Discrete Mathematics 27 (1979) 149–170

SUBGRAPHS WITH TRIANGULAR COMPONENTS*

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Received 4 May 1970

Corrádi and Hajnal [5] showed that if the minimum degree $\delta(G)$ of a graph on p vertices satisfies the inequality $\delta(G) \geq \frac{1}{3}(2p-1)$, then G has a subgraph consisting of $[\frac{1}{3}p]$ triangular components. They gave a class of graphs which shows that their inequality is best possible. In this paper, we characterize the extremal graphs G, and we thereby show that there are two classes of graphs G for which the inequality is best possible.

1. Introduction

Let G and H be graphs on p vertices such that

$$(\Delta(\bar{G})+1)(\Delta(H)+1) \leq p+1, \tag{1.1}$$

where Δ denotes the maximum degree, and $\bar{G}$ denotes the complement of G. (We follow the notation of Harary [7].) We conjectured (see [2] or [3]) that (1.1) is sufficient to insure that H is a subgraph of G, and we gave two classes of graphs to show that if the right side of (1.1) is replaced $p+2$, then the conclusion fails for two classes of graphs. When $\Delta(H)=2$, (1.1) is equivalent to

$$\delta(G) \geq \tfrac{1}{3}(2p-1). \tag{1.2}$$

Corrádi and Hajnal [5] gave the following result.

Theorem 1.1. *Let G and H be graphs on p vertices, such that every component of H is a triangle, except possibly for one component that is either K_1 or K_2. If* (1.2) *holds, then H is a subgraph of G.*

The extremal graphs, i.e., those graphs G for which

$$\delta(G) = \tfrac{2}{3}(p-1) \tag{1.3}$$

and such that H is not a subgraph of G of Theorem 1.1 are the only known extremal graphs in the case $\Delta(H)=2$ of the aforementioned conjecture.

* This work is part of the author's Ph.D. dissertation, done at the Ohio State University under Prof. Neil Robertson.
This result was announced in a paper presented at the 8th Southeastern Conference on Combinatorics.

It is often convenient to work with the complement $\bar{G}$, rather than G, because $\bar{G}$ has fewer edges. Then H is a subgraph of G if and only if H and $\bar{G}$ can be placed on the same vertex set with no overlapping edges.

The graphs in the first class of extremal graphs, referred to as *type 1*, are the graphs $\bar{G}$ consisting of two cliques K_{b+1}, where $p=3b+1$, $b>0$. These cliques are components of $\bar{G}$, since $\delta(G)=\frac{2}{3}(p-1)$ implies $\Delta(\bar{G})=\frac{1}{3}(p-1)=b$. We allow edges on the remaining $p-2(b+1)=b-1$ vertices of a type 1 graph $\bar{G}$. We also denote the class of type 1 graphs on $p=3b+1$ vertices by $C_1(b)$.

The second class of extremal graphs (referred to generally as *type 2* graphs), denoted $C_2(b)$, consists of the graph on $p=3b+1$ vertices, with b odd, such that $\bar{G}$ contains a clique K_{b+1} and a biclique $K_{b,b}$. Since $\delta(G)=\frac{2}{3}(p-1)$, these are separate components of $\bar{G}$. Of course, $|C_2(b)|=1$ for each odd b.

It is readily verified that the graph H, containing $[\frac{1}{3}p]$ triangular components, is not a subgraph of a type 1 or a type 2 graph on $p=3b+1$ vertices.

We shall show that the type 1 and type 2 graphs are the *only* graphs with $\delta(G)\geqslant\frac{2}{3}(p-1)$ which do not contain $[\frac{1}{3}p]$ disjoint triangles.

2. Related literature

The following result of ours was announced in [1] (see footnote, page 226 of [1]) and it appears with proof in [2]. It was independently obtained by Sauer and Spencer [8].

Theorem 2.1. *If two graphs G and H, each on p vertices, satisfy*

$$2\Delta(H)\Delta(\bar{G})\leqslant p-1, \tag{2.1}$$

then H is a subgraph of G.

Theorem 2.1 agrees with conjecture if $\Delta(H)$ or $\Delta(\bar{G})$ is 1, but otherwise the inequalities differ by a factor of almost 2.

Hajnal and Szemeredi [6] generalized Theorem 1.1, and they obtained the following result.

Theorem 2.2. *If two graphs G and H on p vertices satisfy* (1.1) *and if H has $[p/(\Delta(H)+1)]$ components isomorphic to $K_{\Delta(H)+1}$, then H is a subgraph of G.*

The extremal graphs of Theorem 1.1 generalize to extremal graphs for Theorem 2.2: one merely uses components K_{b+1} and possibly a single $K_{b,b}$ with $b=\Delta(\bar{G})$, instead.

Several other related results are mentioned in [3].

