SUPEREULERIAN GRAPHS

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1. <u>Introduction</u>. We have developed a technique for determining whether a graph G is supereulerian (i.e., whether G has a spanning eulerian subgraph), and for studying the structure of G if it is not supereulerian.

We shall follow the notation of Bondy and Murty [4], with minor changes. The order of the graph G is denoted n, and the <u>arboricity</u> of G is a(G). As in [4], G may have multiple edges, the disjoint union of S and T is S Δ T, and $\omega(G)$ is the number of components of G.

For any connected subgraph H of G, the <u>contraction</u> G/H is the graph obtained by replacing H with a vertex denoted v_H, where each $w \in V(G - H)$ is joined in G/H to v_H by as many edges as join w and V(H) in G. When $S \subseteq V(G)$ and H is a subgraph of G, we define S/H to be $S - V(H)$ if $|S \cap V(H)|$ is even, and S/H is $(S - V(H)) \cup \langle v_H \rangle$ if $|S \cap V(H)|$ is odd.

An <u>S-forest</u> is a forest Γ in G such that both

i) $G - E(\Gamma)$ is connected; and

ii) S is the set of odd degree vertices of Γ.

If G has an S-forest for each even subset $S \subseteq V(G)$, then G is said to be <u>collapsible</u>. These concepts were introduced in [7].

Among cycles, only the 3-cycle and 2-cycle are collapsible. A collapsible graph must be 2-edge-connected.

In [7], we collected results of Kundu [15], Nash-Williams [17] and Tutte [18], Jaeger [13], Euler [11], and Harary and Nash-Williams

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[12], to prove:

Theorem 1.1 Each of the following implies the next:

(a) $\kappa'(G) \geq 4$;

(b) $|E| \geq 2(\omega(G - E) - 1)$, for any $E \subseteq E(G)$;

(c) G has two edge-disjoint spanning trees;

(d) G is collapsible;

(e) G is supereulerian;

(f) L(G), the line graph of G, is hamiltonian.

Tutte and Nash-Williams proved (b) $\Leftrightarrow$ (c).

If

(1.1) $$S = \{v \in V(G) \mid d(v) \text{ is odd in } G\},$$

then G is supereulerian if and only if G has an S-forest. Because of this, the significance of collapsible graphs lies in the usefulness of the following result:

Theorem 1.2 The graph H is collapsible if and only if for every supergraph G of H and for every even set $S \subseteq V(G)$, these are equivalent:

(i) G has S-forest;

(ii) G/H has an (S/H)-forest.

Proof: When H is collapsible, Theorem 3 of [7] implies that (i) $\Leftrightarrow$ (ii).

Suppose that H is not collapsible. Then for some even set $S \subseteq V(H)$, H has no S-forest. Hence, (i) fails for S when G = H. However, G = H implies $G/H = K_1$, and K_1 is collapsible, and so (ii) holds. Thus, (i) and (ii) are not equivalent. ■

The next result gives a slightly larger class of collapsible graphs than that given by (c) $\Rightarrow$ (d) of Theorem 1.1

Theorem 1.3 ([7], Theorem 7) If G is at most one edge short of having at least two edge-disjoint spanning trees, then exactly one of the following holds:

(i) G is collapsible;

(ii) G has a cut-edge. ■

The graph $G = K_{2,t-2}$ $(t \geqslant 2)$ is a 2-edge-connected graph that is two edges short of having two edge-disjoint spanning trees. It is not collapsible.

If (1.1) holds, then the set of edges used twice in a shortest Postman's walk (see Kuan [14]) induces an S-forest.

2. Reduced graphs. Let E be a minimal edge set of G such that each component of G - E is collapsible. Denote the components of G - E by $H_1, H_2, \ldots H_c$. The reduction of G is the graph G_1 obtained from G by contracting each H_i ($1 \leqslant i \leqslant c$) to a distinct vertex. Thus, $E(G_1) = E$. If each H_i is a single vertex, then G is said to be reduced.

It can be shown ([7], Theorem 5) that the reduction G_1 of G is reduced (i.e., it contains no nontrivial collapsible subgraph), so that our terminology makes sense. The following theorem can be derived from results in the previous section.

