GRAPHS WITH UNIFORM DENSITY

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The **density** of a graph with e edges and v vertices is $e/(v-1)$. For example, the density of a tree is 1, and the density of the complete graph K_n is $n/2$. A graph which contains no subgraph with greater density is said to have **uniform density**. (Narayanan and Vartak [3] call such graphs "molecular," and Catlin, Hobbs, and Lai [2] have studied them extensively in the more general context of edge-toughness and fractional arboricity.) Examples of graphs with uniform density include trees, complete graphs, cycles (indeed, all edge-transitive graphs), plane triangulations, and K_4 minus an edge. On the other hand, the graph G consisting of the 6-cycle $v_0v_1v_2v_3v_4v_5v_0$ together with the diagonal v_0v_2 does not have uniform density, since the density of the induced subgraph on $\{v_0, v_1, v_2\}$ is $3/2$, which exceeds the density of G, namely $7/5$. It is not hard to see that a graph with at least one edge which has uniform density must be connected (and therefore has uniform density no less than 1).

In this paper we show that given any rational number $r \geq 1$, there exists a simple graph (*i.e.*, one having no parallel edges) with uniform density r.

We begin with an addition lemma for graphs with uniform density.

LEMMA 1. *Let G_1 and G_2 be graphs on the same vertex set, having no edges in common. If G_1 and G_2 have uniform densities d_1 and d_2, respectively, then $G = G_1 \cup G_2$ has uniform density $d_1 + d_2$.*

Proof. Clearly the density of G is $d_1 + d_2$. To see that G has uniform density, suppose H is a subgraph of G, with v vertices. For $i = 1, 2$, let $H_i = H \cap G_i$. Then the density of H is the sum of the densities of H_1 and H_2, which cannot exceed $d_1 + d_2$. ∎

For any integers m and n with $m - 2 \geq n \geq 2$, Catlin *et al.* [1] construct a graph $W(m,n)$ on $n+1$ vertices (a "broken wheel") as follows. Let C_n be the n-cycle $v_0v_1v_2\dots v_{n-1}v_0$ (the "rim"), and let v be a vertex not on C_n. Then $W(m,n)$ is the graph obtained from $C_n \cup \{v\}$ by adding $m-n$ edges of the form vv_k ("spokes"), where $k = \lfloor ni/(m-n) \rfloor$, for all integers i such that $0 \leq i < m-n$. For example, $W(8,5)$ is shown in Figure 1.

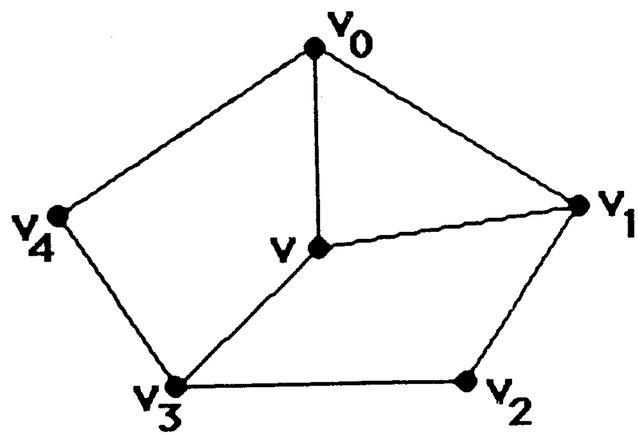

Figure 1. The broken wheel $W(8,5)$, a simple graph with uniform density $8/5$.

The following result is proved in [1].

LEMMA 2. *If $m-2 \geq n \geq 2$, then the graph $W(m,n)$ has uniform density m/n.* ∎

Note that if $m \leq 2n$ and $n \geq 3$, then $W(m,n)$ is a simple graph. This construction (together with the example of the cycles) gives us simple graphs with any specified uniform density between 1 and 2. In order to obtain simple graphs with uniform density greater than 2, we will add spanning trees to broken wheels, and apply Lemma 1.

THEOREM 1. *For every rational number $r \geq 1$, there exists a simple graph G with uniform density r.*

Proof. If r is an integer, then we take $G = K_{2r}$. If $1 < r < 2$, then $W(2m, 2n)$ is the desired simple graph, where $r = m/n$ (the factors of 2 are to insure that the hypothesis of Lemma 2 is met). Otherwise, we write $r = k + \alpha$, where k is a positive integer and $1 < \alpha < 2$. Our plan is to construct a simple graph with uniform density α and then add k pairwise edge-disjoint spanning trees.

