

# ON THE HAJNAL-SZEMERÉDI THEOREM ON DISJOINT CLIQUES\*

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## 1. Introduction.

We follow the notation of Harary [11], except for several changes. For  $x \in V(G)$ ,  $N(x)$  denotes the vertices of  $G$  adjacent to  $x$ , i.e., the neighborhood of  $x$ . For  $x \in V(G)$  and  $V \subseteq V(G)$ ,  $d(x, V)$  denotes the number of edges of  $G$  joining  $x$  and  $V$ . The maximum degree  $\nabla(G)$  will frequently be denoted  $\nabla$  only.

In 1970 Hajnal and Szemerédi [10] proved the following conjecture of Erdős:

**THEOREM 1.** *Let  $n$  be a natural number, and let  $G$  be a graph on  $p$  vertices. If*

$$(1) \quad n(\Delta + 1) \leq p ,$$

*then  $G$  contains  $\Delta + 1$  disjoint independent sets, all with  $n$  vertices.*

Corrádi and Hajnal [9] had earlier proved Theorem 1 in the case  $n = 3$ . If instead of (1), we have

$$(2) \quad n(\Delta + 1) = p + 1 ,$$

then it is easily shown from Theorem 1 that  $G$  contains  $\Delta + 1$  disjoint independent sets, one having  $n - 1$  vertices and the rest having  $n$  vertices.

To see that (1) and (2) cannot be improved, let  $G$  contain  $(n-1)K_{\Delta+1} \cup \bar{K}_{\Delta-1}$  as a spanning subgraph. Such a graph  $G$  will be called a *type 1 graph*. Note that

$$(3) \quad n(\Delta + 1) = p + 2 ,$$

barely in violation of (1), and that there may be edges of  $G$  on the

\* This work is supported in part by the National Science Foundation under Grant No. MCS-7903125. It was presented at the Seventh British Combinatorial Conference, Cambridge, 1979.

$\bar{K}_{\Delta-1}$  portion of the spanning subgraph. Also, note that  $G$  does not contain  $\Delta$  disjoint independent  $n$ -sets. A different example is the graph

$$G = (n-2)K_{\Delta+1} \cup K_{\Delta, \Delta},$$

with  $\Delta$  odd. We call this a *type 2 graph*. Again, (3) holds, and  $G$  does not contain  $\Delta$  disjoint independent  $n$ -sets. The type 1 graphs were given by Hajnal and Szemerédi [10], and in the case  $n = 3$  by Corrádi and Hajnal [9]. The type 2 graphs are due to Catlin [5].

In [5] (see also [8]), we showed that the only extremal graphs for Corrádi and Hajnal's theorem ( $n = 3$  in Theorem 1) are the type 1 and type 2 graphs. For  $n = 2$ , it is easily shown that the type 1 and type 2 graphs are the only extremal graphs. In this paper we show that also in the case  $n = 4$ , the only extremal graphs are those of type 1 or 2.

When two graphs  $G$  and  $H$ , each with  $p$  vertices, can be placed on the same set of  $p$  vertices so that no edges overlap, we say that  $G$  and  $H$  are *mutually placeable*. We prove

**THEOREM 2.** *If a graph  $G$  on  $p$  vertices satisfies*

$$(4) \quad 4\Delta(G) + 2 = p,$$

*then either  $G$  and  $\Delta K_4 \cup K_2$  are mutually placeable, or  $G$  is of type 1 or type 2.*

In our Master's thesis in 1973, we were the first to consider the problem of finding sufficient conditions for two graphs  $G$  and  $H$  to be mutually placeable, depending on only  $\Delta(G)$ ,  $\Delta(H)$ , and  $p$ . This work appeared in 1974, with an announcement of the following stronger, simpler result of ours ([4], footnote, p. 226), which Sauer and Spencer [13] also obtained, independently of us.

**THEOREM 3.** *If*

$$(5) \quad 2\Delta(G)\Delta(H) < p,$$

*then  $G$  and  $H$  are mutually placeable.*

This also appears in our Ph.D. thesis [5]. We proved [5] that it is best possible only when  $\Delta(G)$  or  $\Delta(H)$  is 1. Sauer and Spencer [13] showed, using a probabilistic argument, that this result is not too far from being best possible. Motivated by Theorem 3, Bollobás and Eldridge and this author independently conjectured how to make (5) best possible.