Theorem 2.1 ([7], Theorem 6) Let G_1 be the reduction of G, where $V(G_1) = \{v_1, \ldots, v_c\}$, and the contraction-mapping $G \to G_1$ maps the collapsible subgraph H_i to $v_i \in V(G_1)$ $(1 \leqslant i \leqslant c)$. Then

a) G_1 is simple;

b) G_1 has no K_3;

c) $a(G_1) \leqslant 2$;

d) For any subgraph H of G_1, if $|E(H)| \geqslant 2|V(H)| - 3$, then $H \in \{K_1, K_2\}$;

e) Let $S \subsetneq V(G)$ be an even set, and define $S_1 = \{v_i \in V(G_1):$

$|V(H_i) \cap S|$ is odd). Then G has an S-forest iff G_1 has S_1-forest;

f) G is supereulerian iff G_1 is supereulerian;

g) L(G) is hamiltonian iff G_1 has a closed trail containing at least one vertex of each edge of G_1, and containing each vertex v_i $(1 \leqslant i \leqslant c)$ such that $|V(H_i)| > 1$. ■

3. Applications. Theorem 2.1 has been applied in various ways. Proofs of the following results appear in the papers cited.

Theorem 3.1 (Catlin [8]) Let G be a connected graph of order n and let $p \geqslant 2$ be an integer. If

$$n \geqslant 4p^2 \tag{3.1}$$

and if

$$d(u) + d(v) > \frac{2n}{p} - 2 \tag{3.2}$$

whenever $uv \notin E(G)$, then exactly one of the following holds:

(a) G is collapsible;

(b) The reduction G_1 of G has order less than p;

(c) $p = 2$, and $G - x = K_{n-1}$ for some $x \in V(G)$ with $d(x) = 1$;

(d) $p = 3$, and the reduction G_1 of G is $K_{1,2}$, where some adjacent pair of vertices of $G_1 = K_{1,2}$ each have a singleton preimage in the contraction-mapping $G \to G_1$.

(e) $p = 4$, and the reduction G_1 of G is C_4, where some adjacent pair of vertices of $G_1 = C_4$ each have a singleton preimage in the contraction-mapping $G \to G_1$.

Theorem 3.1, as stated here, is different than Theorem 3 of [8] in two ways: in [8], conclusion (b) is that G_1 is a contraction of G, but here (b) states that G_1 is the reduction of G (a special type of contraction); the conclusion (d) has been added, to allow that change in (b). Corresponding minor changes can be made in the proof of [8], but we omit them here.

By Theorem 1.1, if G is collapsible, then (e) and (f) of Theorem 1.1 hold, and by Theorem 2.1, if G_1 is supereulerian then so is G. Therefore, Theorem 3.1 gives a sufficient condition for (e) and (f) of Theorem 1, and if (e) fails, then Theorem 3.1 shows that G is contractible to a reduced graph G_1 with no spanning eulerian subgraph, where $|V(G_1)| < p$ or (c) of Theorem 3.1 holds.

Let G_1 be a reduced graph of order p, and let $t \geq 3$. Construct a graph G of order $n = pt$ by replacing each vertex of G_1 with a distinct complete graph K_t. Since K_t is collapsible, G_1 is the reduction of G. Except for a few exceptional cases,

$$\delta(G) = t - 1 = \frac{n}{p} - 1,$$

and hence (3.2) barely fails. Since G_1 is reduced, G is not collapsible, and the reduction of G has order p, so b) barely fails. Therefore, (3.2) is best-possible.

On the other hand, (3.1) can be improved a bit.

In the case where (a) of Theorem 3.1 is replaced by "G is supereulerian" (which is justified by (d) $\Rightarrow$ (e) of Theorem 1.1), the case $p = 2$ of Theorem 3.1 was obtained by Lesniak-Foster and Williamson [16], with $n \geq 6$ instead of (3.1); a result similar to the case $p = 3$ was obtained by Benhocine, Clark, Köhler, and Veldman ([3], Theorem 8); and the case $p = 5$ was conjectured by Benhocine, Clark, Köhler and Veldman [3].