We shall need the following theorem (see [4]) to prove our main result. Let

$x_1 \in X_1$ and $x_2 \in X_2$ satisfy the requirement that

$$f(X_1, X_2) = f(X_1 - x_1 + x_2, X_2 - x_2 + x_1),$$

then x_1 and x_2 are said to be *interchangeable.*

Theorem 2.3. *Let G be a graph with $p \geq 2$ and*

$$\delta(G) = c(p-1) \tag{2.2}$$

for some $c \in [0, 1)$. There is a nontrivial partition $X_1 \cup X_2$ of $V(G)$ which maximizes

$$f(X_1, X_2) = \tfrac{1}{2}(1-c)(p_1^2 + p_2^2) - |E(\bar{G}_1)| - |E(\bar{G}_2)|, \tag{2.3}$$

where $G_1 = G[X_1]$ and $G_2 = G[X_2]$ are induced subgraphs, and $p_i = |X_i|$, for $i = 1, 2$. This partition satisfies

$$\delta(G_i) \geq c(p_i - 1). \tag{2.4}$$

Furthermore, suppose $x_1 \in X_1$ and $x_2 \in X_2$ are adjacent in $\bar{G}$ and satisfy

$$\deg_{G_1}(x_1) + \deg_{G_2}(x_2) = \delta(G). \tag{2.5}$$

Then x_1 and x_2 are interchangeable,

$$\deg_G(x_1) = \deg_G(x_2) = c(p-1), \tag{2.6}$$

and the vertices X_{3-i} interchangeable with x_i are adjacent in $\bar{G}_{3-i}$ to x_{3-i}, for $i = 1$ and 2.

3. The main results

We shall prove the following two theorems:

Theorem 3.1. *Let G and H be graphs on p vertices, and suppose that every component of H is isomorphic to either K_1, K_2, or K_3. Let $b = b(H)$ denote the number of triangular components of H, and suppose $b \geq 0$. If*

$$\delta(G) \geq [\tfrac{1}{2}(p+b)],$$

and if H is not a subgraph of G, then either

There is a set S of $b-1$ vertices of G such that $G - S$ is a complete bipartite graph; or (3.1)

There is a set S of $b+1$ vertices, b odd, such that $G - S$ has two components, both isomorphic to K_b, and H has $\frac{1}{3}(p-1)$ triangles. (3.2)

Theorem 3.2. *Let G and H be graphs on p vertices and suppose that every component of H is a triangle K_3, except for one vertex K_1 if $p = 3b+1$, or one edge*

K_2 if $p=3b+2$. If

$$\delta(G)\geq\tfrac{2}{3}(p-1),$$

then H is not a subgraph of G if and only if both

$$\delta(G)=\tfrac{2}{3}(p-1)=2b$$

and G is of type 1 or type 2.

Lemma 3.3. *Let G be a graph with $p=3b+1$ vertices, for some integer b, and with $\delta(G)\geq 2b$. If for some set $S\subseteq V(G)$, with $|S|=b-1$, $G-S$ is bipartite, with bipartition $V_1\cup V_2$, then the following conclusions hold.*

Every vertex of S is adjacent to every vertex of $G-S$;

$$|V_1|=|V_2|;$$

$G-S$ is a complete bipartite graph.
Thus, G is of type 1.

Proof. Without loss of generality, assume that $|V_1|\geq|V_2|$. We have

$$|V_1|\geq\tfrac{1}{2}(p-|S|)=\tfrac{1}{2}(3b+1-(b-1))=b+1.$$

Let $v_1\in V_1$. Since $V_1\cup V_2$ is a bipartition of $G-S$, v_1 is adjacent in $\bar{G}$ to every vertex of V_1-v_1. But

$$\Delta(\bar{G})=p-\delta(G)-1\leq b,$$

and hence we must have $|V_1|=b+1$ and $\delta(G)=2b$. Also, since $\delta(G)\geq 2b$, each $v_1\in V_1$ must be adjacent to every vertex of $G-V_1$, i.e., to every vertex of V_2 and every vertex of S. The conclusions of the lemma follow directly.

Remarks. If G is of type 2, then $p\equiv 4 \pmod 6$, and G is regular of degree $2b=\tfrac{2}{3}(p-1)$. Note that the only graph that is both of type 1 and type 2 is the 4-cycle.

Lemma 3.4. *Let G be a graph with $p=3b+1$ vertices, for some integer b, and with $\delta(G)\geq 2b$. If for some set $S\subseteq V(G)$, with $|S|=b+1$, $G-S$ has two components, then the following conditions hold.*

Every vertex of S is adjacent to every vertex of $G-S$;
$G-S$ has two components, both isomorphic to K_b.
If, furthermore, b pairwise disjoint triangles do not embed in G, then
$p\equiv 4 \pmod 6$;
S is an independent set;
G is of type 1 only if G is a 4-cycle.
Thus, G is of type 2.

