G \in \mathcal{S}(4,5)$, we have

$$|E(G)| \geq 2n - 5, \tag{19}$$

where n is the order of G.

Since $G \neq K_2$ and by (19) and Theorem 9,

$$4n - 10 \leq \sum_{v \in V(G)} d(v) \leq 4n - 8. \tag{20}$$

For $k \in \mathbf{N} \cup \{0\}$, define

$$V_k = \{v \in V(G) | d(v) = k\}.$$

We claim that

$$V_0 \cup V_1 \cup V_2 = \varnothing . \tag{21}$$

By (18), $V_0 \cup V_1 = \varnothing$. Suppose $v \in V_2$ and let $e \in E(G)$ be incident with v. By Lemma 2, $G \in \mathcal{S}(4,5)$ implies $G/e \in \mathcal{S}(4,5)$. Thus, since G is a smallest counterexample, either $G/e \in \mathcal{S}_3$ (in which case $G \in \mathcal{S}_3$, a contradiction); or $\kappa'(G/e) = 1$ (whence $\kappa'(G) = 1$, a contradiction); or G/e is the Petersen graph (in which case $G \notin \mathcal{S}(4,5)$, a contradiction). This proves (21).

Since (21) implies $\delta(G) \geq 3$ we have

$$\sum_{v \in V(G)} d(v) = 4n - 8 \Rightarrow |V_3| \geq 8 \tag{22}$$

and

$$\sum_{v \in V(G)} d(v) = 4n - 10 \Rightarrow |V_3| \geq 10 . \tag{23}$$

By $G \in \mathcal{S}(4,5)$,

$$|V_3| \leq 10 . \tag{24}$$

Since $G \in \mathcal{S}(4,5)$ by hypothesis, it is possible to add a set E_5 (say) of 5 edges to G such that $\kappa'(G + E_5) \geq 4$.

Let ∂E_5 denote the set of vertices of $V(G)$ incident with an edge of E_5. Since $\Sigma d(v)$ is even, (20) implies that $\Sigma d(v) \in \{4n - 8, 4n - 10\}$, and so either (22) or (23) applies. Hence, by (22), (23), and (24),

$$8 \leq |V_3| \leq 10 ,$$

and since $\kappa'(G + E_5) \geq 4$, we must have $V_3 \subseteq \partial E_5$ and hence

$$|\partial E_5 \cap (V(G) - V_3)| = |\partial E_5| - |V_3| \leq 10 - 8 = 2 . \tag{25}$$

We now give a proof (several pages long) that

$$V_4 - \partial E_5 = \varnothing . \tag{26}$$

By way of contradiction, suppose G has a vertex $v \in V_4 - \partial E_5$. Since G is a minimum counterexample to Theorem 14, v is not a cutvertex. Let $\{e_1, e_2, e_3, e_4\}$ denote the set of four edges of $E(G)$ incident with v. It follows from (16) that G is simple, and so we can apply Theorem 8 at v. Since $\kappa'(G + E_5) \geq 4$, Mader's Lifting Theorem (Theorem 8) asserts that $E(G)$ has two edges (e_1 and e_3, say) incident with v that can be lifted such that if G' denotes $G(e_1, e_3)$ then

$$\kappa'_{G'+E_5}(x,y) = \kappa'_{G+E_5}(x,y) \geq 4 \qquad (\forall x, y \in V(G) - v). \tag{27}$$

In G', $d(v) = 2$ and v is incident with e_2 and e_4. Let e_{13} denote the edge of G' created when $e_1, e_3 \in E(G)$ are lifted. Denote by G_0 the graph obtained from G' by dissolving v, and let e_{24} denote the edge of $E(G_0)$ thus created to replace e_2 and e_4 and v. Thus, $e_{13}, e_{24} \in E(G_0)$.

By (27) and the definition of G_0, we have

$$\kappa'(G_0 + E_5) \geq 4.$$

Case A. Suppose $\kappa'(G_0) \geq 2$, i.e., $G_0 \in \mathcal{S}(2,0)$. Since $G_0 \in \mathcal{S}(4,5)$ and since G is a smallest counterexample in $\mathcal{S}(4,5) \cap \mathcal{S}(2,0)$, either $G_0 \in \mathcal{S}_3$ or the $\mathcal{T}$-reduction of G_0 is the Petersen graph.

If $G_0 \in \mathcal{S}_3$ then $G' \in \mathcal{S}_3$, and by Lemma 3 (with $e' = e_1$ and $e'' = e_3$), $G \in \mathcal{S}_3$, contrary to (17).