BOLLOBÁS-CATLIN-ELDRIDGE CONJECTURE. *If*

$$(6) \quad (\Delta(G) + 1)(\Delta(H) + 1) \leq p + 1 ,$$

*then G and H are mutually placeable.*

The Hajnal-Szemerédi theorem is seen to be a special case of our conjecture, as (6) reduces to (2). The type 1 and type 2 graphs thus show that our conjecture is best possible. We know of no other extremal graphs for (6).

The conjecture appears in [2], [5], and [6] and a slightly incorrect version appears in [1]. Also, Sauer and Spencer [13] gave the case  $\Delta(H) = 2$  of this conjecture.

We gave a partial result in [5] for the case  $\Delta(H) = 2$  of the conjecture:

THEOREM 4. *Let G and H be graphs on p vertices, where  $\Delta(H) = 2$ . There is a function  $f(p) = O(p^{2/3})$  such that if*

$$(7) \quad \Delta(G) \leq \frac{p}{3} - f(p) ,$$

*then G and H are mutually placeable.*

The coefficient 1/3 of (7) is best possible, because of the type 1 and type 2 graphs that make the Corrádi-Hajnal theorem best possible.

## 2. *The Proof of Theorem 2.*

In our proof we shall use the following theorem, which appears in [7] as Theorems 2.1 and 2.2. One can also formulate the proof of Theorem 2 using the concept of chains, as in the proofs of Theorem 1 (see [9], [10], or [1]).

THEOREM 5. For any graph  $G$  there is a nontrivial partition  $V_1 \cup V_2$  of  $V(G)$  such that the subgraph  $G_i$  induced by  $V_i$  ( $i = 1, 2$ ) satisfies

$$(8) \quad \Delta(G_i) \leq k_i,$$

where  $p_i = |V_i|$  and

$$(9) \quad k_i = \left\lfloor \frac{\Delta(G)}{p-1} (p_i - 1) \right\rfloor.$$

Furthermore, let

$$M_i = \{v \in V_i : d(v, G_i) = k_i\}.$$

If

$$(10) \quad k_1 + k_2 + 1 = \Delta(G),$$

then for each  $v \in M_i$  ( $i = 1, 2$ ),  $M_{3-i} \cap N(v)$  is a complete graph, and

$$(11) \quad d(v, G) = \Delta(G).$$

It is natural to ask whether or not the size of the sets  $V_1, V_2$  can be specified in Theorem 5, in the sense that the Hajnal-Szemerédi theorem asserts that  $V(G)$  is partitioned into independent sets of equal size. We give some examples to show that this cannot be done in the obvious natural way.

The Petersen graph and the Heawood graph have  $\Delta = 3$ , but there is no partition of either graph satisfying  $|V_1| = |V_2| = \frac{p}{2}$  such that the subgraphs  $G_i$  induced by  $V_i$  have  $\Delta(G_i) \leq 1$  for  $i = 1$  and  $2$ .

For any natural number  $k$  let  $H$  be a  $2k$ -regular graph on  $4k + 1$  vertices. Let  $G = H + \bar{K}_{2k+1}$ . Thus,  $G$  is  $(4k + 1)$ -regular and has  $p = 6k + 2$  vertices. It is routine to show that there is no partition  $V(G) = V_1 \cup V_2$  with  $|V_1| = |V_2| = 3k + 1$  such that  $\Delta(G_i) \leq 2k$  for  $i = 1, 2$ .

These examples also show that a similar theorem of Lovász [12] cannot be extended in the same natural way.

We now proceed with the proof of Theorem 2.

From (8) of Theorem 5, we see that  $\Delta(G_1) \leq s$ ,  $\Delta(G_2) \leq t$  under any of the following three conditions:

- I.  $p_1 = 4s + 3$ ,  $p_2 = 4t + 3$ ,  $s \leq t$ ,  $s + t + 1 = \Delta$  ;
- II.  $p_1 = 4s + 5$ ,  $p_2 = 4t + 5$ ,  $s + t + 2 = \Delta$  ;
- III.  $p_1 = 4s + 2$ ,  $p_2 = 4t + 4$ ,  $s + t + 1 = \Delta$  .

Clearly, these exhaust all of the possibilities in which  $p = p_1 + p_2 = 4\Delta + 2$ . We consider the three cases separately.

We assume as our induction hypothesis that the theorem is true for all graphs on  $4k + 2$  vertices, where  $k < \Delta$ . Since the theorem is trivial for  $k = 0$ , we have a basis for induction.