Suppose $\alpha = m_0/n_0$. We choose n to be a multiple of n_0 so that (i) $n > k/(2-\alpha)$, (ii) the number of positive integers less than $n/2$ and relatively prime to n is greater than k, and (iii) $n > n_0$ if $m_0 = n_0 + 1$. (For example, we can choose n to be $2^c n_0$ for some sufficiently large c; the desired positive integers relatively prime to n are then just the odd primes which do not divide n_0.) Finally, we let $m = n\alpha$. By Lemma 2 the broken wheel $W(m,n)$ has uniform density α. We now add k edge-disjoint spanning trees of the complement of $W(m,n)$, obtained as follows. For each of k distinct integers s strictly between 1 and $n/2$ and relatively prime to n (the existence of which is guaranteed by the choice of n), we form the spanning tree of $\overline{W(m,n)}$ consisting of the path $v_0 v_s v_{2s} \ldots v_{(n-1)s}$ (subscripts read modulo n), together with a unique edge of the form vv_i (the requirement that $n > k/(2-\alpha)$ guarantees that k distinct values of i can be found). It follows from Lemma 1 (since a tree has uniform density 1) that the resulting graph has uniform density $\alpha + k = r$. ∎

Figure 2 shows a graph generated by the proof of Theorem 1 for $r = 13/5$. In this case one spanning tree was added to the graph in Figure 1.

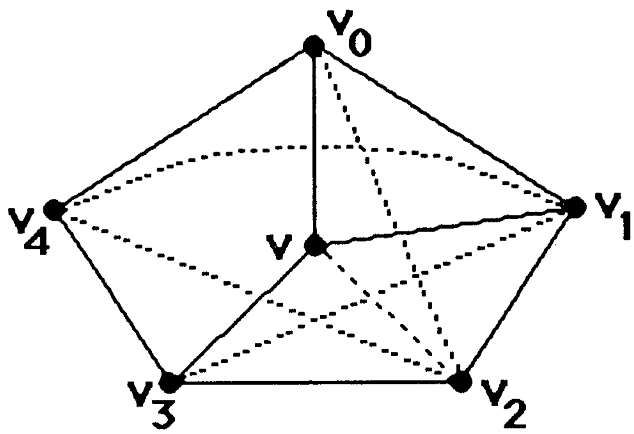

Figure 2. A simple graph with uniform density 13/5.

It is conjectured in [1] that the simple graph G guaranteed by Theorem 1 can be chosen to have $n+1$ vertices, whenever r can be written as the fraction m/n, where $n \leq m \leq \binom{n+1}{2}$. The techniques used in the proof of Theorem 1 can be extended to verify the conjecture in several cases, as we now show.

THEOREM 2. *Let m and n be positive integers, with $n \leq m \leq \binom{n+1}{2}$. Then there exists a simple graph with $n+1$ vertices and m edges having uniform density m/n in each of the following cases:*

(1) *m is a multiple of n;*

(2) *m is a multiple of $n+1$;*

(3) *m is one greater than a multiple of n;*

(4) *neither (1) nor (3) apply, and $k \leq 2n - m_0$, where we write $m/n = k + (m_0/n)$, with k a nonnegative integer and $n+1 < m_0 < 2n$.*

Proof. (1) Suppose m/n is an integer; *i.e.*, $m = tn$ for some positive integer t not exceeding $(n+1)/2$. It is well known that K_{n+1} can be written as the edge-disjoint union of $(n+1)/2$ spanning trees (each with n edges) if n is odd, or as the edge-disjoint union of $n/2$ spanning cycles (each with $n+1$ edges) if n is even. In the former case the union of t of these trees is the desired simple graph with uniform density m/n (the requirement that $m \leq \binom{n+1}{2}$ guarantees that $m/n \leq (n+1)/2$). In the latter case the union of t trees obtained by deleting a single edge from each of t of these cycles is the desired simple graph with uniform density m/n (again, the requirement that $m \leq \binom{n+1}{2}$ guarantees that $m/n \leq (n+1)/2$ and therefore, since n is even, that $m/n \leq n/2$).