When $n \geq 20$, we showed ([7], Theorem 8) that if $\delta(G) \geq \frac{n}{5} - 1$ and if $\kappa'(G) \geq 2$, then either G has a spanning eulerian subgraph, or the reduction of G is $G_1 = K_{2,3}$, where the preimage of each vertex of the contraction $G \to G_1$ is a complete graph on n/5 vertices. This proved a conjecture of Bauer ([1], [2]).

Given a 3-matching $M_3 = \langle u_1v_1, u_2v_2, u_3v_3 \rangle$, we denote

$$\Sigma(M_3) = \sum_{i=1}^{3} d(u_i) + d(v_i).$$

<u>Theorem</u> <u>3.2</u> (Catlin [9]) Let G be a 2-edge-connected simple graph of order n. If for every 3-matching M_3 of G,

$$\Sigma(M_3) \geq 2n + 2, \tag{3.3}$$

then exactly one of the following holds:

i) G is collapsible;

ii) For some collapsible subgraph H, there is an integer t such that

$G/H = K_{2,t}$, and the mapping $G \to G/H$ sends H to a vertex of degree t.

iii) For some $e \in E(G)$,

$$G/e = K_{2,n-3}.$$

iv) G is a bipartite theta graph of order 6. ■

Let $H = K_{1,3}$ and let S be the set of three endvertices of H. Add to H a number of vertices of degree 2, such that each added vertex has its neighborhood in S. Call the resulting graph G. If $\kappa'(G) = 2$ and if iv) fails, then

$$\Sigma(M_3) = 2n + 1$$

for some 3-matching M_3, and all conclusions of Theorem 3.2 fail. Hence, (3.3) is best-possible.

Special cases of Theorem 3.2 with less general inequalities than (3.3) were previously obtained by Lesniak-Foster and Williamson [16], Brualdi and Shanny [5], Clark [10], Veldman ([19], Theorem 5), Benhocine, Clark, Köhler, and Veldman ([3], Theorem 8), and Catlin [6]. These prior results were conditions for the existence of spanning eulerian subgraph or hamiltonian line graphs.

4. Collapsible graphs. An interesting problem is to find collapsible graphs with relatively few edges, compared to the order of the graph.

Theorem 4.1 If G is collapsible, then for any $E \subsetneq E(G)$,

$$|E| \geq \frac{3}{2}(\omega(G - E) - 1). \tag{4.1}$$

Proof: Suppose that G is collapsible. Let $E \subsetneq E(G)$, and let S be a maximum even subset of V(G) such that each component of $G - E$ has at most one member of S. Then

$$|S| \geq \omega(G - E) - 1. \tag{4.2}$$

Let Γ be an S-forest of G. Then by (4.2),

$$|E \cap E(\Gamma)| \geq \frac{1}{2}|S| = \frac{1}{2}(\omega(G - E) - 1). \tag{4.3}$$

By the definition of an S-forest, $G - E(\Gamma)$ is connected, and so

$$|E - E(\Gamma)| \geq \omega(G - E) - 1, \tag{4.4}$$

because a tree on $k = \omega(G - E)$ vertices has $k - 1$ edges. We add (4.3) and (4.4) to get (4.1). ■

The noncollapsible graphs $K_{2,n-2}$ show that the converse of Theorem 4.1 is false. There are numerous other such examples. Compare (4.1) with (b) of Theorem 1.1, which is equivalent to (c).

Corollary 4.2 If G is a collapsible graph on n vertices and m edges, then

$$m \geq \frac{3}{2}(n - 1). \tag{4.5}$$

Proof: Set $E = E(G)$ in Theorem 4.1. ■

The bound in (4.5) is best-possible for any positive odd integer n. Let G be a graph consisting of $t = \frac{n-1}{2} \geq 0$ edge-disjoint triangles, all sharing the same common vertex v, where v is a cutvertex of G. Then

$$n = 2t + 1$$

and

$$|E(G)| = 3t = \frac{3}{2}(n - 1).$$

Thus, (4.5) is best-possible.

We now present a construction of a larger class of graphs satisfying (4.5) with equality, and unlike the previous example, having no nontrivial proper collapsible subgraph.