*Case I.* Suppose

$$p_1 = 4s + 3, \quad p_2 = 4t + 3,$$

where  $\Delta(G_1) \leq s$ ,  $\Delta(G_2) \leq t$ , and

$$(12) \quad \Delta = s + t + 1, \quad s \leq t.$$

We claim that either  $G$  is of type 2 or there is a placement of  $sK_4 \cup K_3$  on  $G_1$ , where the vertices on which the  $K_3$  is placed, denoted

$$X = \{x_1, x_2, x_3\},$$

satisfy the following condition:

$$(13) \quad \text{There is a } z \in V(G_2) \text{ with } N(z) \cap X = \emptyset.$$

Suppose, to the contrary, that (13) fails. Then for all  $z \in V(G_2)$ ,  $N(z) \cap X \neq \emptyset$ , and so

$$(14) \quad \sum_{i=1}^3 d(x_i, G_2) \geq p_2 = 4t + 3,$$

with equality only if the sets  $N(x_i) \cap V_2$  are disjoint for  $i = 1, 2, 3$ . Hence, by (14) and (12),

$$\begin{aligned}
 (15) \quad \sum_1^3 d(x_i, G_1) &= \sum_1^3 d(x_i, G) - \sum_1^3 d(x_i, G_2) \\
 &\leq 3\Delta - (4t + 3) \\
 &= 3s + 3t + 3 - 4t - 3 = 3s - t \leq 2s,
 \end{aligned}$$

with equality only if  $s = t$ , and the sets  $N(x_i) \cap V_2$  are disjoint for  $i = 1, 2, 3$ .

Suppose, by way of contradiction, that there is some  $K_4$  placed on some set  $Y = \{y_1, y_2, y_3, y_4\} \subseteq V_1$  such that there are at most two edges joining  $X$  and  $Y$ , and if there are exactly two, then they are adjacent. We shall contradict our assumption that (13) is false. Without loss of generality, suppose that  $\{x_1 y_1\}$  or  $\{x_1 y_1, x_1 y_2\}$  is the set of edges joining  $X$  and  $Y$ . The case in which no edge joins  $X$  and  $Y$  is considered later.

Let  $V = \{x_2, x_3, y_3, y_4\}$ . If  $z \in V_2$  is adjacent to some vertex of the  $K_3$ , regardless of how  $K_4 \cup K_3$  is placed on  $X \cup Y$ , then either  $x_1 \in N(z)$ , or the  $K_3$  contains  $x_1$ , and so  $|N(z) \cap V| \geq 3$ . Since  $x_1 y_1 \in E(G_1)$ , (12) implies

$$(16) \quad d(x_1, V_2) \leq \Delta - 1 = s + t.$$

By (16),

$$\begin{aligned}
 (17) \quad |V_2 - N(x_1)| &= |V_2| - d(x_1, V_2) \\
 &\geq 4t + 3 - (s + t) = 3t - s + 3.
 \end{aligned}$$

Since each  $z \in V_2 - N(x_1)$  is adjacent to  $3/4$  of the vertices of  $V$ , there is in  $V$  a pair of vertices (say  $x_2, x_3$ ) such that

$$(18) \quad d(x_2, V_2) + d(x_3, V_2) \geq 2\left(\frac{3}{4}\right) |V_2 - N(x_1)|.$$

Hence, by (17) and (18),

$$\begin{aligned}
(19) \quad \sum_1^3 d(x_1, V_2) &\geq d(x_1, V_2) + \frac{3}{2} |V_2 - N(x_1)| \\
&= d(x_1, V_2) + \frac{3}{2} |V_2| - \frac{3}{2} d(x_1, V_2) \\
&= \frac{3}{2} (4t + 3) - \frac{1}{2} d(x_1, V_2) ,
\end{aligned}$$

and so by (12), (19), and (16),

$$\begin{aligned}
\sum_1^3 d(x_1, V_1) &\leq 3\Delta - \sum_1^3 d(x_1, V_2) \\
&\leq 3s + 3t + 3 - (6t + \frac{9}{2}) + \frac{1}{2} d(x_1, V_2) \\
&\leq 3s - 3t - \frac{3}{2} + \frac{1}{2}(s + t) \\
&= s + \frac{5}{2} (s - t) - \frac{3}{2} \leq s - \frac{3}{2} .
\end{aligned}$$

This implies that one of the  $s$   $K_4$ 's placed in  $G_1$  is joined by no edge to the  $K_3$  on  $X$ .