(2) Suppose $m/(n+1)$ is an integer; *i.e.*, $m = t(n+1)$ for some positive integer t not exceeding $n/2$. As in part (1), if n is even, then K_{n+1} can be written as the edge-disjoint union of $n/2$ spanning cycles (each with $n+1$ edges), and the union of t of these cycles is the desired simple graph with uniform density m/n. If n is odd, then K_{n+1} can be written as the edge-disjoint union of $(n-1)/2$ spanning cycles (each with $n+1$ edges) and a 1-factor (with $(n+1)/2$ edges). The union of t of the cycles is the desired simple graph with uniform density m/n (the requirement that $m \leq \binom{n+1}{2}$ guarantees that $m/(n+1) \leq n/2$ and therefore, since n is odd, that $m/(n+1) \leq (n-1)/2$).

(3) Suppose $m = tn+1$ for some positive integer t less than $(n+1)/2$. If n is even, then $t \leq n/2$, and K_{n+1} is the edge-disjoint union of $n/2$ spanning cycles. The desired simple graph is the union of one of these cycles and the trees obtained from $t-1$ of the remaining cycles by deleting a single edge from each. If n is odd, then $t \leq (n-1)/2$, and K_{n+1} is the edge-disjoint union of $(n-1)/2$ spanning cycles and a 1-factor. The desired simple graph is again the union of one of these cycles and the trees obtained from $t-1$ of the remaining cycles by deleting a single edge from each.

(4) We begin by constructing the simple graph $W(m_0, n)$, the rim of which is the cycle $v_0v_1v_2 \ldots v_{n-1}v_0$. The complete graph K_n with vertex set $\{v_0, v_1, v_2, \ldots, v_{n-1}\}$ is the edge-disjoint union of $(n-1)/2$ spanning cycles (if n is odd), or is the edge-disjoint union of $n/2$ spanning trees *which in fact are paths* (if n is even). In the former case, we may assume that $v_0v_1v_2 \ldots v_{n-1}v_0$ is one of the cycles. From each of k of the $(n-3)/2$ other cycles we delete a single edge and adjoin to each a missing spoke of the broken wheel; the union of $W(m_0, n)$ and these k trees is the desired simple graph. In the latter case, we may assume that $v_0v_1v_2 \ldots v_{n-1}v_0$ is one of the paths, together with one edge from one of the other

paths. To each of k of the $(n-4)/2$ remaining paths we adjoin a unique missing spoke of the broken wheel; the union of $W(m_0, n)$ and these k trees is again the desired simple graph. In both cases the hypothesis that $k \leq 2n - m_0$ guarantees that the requisite number of spokes are missing from $W(m_0, n)$; and the hypothesis that $m \leq \binom{n+1}{2}$, together with the parity of n, leads to the required argument that there are enough cycles or paths available. ∎

COROLLARY. *Suppose m and n are positive integers with $n \leq m \leq \binom{n+1}{2}$. If $1 \leq m/n \leq 3$, then there exists a simple graph with $n+1$ vertices having uniform density m/n.*

Proof. If m equals n, $n+1$, $2n$, $2n+1$, or $3n$, then parts (1) or (3) of Theorem 2 apply. Otherwise, if $m/n < 2$, then $W(m, n)$ is the desired simple graph. In the remaining cases part (4) of Theorem 2 applies, with $k = 1$. ∎

REFERENCES

1. Paul A. Catlin, Kurt C. Foster, Jerrold W. Grossman, and Arthur M. Hobbs, "Graphs with specified edge-toughness and fractional arboricity," to appear.

2. Paul A. Catlin, Arthur M. Hobbs, and Hong-Jian Lai, "Matroid unions, edge-toughness, and fractional arboricity," to appear.

3. H. Narayanan and M. N. Vartak, "On molecular and atomic matroids," *Combinatorics and Graph Theory* (Calcutta, 1980), Lecture Notes in Mathematics **885** (1981), Springer-Verlag, 358–364.

ADDED IN PROOF:

The authors have discovered that Andrzej Ruciński and Andrew Vince have proved the conjecture mentioned before Theorem 2 ["Strongly Balanced Graphs and Random Graphs," *J. Graph Theory* **10** (1986), 251–264]. Their proof starts with the same construction as in [1] but proceeds differently from our proof of Theorem 2. This construction is also inherent in a paper by C. Payan ["Graphes Équilibrés et Arboricité Rationnelle," *Europ. J. Combinatorics* **7** (1986), 263–270]. Our earlier paper [1] has been withdrawn.