Proof. Let G and S satisfy the hypotheses. Since $p=3b+1$ and $\delta(G)\geq 2b$, any since $G-S$ has two components, any vertex in the smaller component is adjacent in $\bar{G}$ to at least $\frac{1}{2}|V(G-S)|=b$ vertices in the larger component of $G-S$. But these statements force equality: both components have just b vertices. Also, the first two conclusions of the lemma follow immediately.

If S is not an independent set or if $p\not\equiv 4 \pmod 6$, then either $G[S]$ has an edge, or, since $p=3b+1$, $p\equiv 1 \pmod 6$. In either case, an embedding of b pairwise disjoint triangles is easily found. The rest is easy.

Proof of Theorm 3.1 from Theorm 3.2. Assume without loss of generality that the components of H consist of b triangles K_3, $[\frac{1}{2}(p-3b)]$ edges K_2, and $p-3b-2[\frac{1}{2}(p-3b)]$ vertices K_1. By adding $[\frac{1}{2}(p-3b)]$ vertices to H, each adjacent to both ends of a K_2, we can construct a graph H' on $p+[\frac{1}{2}(p-3b)]$ vertices, where the components of H' consist of $b+[\frac{1}{2}(p-3b)]$ triangles K_3 and $p-3b-2[\frac{1}{2}(p-3b)]$ $(=0$ or $1)$ vertices K_1. By adding an independent set of $[\frac{1}{2}(p-3b)]$ vertices to G, we construct a graph G' in which each added vertex is adjacent to every vertex of G. Thus,

$$|V(G')|=p+[\tfrac{1}{2}(p-3b)]=[\tfrac{3}{2}(p-b)],$$

and

$$\begin{aligned}\delta(G')&\geq \min(p, \delta(G)+[\tfrac{1}{2}(p-3b)]\\&\geq \min(p, [\tfrac{1}{2}(p+b)]+[\tfrac{1}{2}(p-3b)])\\&= \min(p, 2[\tfrac{1}{2}(p-b)])=2[\tfrac{1}{2}(p-b)]\\&=\tfrac{2}{3}(3[\tfrac{1}{2}(p-b)])\geq \tfrac{2}{3}(|V(G')|-1).\end{aligned}$$

Thus, by Theorem 3.2, either H' is a subgraph of G', or G' is a graph of type 1 or type 2. Suppose G' is a graph of type 2. If $G'\neq G$, then G' has a vertex of degree p and $|V(G')|<\frac{3}{2}p$. Hence, G' is not a graph of type 2 unless $G'=G$. Then by Lemma 3.4,

$$|V(G')|\equiv 4 \pmod 6.$$

In this case $[\frac{1}{2}(p-3b)]=0$ vertices were added to G to get G', whence $p-3b=1$, and H has $b=\frac{1}{3}(p-1)$ triangles, and we have the second case of Theorem 3.1.

Suppose G' is a graph of type 1. Then H' has

$$b'=b+[\tfrac{1}{2}(p-3b)]=[\tfrac{1}{2}(p-b)]$$

triangles. Moreover, $|V(G')|=3b'+1$, and there is a set $S'\subseteq V(G')$, with $|S'|=b'-1$, whose removal leaves a complete bipartite graph $G'-S'=K_{b'+1,\,b'+1}$. We have

$$\delta(G')\geq 2[\tfrac{1}{2}(p-b)]=2b'.$$

We claim that $V(G')=V(G)\cup S'$. To prove this, suppose that $V(G)\cup S'$ does not contain a vertex $v\in V(G')-V(G)$. However, $V(G')-V(G)$ has only

$[\frac{1}{2}(p-3b)]$ vertices, and so some vertex w of G lies on the same side of the bipartition as v. But v is adjacent to all vertices of G, and in particular to w, and we have a contradiction, which proves the claim.

Let $S=V(G)\cap S'$. Then by the claim,

$$|S|=|S'|-(|V(G')|-|V(G)|)$$
$$=(b+[\tfrac{1}{2}(p-3b)]-1)-[\tfrac{1}{2}(p-3b)]=b-1,$$

and $G-S$ is bipartite. This is a conclusion of 3.1.

The remaining possibility is that H' is a subgraph of G'. There is an embedding of H' into G' which extends an embedding of H into G. This proves Theorem 3.1.