Now, suppose that no edges join  $Y$  and  $X$ . We imitate the previous argument. Since (13) holds for no placement of  $K_4 \cup K_3$  in  $X \cup Y$ , each  $z \in V_2$  has

$$|N(z) \cap (X \cup Y)| \geq 5 .$$

Since every  $z \in V_2$  is joined to  $5/7$  of  $X \cup Y$ , some  $v \in X \cup Y$  is joined to  $\frac{5}{7} p_2 \geq \frac{5}{14} p > \Delta(G)$  vertices of  $V_2$ , a contradiction.

Hence, every  $K_4$  placed in  $G_1$  is joined to the vertices on which  $K_3$  is placed by a pair of independent edges, regardless of how  $sK_4 \cup K_3$  is placed in  $G_1$ . Thus, equality holds in (15); exactly two edges join each  $K_4$  to  $X$ ;

$$(20) \quad s = t ;$$

and

$$(21) \quad N(x_i) \cap V_2 \text{ are disjoint for } i = 1, 2, 3 .$$

Each  $K_4$  placed in  $G_1$  is joined to the  $K_3$  on  $X$  by two independent edges, and so we can select any pair of  $K_4$ 's, say one on  $Y = \{y_1, y_2, y_3, y_4\}$  and one on  $W = \{w_1, w_2, w_3, w_4\}$ . Without loss of generality, suppose that  $x_1y_1, x_2y_2 \in E(G)$ .

A contradiction follows if we suppose that one of the two  $W$ - $X$  edges is incident with  $X$  at  $x_3$ : suppose  $x_1w_1, x_3w_2 \in E(G)$ . Redefine the placement of the  $K_3$  to be  $\{y_1, x_2, x_3\}$ , and let  $W$  and  $\{x_1, y_2, y_3, y_4\}$  be the vertices where the two  $K_4$ 's are placed. There must be two edges from  $\{y_1, x_2, x_3\}$  to  $W$ , and so  $y_1$  is adjacent to a vertex of  $W$ . A similar argument shows that  $y_2$  is adjacent to a vertex of  $W$ . Next redefine the placement of the cliques to be  $\{x_3, y_1, y_2\}$ ,  $\{x_1, x_2, y_3, y_4\}$ , and  $W$ . Now the  $K_3$  is joined to  $W$  by three edges of  $G$ , contrary to the fact that exactly two edges of  $G_1$  join each  $K_4$  to  $K_3$ .

Therefore, the two  $W$ - $X$  edges are incident with  $X$  at  $x_1$  and  $x_2$ , and we denote them by  $x_1w_1, x_2w_2$ . In general, all pairs of edges joining a  $K_3$  to a  $K_4$  are incident at the same two vertices on the  $K_3$ .

Since  $K_3$  and  $K_4$  from  $X \cup Y$  may be placed on  $\{y_1, y_2, y_3\}$  and  $\{x_1, x_2, x_3, y_4\}$ , the two  $W$ - $Y$  edges are incident with  $Y$  at  $y_1, y_2$ . Similarly, the two  $W$ - $Y$  edges are incident with  $W$  at  $w_1, w_2$ . The  $W$ - $Y$  edges cannot be  $w_1y_2, w_2y_1$ , for in that case  $2K_4 \cup K_3$  can be placed on  $\{w_1, y_1, x_2\}$ ,  $\{w_2, y_2, x_1, y_3\}$ ,  $\{w_3, w_4, x_3, y_4\}$ , so the  $K_3$  is not joined to the last set by any edge. Therefore, the subgraph of  $G_1$  induced by  $W \cup X \cup Y$  is  $2K_3 \cup 3K_1$ , for any  $W, Y$  on which two  $K_4$ 's are placed.

Since  $W$  and  $Y$  are arbitrarily chosen among sets in  $G_1$  on which  $K_4$ 's are placed, we see that  $x_1$  and  $x_2$  lie in disjoint cliques  $K_{s+1}$ , each with one vertex in each  $K_4$  placed in  $G_1$ . Hence,  $G_1 = 2K_{s+1} \cup \bar{K}_{2s+1}$ .

By (20),  $s = t$ , and so the same argument applies to  $G_2$ . Therefore,  $G_2 = 2K_{t+1} \cup \bar{K}_{2t+1}$ .