<u>Theorem 4.3</u> Let H be the bipartite theta graph of order 6, with vertices labelled in the cyclic order u,v,w,z,y,x,u, where vy is the chord of H. Let G' be a graph with a distinguished edge $u'z'$. Define G to be the graph obtained from $(G' - u'z') \cup H$ by setting $u' = u$ and $z' = z$. Then

(4.6) $$|V(G)| = |V(G')| + 4,$$

(4.7) $$|E(G)| = |E(G')| + 6,$$

and G is collapsible if and only if G' is collapsible.

<u>Proof</u>: Define the set

$$T = \langle v,w,x,y \rangle.$$

Since $V(G) - V(G') = T$ and since $|E(H)| = 7$, (4.6) and (4.7) are clear. Assume $u' = u$ and $z' = z$.

Suppose that G is collapsible. Let $S' \subsetneq V(G')$ be an even set. Thus, $S' \subsetneq V(G) - T$, and G has an S'-forest Γ. We must show that G' has an S'-forest. If Γ has no (u,z)-path through H, then $E(\Gamma) \cap E(H) = \emptyset$, and hence Γ is an S'-forest of G'. Suppose, instead, that Γ contains a (u,z)-path through H. Then $H - E(\Gamma)$ is not connected, and since $G - E(\Gamma)$ is connected, $G' - uz - E(\Gamma)$ must be connected.

Therefore, $\Gamma' = \Gamma - T + uz$ is an S'-forest of G'.

Conversely, suppose that G' is collapsible. Let S be an even subset of V(G). To complete the proof, it suffices to show that G has an S-forest. Now, $S \cap T$ is one of the entries of column 1 or column 5 of Table 1. If $S \cap T$ is in column 5, then by the radial symmetry of H, it is equivalent, by a relabelling of V(H), to the entry of column 1 of the same row. Hence, it suffices to suppose that $S \cap T$ is the entry of some row of column 1. Let S' be the entry of column 2 from the same row. Since G' is collapsible and $S' \subsetneq V(G')$ is even, there is an S'-forest Γ' of G'. Either $uz \in E(\Gamma)$ or $uz \notin E(\Gamma')$. If $uz \notin E(\Gamma')$, then we can obtain an S-forest of G by adding to Γ' the edges of column 3, in the same row. If $uz \in E(\Gamma')$, then $\Gamma' - uz$ is an $(S' \Delta \{u,z\})$-forest of $G' - uz$. In that case, form an S-forest of G by adding to $\Gamma' - uz$ the edges from column 4, in the same row. ■

<u>Theorem</u> <u>4.4</u> Let H be the graph of order 8 and size 10 containing an 8-cycle with vertices labelled with the cyclic order s,t,u,v,z,y,x,w,s, and with chords ux and vw. Let G' be a graph with a distinguished edge s'z'. Define G to be the graph obtained from $(G'-s'z') \cup H$ by setting $s' = s$ and $z' = z$. Then

(4.8) $$|V(G)| = |V(G')| + 6,$$

(4.9) $$|E(G)| = |E(G')| + 9,$$

and if G' collapsible, then G is collapsible.

<u>Proof</u>: Define $T = \{t,u,v,w,x,y\}$. Assume $s' = s$ and $z' = z$.

Repeat the relevant parts of the proof of Theorem 4.3, except that u and u' of Theorem 4.3 are replaced by s and s', H is the graph of Theorem 4.4, and Table 2 is used instead of Table 1. The part of the proof of Theorem 4.3 that is not relevant is the portion in which it is assumed that G is collapsible. ■

<u>Theorem</u> <u>4.5</u> For each odd integer $n \geq 1$, except for $n = 5$, there is

a collapsible graph G of order n with $m = \frac{3}{2}(n - 1)$ edges, and with no proper nontrivial collapsible subgraph.

Proof: As a basis for induction, note that Theorem 4.5 is true for $n = 1$, because of $G = K_1$, for $n = 3$, because of $G = K_3$, and for $n = 9$, because of the graph G derived from $G' = K_3$ by the construction of the hypothesis of Theorem 4.4. In each case, inspection (and Corollary 4.2) shows that G has no proper nontrivial collapsible subgraph. Inspection shows that if $n = 5$, no collapsible graph has 6 edges, unless it contains K_3 as a collapsible subgraph.