Lemma 3.5. *Let G be a graph, and $X_1\cup X_2$ be partition of $V(G)$ of the type described in Theorem 2.3 for which*

$$\delta(G_1)+\delta(G_2)=\delta(G). \tag{3.3}$$

Suppose that sets $Y_3\subseteq X_1$, $V_3\subseteq X_2$ exist such that

G_2-V_3 contains a spanning complete bipartite subgraph with nontrivial bipartition $V_1\cup V_2$; (3.4)

G_1-Y_3 is a complete bipartite graph with nontrivial bipartition $Y_1\cup Y_2$: (3.5)

If $v\in V_1\cup V_2$ the $\deg_{G_2}(v)=\delta(G_2)$; (3.6)

If $y\in Y_1\cup Y_2$ then $\deg_{G_1}(y)=\delta(G_1)$. (3.7)

Then any vertex of $Y_1\cup Y_2$ is adjacent to every vertex in V_j, for some $j\in\{1,2\}$. Suppose further that

No vertex of $Y_1\cup Y_2$ is adjacent to vertices in both V_1 and V_2. (3.8)

Then $G-(Y_3\cup V_3)$ is a complete bipartite graph.

Proof. By (3.3), (3.6), and (3.7), the latter part of Theorem 2.3 may be applied to the vertices of $V_1\cup V_2\cup Y_1\cup Y_2$.

Suppose that the first conclusion of the lemma is false for some $y\in Y_1\cup Y_2$. Thus, y is not adjacent in G to a vertex v_1 of V_1 and a vertex v_2 of V_2. By Theorem 2.3, v_1 and v_2 are interchangeable with y, and are thus not adjacent in G. But, by (3.4), v_1 is adjacent to v_2. This contradiction proves the first part of the lemma.

By (3.8), any vertex $y\in Y_1\cup Y_2$ is adjacent in $\bar{G}$ to every vertex V_j for some $j\in\{1,2\}$. By the first part of the lemma, which was just proved, y is adjacent in G to every vertex of V_{3-j}.

Thus, the vertices of $Y_1\cup Y_2$ fall into two classes: those, the set of which we denote Y, which are adjacent (in G) to vertices of V_1 but not V_2; and those, the

Let $y_4 \in Y_4$. Since y_4 is interchangeable with, but not adjacent to, each vertex of V_2, Theorem 2.3 shows that V_2 is actually an independent set. Similarly, so is V_1. Therefore $V_1 \cup V_2$ induces a complete bipartite graph in G_2.

We claim that $\{Y_4, Y_5\} = \{Y_1, Y_2\}$. To see this, suppose that $Y_4 \cap Y_1$ and $Y_4 \cap Y_2$ are both nonempty. Then any vertex $v_2 \in V_2$ is not adjacent in G to a vertex $y_1 \in Y_4 \cap Y_1$, nor to a vertex $y_2 \in Y_4 \cap Y_2$. By Theorem 2.3, y_1 and y_2 are interchangeable with v_2 and are thus not adjacent. However, (3.5) implies that y_1 and y_2 are adjacent. This contradiction shows that either $Y_4 \cap Y_1$ or $Y_4 \cap Y_2$ is empty. Similarly, either $Y_5 \cap Y_1$ or $Y_5 \cap Y_2$ is empty. Since $V_1 \cup V_2$ and $Y_1 \cup Y_2$ are nontrivial, and

$$Y_4 \cup Y_5 = Y_1 \cup Y_2,$$

the claim must follow.

In either case of this claim, there is a $j \in \{1, 2\}$ such that $(V_1 \cup Y_j) \cup (V_2 \cup Y_{3-j})$ is a bipartition of $G - (Y_3 \cup V_3)$, and this bipartite graph is complete. This proves Lemma 3.5.

We define for $X_j' \subseteq V(G)$

$$G_j' = G[X_j'] \quad j = 1, 2,$$

and

$$p_j' = |X_j'|, \quad j = 1, 2.$$

A vertex v of G, G_j or G_j' is *critical* in G, G_j, G_j', if

$$\deg_G(x) - 1 < \tfrac{1}{3}(p-1),$$
$$\deg_{G_j}(x) - 1 < \tfrac{1}{3}(p_j - 1),$$

or

$$\deg_{G_j'}(x) - 1 < \tfrac{1}{3}(p_j' - 1),$$

respectively.

Lemma 3.6. *Suppose Theorem* 3.2 *is valid for all graphs with fewer than p vertices. Suppose*

$$p \equiv 1 \pmod 3$$

and that $X_1 \cup X_2$ is a partition of $V(G)$ which satisfies the conditions of Theorem 2.3 *with $c = \frac{2}{3}$. For $\{z, z'\} \subseteq V(G)$, write*

$$X_j' = X_j - \{z, z'\} \quad \textit{for } j = 1, 2,$$

and assume that

$$p_j' \equiv 1 \pmod 3 \quad \textit{for } j = 1, 2,$$

that

$$\delta(G_j') \geq \tfrac{2}{3}(p_j' - 1) \quad \textit{for } j = 1, 2, \tag{3.9}$$

and that $p_j \equiv 0 \pmod 3$ for $j \in \{1, 2\}$ implies that $z \in X_j$ and that there exist critical vertices $x_3, x_4 \in X_{3-j}$ such that $G[z, x_3, x_4]$ is a triangle. Then, if $\frac{1}{3}(p_j' - 1)$ pairwise disjoint triangles cannot be embedded in G_j', for $j = 1$ and $j = 2$, both G_1' and G_2' are of type 1.