By (21), the three neighborhoods of the vertices on which the  $K_3$  is placed do not intersect. For any  $Y = \{y_1, y_2, y_3, y_4\}$  on which a  $K_4$  is placed, where  $x_1y_1$  and  $x_2y_2$  are the  $X$ - $Y$  edges, we can place the



$K_3$  and  $K_4$  on  $X \cup Y$  so that the  $K_3$  is on  $\{x_1, x_2, x_3\}$ ,  $\{x_1, x_2, y_3\}$ , or  $\{x_1, x_2, y_4\}$ . Since the neighborhoods do not overlap, and since each  $z \in V_2$  is adjacent to a vertex on which the  $K_3$  is placed,  $N(x_3) \cap V_2 = N(y_3) \cap V_2 = N(y_4) \cap V_2$ . Letting  $Y$  roam over all  $s$   $K_4$ 's placed in  $G_1$ , where  $y_3$  and  $y_4$  are not adjacent to  $X$ , we obtain  $2s + 1 = s + t + 1 = \Delta$  vertices ( $x_3$ ,  $s$  values of  $y_3$ ,  $s$  values of  $y_4$ ), with identical neighborhoods in  $V_2$ . Now, since  $x_1$  and  $x_2$  are each adjacent to  $s$  vertices in  $G_1$ , they are each adjacent to at most  $\Delta - s = t + 1$  vertices in  $V_2$ , and so by the conditions of Case I and by (21),

$$\Delta \geq |N(x_3) \cap V_2| \geq |V_2| - 2(t + 1) = 2t + 1 = \Delta .$$

Thus, equality holds, and  $x_3$  is contained in a  $K_{\Delta, \Delta}$  in  $G$ . It also follows that  $(N(x_1) \cap V_2) \cup (N(x_2) \cap V_2) = 2K_{t+1}$ , because  $G_2 = 2K_{t+1} \cup \bar{K}_{2t+1}$ , and the  $K_{\Delta, \Delta}$  of  $G$  uses the vertices of  $\bar{K}_{2t+1}$  in  $G_2$ . By the second part of Theorem 5, and since (12) implies (10),  $N(x_1) \cap V_2$  and  $N(x_2) \cap V_2$  are the two  $K_{t+1}$ 's of  $G_2$ . Also, by the same part of Theorem 5, for any  $z \in N(x_i) \cap V_2$  ( $i = 1, 2$ ),  $N(z) \cap V_1$  is a  $K_{s+1}$  in  $G_1$ . For any  $x'_i$  in the same  $K_{s+1}$  of  $G_1$  as  $x_i$ ,  $N(x'_i) \cap V_2$  is the  $K_{t+1}$  of  $G_2$  containing  $z$ , and thus  $N(x'_i) \cap V_2 = N(x_i) \cap V_2$ . By (12),

$$G = 2K_{s+1+t+1} \cup K_{\Delta, \Delta} = 2K_{\Delta+1} \cup K_{\Delta, \Delta} ,$$

and since (12) and (20) imply that  $\Delta$  is odd,  $G$  is of type 2.

Suppose that (13) holds.

If  $s = 0$ , then each vertex of  $V_1$  is adjacent to  $\Delta = t + 1$  vertices of  $G_2$ . Since  $|V_1| = 3$ , at least  $t$  vertices of  $V_2$  are adjacent to no vertex of  $V_1$ . Any one of these  $t$  vertices may be put into  $V_1$ , whence  $|V_1| = 4$ ,  $|V_2| = 4t + 2$ , which is essentially Case III.

Suppose  $s \geq 1$ . If  $tK_4 \cup K_2$  can be placed in  $G_2 - z$ , then we are done. Otherwise,  $G_2 - z$  is of type 1 or type 2, and contains three  $t$ -regular components. Since  $\Delta(G_2) = t$ ,  $z$  is adjacent to no vertex in a  $t$ -regular component of  $G_2 - z$ . Hence,  $tK_4 \cup K_3$  can be placed in  $G_2$ , where the  $K_3$  is placed on any set  $Y \subseteq V_2$  having one vertex in each  $t$ -regular component of  $G_2 - z$ .