By way of induction, suppose that G' is a collapsible graph of odd order $n' \geq 3$ $(n' \neq 5)$, with

$$m' = \frac{3}{2}(n' - 1) \tag{4.10}$$

edges and with no proper collapsible subgraph. Let $u'z' = uz \in E(G')$, and let G and H be the graphs defined by the hypothesis of Theorem 4.3. Thus,

$$m' + 6 = m, \quad n' + 4 = n, \tag{4.11}$$

and by Theorem 4.3, G is collapsible.

Let G" be a minimal nontrivial collapsible subgraph of G. It remains to show that G = G". Since G' has no proper nontrivial collapsible subgraph, $G'' \cap H \neq \emptyset$. If G" has no (u,z)-path through H, then either u or z is a cutvertex of G" that separates collapsible blocks, contrary to the minimality of G". Hence, G" has a (u,z)-path P through H.

Define

$$E = E(H) - E(G''), \quad E' = E(G' - uz) - E(G''),$$
$$V = V(H) - V(G''), \quad V' = V(G') - V(G'').$$

By the definition of V and V',

$$|V| = \omega(G - E) - 1; \ |V'| = \omega(G - E') - 1. \tag{4.12}$$

Now, G and G" are collapsible. Hence, by Theorem 4.1 and (4.12),

(4.13) $$|E| \geqslant \frac{3}{2}(\omega(G - E) - 1) = \frac{3}{2}|V|,$$

(4.14) $$|E'| \geqslant \frac{3}{2}(\omega(G - E') - 1) = \frac{3}{2}|V'|,$$

and by Corollary 4.2,

(4.15) $$|E(G'')| \geqslant \frac{3}{2}(|V(G'')| - 1).$$

We add (4.13), (4.14) and (4.15), and we use (4.11) to get

$$m' + 6 = m = |E(G'')| + |E| + |E'| \geqslant \frac{3}{2}\,[|V(G'')| - 1 + |V| + |V'|]$$

$$= \frac{3}{2}(n - 1) = \frac{3}{2}(n' + 3).$$

By (4.10), this implies equality in (4.13), (4.14), and (4.15). Therefore, $|V|$ is even. Since P exists, $|V| \leqslant 2$. If $|V| = 2$, then the structure of H forces $|E| = 4$, contrary to equality in (4.13). Hence $|V| = 0$, and since (4.13) holds with equality, H is a subgraph of G". By Theorem 4.3, with G" in place of G, the subgraph G_0 of G', obtained from G" by replacing H with an edge uz, is a collapsible subgraph of G'. By the induction hypothesis, $G_0 = G'$. Hence, G" = G, and so G has no proper nontrivial collapsible subgraph. ■

We define a minimally collapsible graph to be a collapsible graph G such that any subdivision of G yields a noncollapsible graph.

Conjecture If G is a minimally collapsible graph, then equality holds in (4.5).

TABLE 1

1. S ∩ T	2. S'	3. uz ∉ E (Γ')	4. uz ∈ E(Γ')	5. S ∩ T
∅	S − T	∅	uv,vy,yz	
⟨w,x⟩	(S − T)Δ⟨u,z⟩	ux,wz	vw,vy,yz	
⟨w,y⟩	S− T	vw,vy	uv,vw,yz	⟨v,x⟩
⟨v,y⟩	S − T	vy	uv,yz	
⟨v,w⟩	S − T	vw	uv,wz	⟨x,y⟩
T	S − T	vw,xy	uv,wz,xy	
⟨x⟩	(S − T)Δ⟨u⟩	ux	xy,yz	⟨w⟩
⟨v⟩	(S − T)Δ⟨u⟩	uv	vy,yz	⟨y⟩
⟨v,w,x⟩	(S − T)Δ⟨u⟩	ux,vw	vw,xy,yz	⟨w,x,y⟩
⟨v,w,y,⟩	(S − T)Δ⟨z⟩	vy,wx	uv,wz,yz	⟨y,x,y⟩