Hence, $G_0/\mathcal{T}$ is the Petersen graph. Since $G_0 + E_5 \in \mathcal{S}(4,0)$ and since $\mathcal{S}(4,0)$ is closed under contraction (by Lemma 2),

$$(G_0 + E_5)/\mathcal{T} \in \mathcal{S}(4,0),$$

and so each vertex of the Petersen graph $G_0/\mathcal{T}$ is incident with exactly one edge of E_5. Let $H_1, H_2, \ldots, H_{10}$ denote the ten maximal subgraphs of G_0 that lie in $\mathcal{T}$ and that are each contracted to obtain from G_0 the ten vertices of $G_0/\mathcal{T}$. We have

$$|V(H_i) \cap \partial E_5| = 1 \quad (1 \leq i \leq 10), \tag{28}$$

and each H_i is incident with exactly three edges of $G_0/\mathcal{T}$ that have exactly one end in $V(H_i)$. Each H_i not containing e_{13} or e_{24} is a subgraph of G, and thus is $\mathcal{T}$-reduced, by (15) and since all subgraphs of a reduced graph are reduced. By Theorem 5, $\mathcal{T} \cap \mathcal{T}^R = \{K_1\}$, and so such a subgraph H_i (not containing e_{13} or e_{24}) is K_1.

Let $E = E(G_0/\mathcal{T})$. Then $E \subseteq E(G_0)$.

Subcase A1. Suppose $e_{13}, e_{24} \in E$. Then each H_i $(1 \leq i \leq 10)$ is a K_1, by our prior remarks, and so G_0 is the Petersen graph. By the construction of G_0 from G, the graph G has order 11, and $\delta(G) = 3$, and G has a single vertex v of degree $\Delta(G) = 4$. But this violates Lemma 4, since G is $\mathcal{T}$-reduced.

Subcase A2. Suppose that some H_i $(1 \leq i \leq 10)$ contains exactly one member of $\{e_{13}, e_{24}\}$, say $E(H_1) \cap \{e_{13}, e_{24}\} = \{e_{13}\}$. Since the Petersen graph $G_0/\mathcal{T}$ is 3-regular, there are exactly three edges of G_0 with just one end in $V(H_1)$. By (28), $|V(H_1) \cap \partial E_5| = 1$, and so there are exactly four edges of $G_0 + E_5$ with exactly one end in $V(H_1)$. By $\kappa'(G_0 + E_5) \geq 4$,

$$\sum_{v \in V(H_1)} d_{G_0+E_5}(v) \geq 4|V(H_1)|$$

and so

$$\sum_{v \in V(H_1)} d_{H_1}(v) \geq 4|V(H_1)| - 4 .$$

Hence, $|E(H_1)| \geq 2|V(H_1)| - 2$. Let H_1' denote the graph obtained from H_1 by replacing e_{13} with $\{e_1, e_3, v_{13}\}$ so that H_1' is the corresponding subgraph of G' and of G (where $v = v_{13}$). Then

$$|E(H_1')| \geq 2|V(H_1')| - 3 .$$

But this violates Theorem 9, for since G is $\mathcal{T}$-reduced, so is its subgraph H_1'.

Subcase A3. Suppose that some H_i ($1 \leq i \leq 10$), say $H_i = H$, contains both e_{13} and e_{24}. Since the Petersen graph $G_0/\mathcal{T}$ is 3-regular, there are exactly three edges of G_0 with exactly one end in $E(H)$. By (28), $|V(H) \cap \partial E_5| = 1$, and so there are exactly four edges of $G_0 + E_5$ with exactly one end in $V(H)$. Hence, $\kappa'(G_0 + E_5) \geq 4$ gives

$$\sum_{v \in V(H)} d_{G_0+E_5}(v) \geq 4|V(H)|$$

and so

$$\sum_{v \in V(H)} d_H(v) \geq 4|V(H)| - 4 .$$

Thus, $|E(H)| \geq 2|V(H)| - 2$. Let H'' be the graph obtained from H by replacing $\{e_{13}, e_{24}\}$ with $\{e_1, e_2, e_3, e_4, v\}$, so that H'' is the subgraph of G corresponding to H. Then $|E(H'')| \geq 2|V(H'')| - 2$. This violates Theorem 9, for H'' is $\mathcal{T}$-reduced, since it is a subgraph of the $\mathcal{T}$-reduced graph G.

These subcases exhaust the possibilities and always yield contradictions, and so Case A is complete.

Case B. Suppose $\kappa'(G_0) < 2$. By (18), $\kappa'(G) \in \{2, 3\}$.