Since each  $y \in Y$  is adjacent to  $t$  vertices in  $G_2$ , (12) implies  $|N(y) \cap V_1| \leq s + 1$ . Thus, there is a set

$$S = \{x \in V_1 : N(x) \cap Y = \emptyset\}$$

with  $|S| = |V_1| - 3|N(y) \cap V_1| \geq s$ . For any  $x \in S$ ,  $(t+1)K_4$  can be placed on  $G - (G_1 - x)$ , and so either  $sK_4 \cup K_2$  can be placed on  $G_1 - x$ , in which case we are done, or  $G_1 - x$  is of type 1 or 2 (for each  $x \in S$ ). Suppose the latter. Thus, if  $G_1 - x$  is of type 2, i.e., if  $G_1 - x$  is  $s$ -regular for all  $x \in S$ , then  $s \leq |S| = 1$ , and  $G_1 - x = 3K_2$  is also of type 1. Therefore,  $G_1 - x$  is of type 1, and the  $3K_{s+1}$  of  $G_1 - x$  must be disjoint from  $S \subseteq V(G_1)$ , whence  $|S| = s$  and  $G_1 - S$  is  $3K_{s+1}$ . Since  $Y = \{y_1, y_2, y_3\}$  is an arbitrary 3-set in  $V_2 - z$ , with one vertex on each  $t$ -regular component, since

$$V_1 - S = V_1 \cap (N(y_1) \cup N(y_2) \cup N(y_3))$$

must be the vertices of  $3K_{s+1}$  in  $G_1$ , and since  $G_1$  cannot have more than three  $K_{s+1}$ 's, each vertex in any  $t$ -regular component of  $G_2 - z$  is adjacent to  $s+1$  vertices of the  $3K_{s+1}$  in  $G_1$ . By Theorem 5, for any  $y \in Y$ ,  $N(y) \cap V_1$  is a  $K_{s+1}$  of  $G_1$ . By Theorem 5, for a given  $K_{s+1}$ , say  $C$ , in  $G_1$ ,

$$\{y \in V_2 : d(y, G_2) = t \text{ and } N(y) \cap V_1 = V(C)\}$$

is a complete graph in  $G_2$ , and hence  $G_2$  contains  $3K_{t+1}$  which together with  $3K_{s+1}$  of  $G_1$  forms  $3K_{\Delta+1}$  in  $G$ . Hence,  $G$  is of type 1.

*Case II.* Suppose

$$p_1 = 4s + 5, \quad p_2 = 4t + 5,$$

where  $\Delta(G_1) \leq s$  and  $\Delta(G_2) \leq t$ , and

$$s + t + 2 = \Delta(G).$$

Let  $x_i \in V_i$  ( $i = 1, 2$ ) be chosen nonadjacent. For  $i = 1$  or  $2$ ,  $G_i - x_i$  has  $4s + 4$  or  $4t + 4$  vertices, respectively, and maximum degree at most  $s$  or  $t$ , respectively. By Theorem 1,  $(s+1)K_4$  can be

$K_3$  and  $K_4$  on  $X \cup Y$  so that the  $K_3$  is on  $\{x_1, x_2, x_3\}$ ,  $\{x_1, x_2, y_3\}$ , or  $\{x_1, x_2, y_4\}$ . Since the neighborhoods do not overlap, and since each  $z \in V_2$  is adjacent to a vertex on which the  $K_3$  is placed,  $N(x_3) \cap V_2 = N(y_3) \cap V_2 = N(y_4) \cap V_2$ . Letting  $Y$  roam over all  $s$   $K_4$ 's placed in  $G_1$ , where  $y_3$  and  $y_4$  are not adjacent to  $X$ , we obtain  $2s + 1 = s + t + 1 = \Delta$  vertices ( $x_3$ ,  $s$  values of  $y_3$ ,  $s$  values of  $y_4$ ), with identical neighborhoods in  $V_2$ . Now, since  $x_1$  and  $x_2$  are each adjacent to  $s$  vertices in  $G_1$ , they are each adjacent to at most  $\Delta - s = t + 1$  vertices in  $V_2$ , and so by the conditions of Case I and by (21),