Proof. Since Theorem 3.2 holds for graphs on fewer than p vertices, since (3.9) holds, and since $\frac{1}{3}(p_j' - 1)$ triangles cannot be embedded in G_j', $j = 1, 2$, it follows that G_1' is of type 1 or type 2, and G_2' is of type 1 or type 2. Thus

$$\delta(G_j') = \tfrac{2}{3}(p_j' - 1), \qquad j = 1, 2,$$

whence

$$\delta(G_1') + \delta(G_2') = \tfrac{2}{3}(p_1' + p_2' - 2) = \tfrac{2}{3}(p - 1) - 2. \tag{3.10}$$

Moreover, by Theorem 2.3,

$$\delta(G_1) + \delta(G_2) \geq \tfrac{2}{3}(p - 1) - \tfrac{2}{3}.$$

The left side is an integer and $p \equiv 1 \pmod 3$, whence

$$\delta(G_1) + \delta(G_2) \geq \tfrac{2}{3}(p - 1).$$

So that (3.10) also holds, it follows that *if z or z', respectively, is in X_j, for $j \in \{1, 2\}$, then z or z' is adjacent in G to every critical vertex of G_j'.* In fact

$$\delta(G_1) + \delta(G_2) = \tfrac{2}{3}(p - 1), \tag{3.11}$$

and vertices critical in G_j' are critical in G_j, for $j = 1, 2$. Also, since critical vertices of G_1 and critical vertices of G_2 are interchangeable if they are adjacent in $\bar{G}$, critical vertices of G_1' and critical vertices of G_2' are also interchangeable if they are adjacent in $\bar{G}$. By (2.6) of Theorem 2.3, such vertices are also critical in G.

Without loss of generality, it suffices to show that G_2' is of type 1.

Let y_1, y_2 be any pair of adjacent critical vertices of G_1'. If G_1' is of type 2, then every vertex of G_1' is critical in G_1', whence, any adjacent pair suffices. If G_1' is of type 1, then $p_1' \geq 4$, and there is a set $Y_3 \subseteq X_1'$ with

$$|Y_3| = \tfrac{1}{3}(p_1' - 1) - 1,$$

such that $G_1' - Y_3$ is a complete bipartite graph with bipartition $Y_1 \cup Y_2$, where

$$|Y_1| = |Y_2| = \tfrac{1}{3}(p_j' - 1) + 1.$$

Since $Y_1 \cup Y_2$ is the set of critical vertices in G_1' if $y_1 \in Y_1$ and $y_2 \in Y_2$, then y_1 and y_2 are adjacent critical vertices of G_1'.

Suppose by way of contradiction that G_2' is of type 2 and not of type 1. Then every vertex v of G_2 is critical in G_2, and hence interchangeable with $y_i (i = 1, 2)$ if y_i is adjacent in $\bar{G}$ to v.

Since y_i is critical in G_1' and in G,

$$\begin{aligned}|E(y_i, X_2')| &= \deg_G(y_i) - \deg_{G_1'}(y_i) - |E(y_i, \{z, z'\})| \\ &= \tfrac{2}{3}(p-1) - \tfrac{2}{3}(p_1'-1) - |E(y_i, \{z, z'\})| \\ &= \tfrac{2}{3}(p_2'+2) - |E(y_i, \{z, z'\})|.\end{aligned}$$

Hence, the number of vertices of G_2' adjacent in $\bar{G}$ to y_i is at least

$$p_2' - |E(y_i, X_2')| \geq \tfrac{1}{3}(p_2'-1) + |E(y_i, \{z, z'\})| - 1,$$

and these vertices are interchangeable with y_i and thus form an independent set.

We have two cases: when $\{z, z'\} \cap X_1$ is not empty, and when $\{z, z'\} \subseteq X_2$. In the first case, without loss of generality, suppose $z \in X_1$. In G_1, z is adjacent to every critical vertex of G_1', including $y_1, y_2 \in X_1'$. Hence, $|E(y_i, \{z, z'\})| \geq 1$. In the second case, by the hypotheses of the lemma, $p_2 \equiv 0 \pmod 3$ and z lies in a triangle $G[z, x_3, x_4]$, where x_3 and x_4 are adjacent critical vertices of G_1. Pick y_1, y_2 so that $\{y_1, y_2\} = \{x_3, x_4\}$, which is possible because $G_1 = G_1'$ here. Then $|E(y_i, \{z, z'\})| \geq 1$. Therefore, in either case *there are at least* $\frac{1}{3}(p_2'-1)$ *critical vertices of* G_2' *interchangeable with* y_i $(i = 1, 2)$.