First we dispose of the case $\kappa'(G) = 2$. Let $\{e, e'\}$ be a 2-edge-cutset of G, and let G_1 and G_2 denote the components of $G - \{e, e'\}$. Define $n_i = |V(G_i)|$, for $i = 1, 2$. By (21), $\delta(G) \geq 3$, and so $G_i \notin \{K_1, K_2\}$. Hence by Theorem 9,

$$|E(G_i)| \leq 2n_i - 4 \qquad (i = 1, 2) .$$

By this and (19),

$$\begin{aligned}2n - 5 \le |E(G)| &= |E(G_1)| + |E(G_2)| + |\{e, e'\}| \\ &\le (2n_1 - 4) + (2n_2 - 4) + 2 = 2n - 6\,,\end{aligned}$$

a contradiction. Therefore, we may assume $\kappa'(G) = 3$.

Since $\kappa'(G) = 3$ and $\kappa'(G_0) < 2$, the derivation of G_0 from G implies that G_0 has a cut-edge e (that is not a cut-edge of G), and that one component of $G_0 - e - \{e_{13}, e_{24}\}$ (say G_1) contains both ends of e_{13}, while the other component of $G_0 - e - \{e_{13}, e_{24}\}$ (say G_2) contains both ends of e_{24}. (See Figure 1.) For $i \in \{1, 2\}$, let $G_i(v)$ denote $G[V(G_i) \cup \{v\}]$. In $G_i(v)$, v is incident with e_i and e_{i+2} only. By Lemma 2, $\mathcal{S}(4, 5)$ is closed under contraction, and so

$$G/G_i(v) \in \mathcal{S}(4,5) \qquad (i = 1, 2)$$

and by the minimality of G, both $G/G_1(v)$ and $G/G_2(v)$ satisfy a conclusion of Theorem 14.

By (18) and since $\mathcal{S}(2, 0)$ is closed under contraction (Lemma 2), neither $G/G_1(v)$ nor $G/G_2(v)$ has a cut-edge. We claim that neither has the Petersen graph as a $\mathcal{T}$-reduction, either.

Let $i \in \{1, 2\}$. By way of contradiction, suppose that the $\mathcal{T}$-reduction of $G/G_i(v)$ is the Petersen graph, and let E_{15} be the set of 15 edges of this Petersen graph. Then $E_{15} \subseteq E(G)$ and $G - E_{15}$ consists of 10 components, say $H_1, H_2, \ldots, H_{10}$, one of which contains $G_i(v)$. By (15) and since subgraphs of reduced graphs are reduced, Theorem 9 implies

$$|E(H_j)| \le 2|V(H_j)| - 2 \qquad (1 \le j \le 10)\,, \tag{29}$$

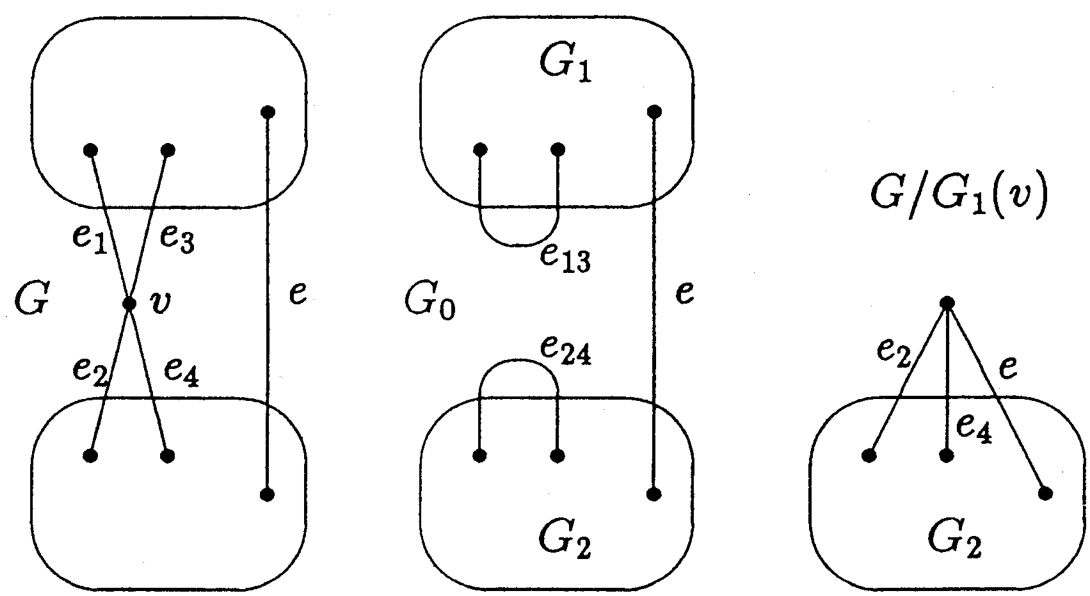

FIGURE 1. Three graphs of Case B.

with equality if and only if $H_j = K_1$. Hence, by (19) and (29),

$$2n - 5 \le |E(G)| = |E_{15}| + \sum_{j=1}^{10} |E(H_j)|$$

$$\le 15 + \sum_{j=1}^{10} (2|V(H_j)| - 2) = 2n - 5 ,$$

and so equality holds in (29) for $1 \le j \le 10$. But then the component H_j containing the nontrivial subgraph $G_i(v)$ is trivial, a contradiction.