$$\Delta \geq |N(x_3) \cap V_2| \geq |V_2| - 2(t + 1) = 2t + 1 = \Delta .$$

Thus, equality holds, and  $x_3$  is contained in a  $K_{\Delta, \Delta}$  in  $G$ . It also follows that  $(N(x_1) \cap V_2) \cup (N(x_2) \cap V_2) = 2K_{t+1}$ , because  $G_2 = 2K_{t+1} \cup \bar{K}_{2t+1}$ , and the  $K_{\Delta, \Delta}$  of  $G$  uses the vertices of  $\bar{K}_{2t+1}$  in  $G_2$ . By the second part of Theorem 5, and since (12) implies (10),  $N(x_1) \cap V_2$  and  $N(x_2) \cap V_2$  are the two  $K_{t+1}$ 's of  $G_2$ . Also, by the same part of Theorem 5, for any  $z \in N(x_i) \cap V_2$  ( $i = 1, 2$ ),  $N(z) \cap V_1$  is a  $K_{s+1}$  in  $G_1$ . For any  $x'_i$  in the same  $K_{s+1}$  of  $G_1$  as  $x_i$ ,  $N(x'_i) \cap V_2$  is the  $K_{t+1}$  of  $G_2$  containing  $z$ , and thus  $N(x'_i) \cap V_2 = N(x_i) \cap V_2$ . By (12),

$$G = 2K_{s+1+t+1} \cup K_{\Delta, \Delta} = 2K_{\Delta+1} \cup K_{\Delta, \Delta} ,$$

and since (12) and (20) imply that  $\Delta$  is odd,  $G$  is of type 2.

Suppose that (13) holds.

If  $s = 0$ , then each vertex of  $V_1$  is adjacent to  $\Delta = t + 1$  vertices of  $G_2$ . Since  $|V_1| = 3$ , at least  $t$  vertices of  $V_2$  are adjacent to no vertex of  $V_1$ . Any one of these  $t$  vertices may be put into  $V_1$ , whence  $|V_1| = 4$ ,  $|V_2| = 4t + 2$ , which is essentially Case III.

Suppose  $s \geq 1$ . If  $tK_4 \cup K_2$  can be placed in  $G_2 - z$ , then we are done. Otherwise,  $G_2 - z$  is of type 1 or type 2, and contains three  $t$ -regular components. Since  $\Delta(G_2) = t$ ,  $z$  is adjacent to no vertex in a  $t$ -regular component of  $G_2 - z$ . Hence,  $tK_4 \cup K_3$  can be placed in  $G_2$ , where the  $K_3$  is placed on any set  $Y \subseteq V_2$  having one vertex in each  $t$ -regular component of  $G_2 - z$ .

placed in  $G_1 - x_1$ , and  $(t+1)K_4$  can be placed in  $G_2 - x_2$ . Since  $K_2$  can be placed on  $\{x_1, x_2\}$ , it follows that  $\Delta K_4 \cup K_2$  can be placed in  $G$ .

*Case III.* Suppose

$$p_1 = 4s + 2, \quad p_2 = 4t + 4,$$

where  $\Delta(G_1) \leq s$  and  $\Delta(G_2) \leq t$ , and

$$(22) \quad s + t + 1 = \Delta(G).$$

If  $sK_4 \cup K_2$  can be placed in  $G_1$ , then we can place  $\Delta K_4 \cup K_2$  on  $G$ , because the theorem of Hajnal and Szemerédi ensures that  $(t+1)K_4$  can be placed on  $G_2$ . Otherwise,  $s \geq 1$ , and by our induction hypothesis, either  $G_1$  is  $2K_{s+1} \cup K_{s,s}$ , or  $G_1$  contains  $3K_{s+1}$ . In either case, we can place  $(s-1)K_4 \cup 2K_3$  on  $G_1$ , where the  $K_3$ 's are placed on  $X = \{x_1, x_2, x_3\}$  and  $Y = \{y_1, y_2, y_3\}$ . We allow  $X$  and  $Y$  to be arbitrary disjoint 3-sets, with one vertex from each  $s$ -regular component of  $G_1$ . Note that for any such sets  $X$  and  $Y$ , such a placement is always possible.

Since

$$d(x_i, G_1) = d(y_i, G_1) = s \quad \text{for } i = 1, 2, 3,$$

it follows from (22) that

$$(23) \quad \sum_1^3 d(x_i, V_2) \leq 3(\Delta - s) = 3t + 3, \quad \sum_1^3 d(y_i, V_2) \leq 3t + 3,$$

with equality only if  $d(x_i, V_2) = d(y_i, V_2) = t + 1$ ,  $i = 1, 2, 3$ . Let

$$S_X = \{v \in V_2 : N(v) \cap X = \emptyset\}$$

and

$$S_Y = \{v \in V_2 : N(v) \cap Y = \emptyset\}.$$

By (23),

$$(24) \quad |S_X| = p_2 - \left| \bigcup_1^3 N(x_i, V_2) \right| \geq p_2 - (3t + 3) = t + 1,$$