Since G_2' is of type 2, $p_2' \equiv 4 \pmod 6$, and since G_2' is not of both type 1 and type 2, it follows that $p_2' \geq 10$. Hence, at least $\frac{1}{3}(p_2'-1) \geq 3$ critical vertices of G_2' are interchangeable with $y_i (i = 1, 2)$. By Theorem 2.3, this set of $\frac{1}{3}(p_2'-1)$ vertices is an independent set. But since G_2' is of type 2, there is only one maximal independent set S_2 of more than 2 vertices, and S_2 has $\frac{1}{3}(p_2'-1)+1$ vertices. Therefore, each $y_i (i = 1, 2)$ is interchangeable with all but at most one vertex of S_2. Since $|S_2| \geq 3$, there is a critical vertex $v \in S_2$, critical in G_2, interchangeable with both vertices y_1, y_2 critical in G_1. By Theorem 2.3 y_1 and y_2 are not adjacent, contrary to the choice of y_1 and y_2. Hence, G_2' is of type 1, and the lemma is proved.

We leave to the reader the proofs of the next two lemmas.

Lemma 3.7. *Let* G_0 *be a graph of type* 1 *on* $3b_0+1$ *vertices. Let* S_0 *be the set of* b_0-1 *vertices whose removal leaves* $G - S_0 = K_{b_0+1, b_0+1}$. *Any embedding of* b_0-1 *pairwise disjoint triangles into* G_0 *uses all but four vertices* $v_1, v_2, v_3, v_4 \in V(G_0) - S_0$, *and these four vertices induce a* 4-*cycle in* G_0. *Furthermore,* v_1, v_2, v_3, v_4 *may be chosen to be any four vertices of* $G_0 - S_0$ *that induce a* 4-*cycle in* G.

Lemma 3.8. *Let* G_0 *be a graph of type* 2 *on* $3b_0+1$ *vertices. Let* S_0 *be the independent set of* b_0+1 *vertices such that* $G_0 - S_0$ *consists of two components, each* K_{b_0}. *Any embedding of* b_0-1 *pairwise disjoint triangles into* G_0 *uses all but four vertices, two in* S_0, *and one in each* K_{b_0}, *and these four vertices induce a* 4-*cycle in* $\bar{G}_0$. *Furthermore, for any four vertices of* $V(G_0)$ *with two in* S_0 *and one in each* K_{b_0}, *there is an embedding of* b_0-1 *pairwise disjoint triangles into the remaining* $3b_0-3$ *vertices of* G_0.

To save work, we assume without proof Theorem 1.1 of Corrádi and Hajnal [5]:

Proof of Theorem 3.2. By Theorem 1.1, it suffices to consider graphs G for which

$$\delta(G)=\tfrac{2}{3}(p-1).$$

Equality implies that

$$p\equiv 1 \pmod 3.$$

Thus, we can assume that H is a graph with b triangles and one isolated vertex, and that

$$p=3b+1, \qquad \delta(G)=\tfrac{2}{3}(p-1)=2b.$$

By Theorem 2.3 there are disjoint nonempty sets X_1, X_2 such that $V(G)=X_1\cup X_2$ and the induced subgraphs G_i, for $G_i=G[X_i]$, $i=1,2$, satisfy

$$\delta(G_i)\geqslant\tfrac{2}{3}(p_i-1), \tag{3.12}$$

where $p_i=|X_i|$.

Assume inductively that Theorem 3.2 is true for graphs smaller than G, and suppose that H is not a subgraph of G. Theorem 3.2 is true for $p\leqslant 4$, and so we have a basis for induction. We have two cases: either one of the sets X_i has cardinality a multiple of 3, or neither do. In one subcase (Subcase IIA), we show that if H is not a subgraph of G, then G is of type 2. In other subcases, we verify the hypotheses of Lemma 3.5, and hence there is a subset $S=Y_3\cup V_3$ of $V(G)$, with $|S|=b-1$, such that $G-S$ is a bipartite graph. Thus, by Lemma 3.3, G is of type 1. We consider each case below.