Therefore, $G/G_1(v)$ and $G/G_2(v)$ must satisfy the first conclusion of Theorem 14: both are in $\mathcal{S}_3$ and both have 3-edge-colorings such that the union of any two color classes is an even graph. For $i \in \{1,2\}$, the vertex v has degree 3 in $G/G_i(v)$, and so the three edges incident with v (namely, $\{e_1, e_3, e\}$ and $\{e_2, e_4, e\}$ in $G/G_2(v)$ and $G/G_1(v)$, respectively) have different colors. These two edge-colorings can be joined (so that e has the same color in both) to give a 3-edge-coloring of $E(G)$ that proves $G \in \mathcal{S}_3$. This contradicts (17). Case B is complete and (26) is proved.

If G is 3-regular, then (24) and Lemma 4 imply that G is the Petersen graph and thus violates (17). Hence, $\Delta(G) \ge 4$. By (21), $\delta(G) \ge 3$. Set

$$W = \bigcup_{i=5}^{\infty} V_i .$$

By (20), (21), and (24), $|W| \le 2$. By (25) and (26), $|V_4| \le 2$. Since $|V_3| \le 10$, either $W = \emptyset$ and $|V(G)| = |V_3 \cup V_4| \le 12$, or $W \ne \emptyset$ and

$$|V(G)| = |V_3 \cup V_4 \cup W| \le 14 ,$$

and either case contradicts Lemma 4, for G is $\mathcal{T}$-reduced with $\delta(G) \ge 3$ by (15) and (21).

Since every case leads to a contradiction, there is no smallest counterexample G, and Theorem 14 is proved. ∎

7. SOME CONJECTURES

Conjecture 2. If $G \in \mathcal{S}_{2,2}$ then exactly one of the following holds:

(a) $G \in \mathcal{CL}$.
(b) The $\mathcal{CL}$-reduction of G is either $2K_1$ or K_2 or $K_{2,t}$ for some $t \in \mathbf{N}$.

Conjecture 3. If $G \in \mathcal{S}_{2,2}$ then exactly one of the following holds:

(a) $G \in \mathcal{S}_3^O$.
(b) The $\mathcal{S}_3^O$-reduction of G is $2K_1$ or K_2 or $K_{1,2}$.

Conjecture 3 would follow from Conjecture 2. Suppose $G \in \mathcal{S}_{2,2}$ and suppose Conjecture 2 is true. If $G \in \mathcal{CL}$, then (a) of Conjecture 3 holds, by Corollary 13A. Since $\mathcal{CL} \subset \mathcal{S}_3^O$ (Corollary 13A), the $\mathcal{S}_3^O$-reduction of G is obtained from a $\mathcal{CL}$-reduction of G, by contracting subgraphs in $\mathcal{S}_3^O$. If $G/\mathcal{CL} \in \{2K_1, K_{2,1}, K_2\}$, then $G/\mathcal{CL}$ is $\mathcal{S}_3^O$-reduced, but if $G/\mathcal{CL} = K_{2,t}$ $(t \geq 2)$, then from Theorem 13 we get $G/\mathcal{CL} \in \mathcal{S}_3^O$, whence $G \in \mathcal{S}_3^O$.

Conjecture 4. If $G \in \mathcal{S}_{2,3}$, then exactly one of the following holds:

(a) $G \in \mathcal{S}_3^O$.
(b) G has at least one cut-edge.
(c) G is contractible to the Petersen graph.

It follows from Theorem 7 that $\mathcal{S}(4,5) \subseteq \mathcal{S}_{2,3}$, and so the hypothesis of Conjecture 4 is weaker than the hypothesis of Theorem 14. By Theorem 4, $\mathcal{S}_3^O \subseteq \mathcal{S}_3$, and since this containment is proper (large cycles are not in $\mathcal{S}_3^O$), (a) of the conclusion of Conjecture 4 is stronger than (a) of Theorem 14.

The Blanuša snark [2] is a 3-regular graph of order 18 and girth 5 that does not satisfy any conclusion of Theorem 14. It is 9 edges short of being 4-edge-connected. We know of no graph at most 8 edges short of being 4-edge-connected that does not satisfy a conclusion of Theorem 14.

In [6], we conjectured that $\mathcal{CL} = \mathcal{SL}^O$. By Theorem 13, the analogous conjecture $\mathcal{C}_3 = \mathcal{S}_3^O$ is false, because $\mathcal{C}_3$ does not contain the four-cycle.

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