and likewise,

$$(25) \quad |S_Y| \geq t + 1,$$

with equality in both (24) and (25) only if

$$(26) \quad d(x_i, V_2) = d(y_i, V_2) = t + 1, \quad i = 1, 2, 3,$$

only if

$$(27) \quad N(x_i) \cap V_2, \quad i = 1, 2, 3, \quad \text{are disjoint}$$

and only if

$$(28) \quad N(y_i) \cap V_2, \quad i = 1, 2, 3, \quad \text{are disjoint.}$$

If  $t = 0$  and for some  $v \in V_2$ ,  $S_X = S_Y = \{v\}$  for any choice of  $X$  and  $Y$ , then each  $v' \in V_2 - v$  is adjacent to  $s + 1$  of the  $3s + 3$  vertices of degree  $s$  in  $G_1$ . By Theorem 5, for each  $v'$ ,  $N(v') = K_{s+1}$ , and  $G$  contains  $3K_{s+2} = 3K_{\Delta+1}$ . Otherwise,  $|S_X \cup S_Y| > 1$ .

For any  $x_4 \in S_X$  and  $y_4 \in S_Y$ , with  $x_4 \neq y_4$ ,  $(s + 1)K_4$  can be placed in  $G - (G_2 - \{x_4, y_4\})$ . If  $tK_4 \cup K_2$  can be placed in  $G_2 - \{x_4, y_4\}$  for some  $x_4 \in S_X$  and  $y_4 \in S_Y$  and for some  $X$  and  $Y$ , then we are done. Otherwise, by the induction hypothesis,  $G_2 - \{x_4, y_4\}$  is  $2K_{t+1} \cup K_{t,t}$  ( $t$  odd) or  $G_2 - \{x_4, y_4\}$  contains  $3K_{t+1}$  for any disjoint  $X$  and  $Y$ , each with one vertex from each  $s$ -regular component of  $G_1$ , and for any  $x_4 \in S_X$ ,  $y_4 \in S_Y$ . Also, if  $tK_4 \cup K_2$  cannot be placed, then  $t \geq 1$ .

If  $t = 1$  and  $G_2 - \{x_4, y_4\} = 2K_{t+1} \cup K_{t,t}$ , then  $G_2 - \{x_4, y_4\}$  contains  $3K_{t+1}$ . Otherwise,  $t \geq 2$ , and  $|S_X| \geq t + 1 \geq 3$ . In that case, choose  $x'_4 \in S_X - \{x_4, y_4\}$ . If  $G_2 - \{x_4, y_4\}$  is of type 2, then  $G_2 - \{x'_4, y_4\}$  has vertices of degree  $t - 1$ , namely,  $N(x'_4) \cap V_2$ , and

so it cannot be of type 2. Therefore, for  $t \geq 1$ ,  $G_2 - \{x_4, y_4\}$  is of type 1, containing  $3K_{t+1}$ , regardless of the choice of  $X, Y$ , and  $x_4 \in S_X$ ,  $y_4 \in S_Y$ .

Suppose that for some  $X, Y$ ,

$$|S_X \cup S_Y| > t + 1 .$$

Then, since  $p_2 = 4(t + 1)$  and  $G_2$  has  $3K_{t+1}$ ,  $S_X \cup S_Y$  has a vertex, say  $x_4 \in S_X$ , inside a  $K_{t+1}$ , and a vertex, say  $y_4 \in S_Y$ , outside that same  $K_{t+1}$ . Therefore,  $G_2 - \{x_4, y_4\}$  has  $K_t$  as a component, and is neither type 1 nor type 2, a contradiction. Therefore,  $|S_X \cup S_Y| \leq t + 1$ , and so equality holds in (24) and (25), and as a consequence, we have (26), (27), and (28). By (22), we can use the latter part of Theorem 5, and thus conclude that each  $N(x_i) \cap V_2$  and each  $N(y_i) \cap V_2$  is a  $K_{t+1}$  of  $G_2 - S_X$ . It also follows that the three  $K_{t+1}$ 's of  $G_2$  pair with three  $K_{s+1}$ 's of  $G_1$  to form  $3K_{\Delta+1}$  in  $G$ . Hence,  $G$  is of type 1, and the proof is complete.

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*Received November 6, 1979.*