TABLE 2

1. S ∩ T	2. S'	3. sz ∉ E(Γ')	4. sz ∈ E(Γ')	5. S ∩ T
∅	S − T	∅	sw,vw,vz	
⟨t,u⟩	S − T	tu	st,uv,vz	⟨x,y⟩
⟨t,v⟩	S − T	tu,uv	st,vz	⟨w,y⟩
⟨t,w⟩	S − T	tu,uv,vw	st,vw,vz	⟨v,y⟩
⟨t,x⟩	S − T	tu,ux	vw,wx,vz,st	⟨u,y⟩
⟨t,y⟩	S − T	tu,ux,xy	st,yz	
⟨u,v⟩	S − T	uv	sw,wx,xu,vz	⟨w,x⟩
⟨u,w⟩	S − T	ux,xw	uv,vz,sw	⟨v,x⟩
⟨u,x⟩	S − T	ux	ux,sw,wv,vz	
⟨v,w⟩	S − T	vw	sw,vz	
⟨t,u,v,w⟩	S − T	tu,vw	sw,tu,vz	⟨v,w,x,y⟩
⟨t,u,v,x⟩	(S − T)Δ⟨s,z⟩	st,ux,vz	tu,vw,wx	⟨u,w,x,y⟩
⟨t,u,v,y⟩	(S − T)Δ⟨s,z⟩	st,uv,yz	tu,vw,wx,xy	⟨t,w,x,y⟩
⟨t,u,w,x⟩	S − T	tu,wx	st,uv,vz,wx	⟨u,v,x,y⟩

⟨t,u,w,y⟩	S − T	tu,wx,xy	sw,tu,zy	⟨t,v,x,y⟩
⟨t,u,x,y⟩	S − T	tu,xy	st,ux,yz	
⟨t,v,w,x⟩	S − T	tu,ux,vw	st,wx,vz	⟨u,v,w,y⟩
⟨t,v,w,y⟩	(S − T)Δ⟨s,z⟩	st,vw,yz	tu,ux,xy,vw	
⟨u,v,w,x⟩	S − T	uv,wx	sw,ux,vz	
T	S − T	tu,vw,xy	st,ux,vw,yz	
⟨t⟩	(S − T)Δ⟨s⟩	st	tu,uv,vz	⟨y⟩
⟨u⟩	(S − T)Δ⟨z⟩	uv,vz	ux,xw,ws	⟨x⟩
⟨v⟩	(S − T)Δ⟨z⟩	vz	vw,ws	⟨w⟩
⟨t,u,v⟩	(S − T)Δ⟨s⟩	st,uv	tu,vz	⟨w,x,y⟩
⟨t,u,w⟩	(S − T)Δ⟨z⟩	tu,wv,vz	sw,tu	⟨v,x,y⟩
⟨t,u,x⟩	(S − T)Δ⟨s⟩	st,ux	tu,xw,wv,vz	⟨u,x,y⟩
⟨t,u,y⟩	(S − T)Δ⟨z⟩	tu,yz	st,ux,xy	⟨t,x,y⟩
⟨t,v,w⟩	(S − T)Δ⟨s⟩	st,vw	tu,ux,xw,vz	⟨v,w,y⟩
⟨t,v,x⟩	(S − T)Δ⟨z⟩	tu,ux,vz	st,vw,wx	⟨u,w,y⟩
⟨t,v,y⟩	(S − T)Δ⟨z⟩	tu,uv,yz	st,vw,wx,xy	⟨t,w,y⟩
⟨t,w,x⟩	(S − T)Δ⟨s⟩	st,wx	tu,uv,wx,vz	⟨u,v,y⟩
⟨u,v,w⟩	(S − T)Δ⟨s⟩	sw,uv	ux,xw,vz	⟨v,w,x⟩
⟨u,v,x⟩	(S − T)Δ⟨z⟩	ux,vz	sw,wx,uv	⟨u,w,x⟩
T − t	(S − T)Δ⟨s⟩	sw,uv,xy	uv,wx,yz	T − y
T − u	(S − T)Δ⟨s⟩	st,vw,xy	tu,ux,vw,yz	T − x
T − v	(S − T)Δ⟨z⟩	tu,wx,yz	sw,tuxy	T − w

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