Case I. Suppose that

$$p_1\equiv 0 \pmod 3 \quad \text{and} \quad p_2\equiv 1 \pmod 3.$$

Since $\delta(G_1)$ is an integer and $p_1\equiv 0 \pmod 3$, (3.12) gives

$$\delta(G_1)\geqslant\tfrac{2}{3}(p_1-1)+\tfrac{2}{3}=\tfrac{2}{3}p_1,$$

and Theorem 1.1 implies that $\tfrac{1}{3}p_1$ triangles can be embedded in G_1. Write $b_1=\tfrac{1}{3}p_1$ and

$$b_2=\tfrac{1}{3}(p_2-1), \tag{3.13}$$

and note that

$$b_1+b_2=b,$$

and that the $b_1\geqslant 1$ triangles embedded in G_1 use each vertex of G_1. Since b triangles are assumed not to embed in G, it follows that b_2 triangles do not embed in G_2. By the induction hypothesis, either G_2 is of type 1, and there is a set

$V_3 \subseteq X_2$ with

$$|V_3| = b_2 - 1 \tag{3.14}$$

such that

$$G_2 - V_3 = K_{b_2+1,\, b_2+1},$$

or G_2 is of type 2 and there is an independent set

$$S_2 \subseteq X_2 \tag{3.15}$$

on b_2+1 vertices, such that $G_2 - S_2$ has two components, each a clique on b_2 vertices.

If G_2 is of type 2, each vertex $v \in X_2$ has degree

$$\deg_{G_2}(v) = 2b_2 = \tfrac{2}{3}(p_2 - 1).$$

If this alternative applies, write

$$V_2 = S_2, \qquad V_1 = G_2 - S_2, \tag{3.16}$$

and note that (3.4) and (3.6) of Lemma 3.5 hold with V_3 empty.

If G_2 is of type 1, let

$$V_1 \cup V_2 \text{ denote the bipartition of } G_2 - V_3. \tag{3.17}$$

Then (3.4) of Lemma 3.5 holds,

$$|V_1| = |V_2| = b_2 + 1,$$

and so (3.12) and (3.13) give

$$\delta(G_2) \geqslant 2b_2,$$

which allows us to apply Lemma 3.3. Also, by Lemma 3.3, each vertex of $V_j (j = 1, 2)$ is adjacent to every vertex of V_{3-j} and to every vertex of V_3, and if $v \in V_1 \cup V_2$,

$$\deg_{G_2}(v) = 2b_2 = \tfrac{2}{3}(p_2 - 1)$$

whence (3.6) holds if G_2 is of type 1.

It follows in either alternative (either G_2 of type 1 or type 2) that there must be at least

$$\deg_G(v) - \deg_{G_2}(v) \geqslant \tfrac{2}{3}(p-1) - \tfrac{2}{3}(p_2 - 1) = \tfrac{2}{3}p_1$$

vertices in X_1 adjacent to a given vertex $v \in V_1 \cup V_2$, and that (3.4) and (3.6) hold.

Denote by $N(v_1, v_2)$ the vertices of X_1 that are adjacent to both $v_1 \in V_1$ and $v_2 \in V_2$. We have

$$|N(v_1, v_2)| \geqslant 2(\tfrac{2}{3}p_1) - p_1 = b_1. \tag{3.18}$$

Since G_2 is of type 1 or type 2, b_2 disjoint triangles do not embed in G_2. By Lemmas 3.7 and 3.8, there is an embedding of $b_2 - 1$ pairwise disjoint triangles into G_2 such that the four remaining vertices induce a 4-cycle in G_2, with two of its vertices in V_1 and the other two in V_2. Let $\{v_1, v_2\}$ and $\{v_1', v_2'\}$ be disjoint edges of this 4-cycle, where $v_1, v_1' \in V_1$ and $v_2, v_2' \in V_2$.

In the two subcases below, we establish that the hypotheses of Lemma 3.5 apply to G_1 and G_2. We have already established (3.4) and (3.6), and it remains to establish (3.3), (3.5), (3.7), and (3.8).

Subcase IA. Suppose that distinct vertices $v_1, v_1' \in V_1$ and $v_2, v_2' \in V_2$ exist such that $\{N(v_1, v_2), N(v_1', v_2')\}$ possesses a transversal $\{y, y'\}$ in X_1, i.e., distinct $y, y' \in X_1$ such that

$$y \in N(v_1, v_2), \qquad y' \in N(v_1', v_2')$$

Since

$$\delta(G_1) \geqslant \tfrac{2}{3}p_1,$$

we have

$$\delta(G_1 - \{y, y'\}) \geqslant \tfrac{2}{3}p_1 - 2 = \tfrac{2}{3}(p_1 - |\{y, y'\}| - 1).$$

Since b pairwise disjoint triangles do not embed in G, and since $(b_2 - 1) + 2$ triangles can be embedded in $G[X_2 \cup \{y, y'\}]$, we cannot embed

$$b - (b_2 + 1) = b_1 - 1$$

triangles in $G_1 - \{y, y'\}$. By the induction hypotheses, $G_1 - \{y, y'\}$ is a graph of type 1 or of type 2, and by Lemma 3.6 with $\{y, y'\} = \{z, z'\}$, and with $\{v_1, v_2\} = \{x_3, x_4\}$, both $G_1 - \{y, y'\}$ and G_2 are of type 1. Therefore, there is a set Y_3' of $b_1 - 2$ vertices such that $G_1 - Y_3$ is bipartite, where

$$Y_3 = Y_3' \dot{\cup} \{y, y'\}. \tag{3.19}$$

Let $Y_1 \cup Y_2$ be the bipartition of $G_1 - Y_3$. By definition,

$$|Y_1| = |Y_2| = b_1 = \tfrac{1}{3}p_1,$$

and by Lemma 3.3, any vertex y_j of $Y_j (j = 1, 2)$ is adjacent to every vertex of $Y_{3-j} \cup Y_3'$ and has degree $\tfrac{2}{3}p_1 - 2$ in $G_1 - \{y, y'\}$. Thus, (3.5) holds. Since

$$\deg_{G_1}(y_j) \geqslant \delta(G_1) = \tfrac{2}{3}p_1,$$

each vertex of Y_j is also adjacent to y and y', and hence has degree $\delta(G_1)$ in G_1, whence we have (3.7). Therefore

$$\delta(G_1) + \delta(G_2) = \tfrac{2}{3}p_1 + \tfrac{2}{3}(p_2 - 1) = \tfrac{2}{3}(p - 1) = \delta(G),$$

which is (3.3). We have thus proved (3.3) through (3.7) of Lemma 3.5.

We prove (3.8) by assuming that it is false: thus, there is a vertex $y'' \in Y_1 \cup Y_2$ adjacent to some $v_1'' \in V_1$ and to some $v_2'' \in V_2$. By the first conclusion of Lemma 3.5, y'' is adjacent to all vertices in either V_1 or in V_2. Without loss of generality, suppose that y'' is adjacent to all vertices in V_2.

By reconsidering $\{N(v_1, v_2), N(v_1', v_2')\}$ of this subcase, we shall produce a pair of such sets which have either $\{y, y''\}$ or $\{y', y''\}$ as a transversal. If $v_1''=v_1$, then choose $v_2''=v_2 \in V_2$ as another vertex to which y'' is adjacent. Then $\{y, y''\} \subseteq N(v_1, v_2)$, and $y'' \in N(v_1, v_2)$, $y' \in N(v_1', v_2')$ will be a transversal. On the other hand, if $v_1'' \neq v_1$, then choose $v_2'' \neq v_2$, $v_2'' \in V_2$. Then $y \in N(v_1, v_2)$, $y'' \in N(v_1'', v_2'')$ will be a transversal.

There is no loss of generality in using y', y'' for our transversal in the following argument. We shall produce a vertex y^* whose degree in G_1 is less than $\delta(G_1)=\frac{2}{3}p_1$, and this will be our desired contradiction.

Using the transversal $\{y', y''\}$, we repeat the argument of this subcase. Thus, there is a set Y_3'', instead of Y_3, such that $|Y_3''|=b_1$, G_1-Y_3'' is bipartite, and $y', y'' \in Y_3''$. Since $y'' \notin Y_3$ and since $|Y_3|=|Y_3''|$, there is a vertex $y^* \in Y_3-Y_3''$. Note that $G_1-\{y', y''\}$ is of type 1, just like $G_1-\{y, y'\}$.

By previous remarks of this subcase,

$$\deg_{G_1}(y_j)=\delta(G_1)=\tfrac{2}{3}p_1,$$

for every $y_j \in Y_j$, $j=1$ or 2. Since $y'' \in Y_1 \cup Y_2$, $\deg_{G_1} y''=\frac{2}{3}p_1$, and when we repeat the argument of this subcase using the transversal $\{y', y''\}$, the vertex $y^* \in Y_3-Y_3''$ has the same property:

$$\deg_{G_1} y^*=\tfrac{2}{3}p_1.$$

However, y^*, being in Y_3, is adjacent to all $\frac{2}{3}p_1$ members of $Y_1 \cup Y_2$, and hence is not adjacent to $y' \in Y_3$. Thus, $y^* \in V(G_1-Y_3'')$ is not adjacent to all $b_1=\frac{1}{3}p_1$ members of its own side of the bipartition of G_1-Y_3, and is not adjacent to $y' \in Y_3''$, whence $\deg_{G_1} y^*<\frac{2}{3}p_1$, a contradiction. Thus, (3.8) must hold.

Subcase IB. Suppose that there is no pair of disjoint vertices $v_1, v_1' \in V_1$, $v_2, v_2' \in V_2$ in $G_2[V_1 \cup V_2]$ such that $\{N(v_1, v_2), N(v_1', v_2')\}$ possesses a transversal.

Since $p_1>0$, (3.18) implies that $b_1 \geq 1$ and that $N(v_1, v_2)$ and $N(v_1', v_2')$ are nonempty. Since $\{N(v_1, v_2), N(v_1', v_2')\}$ possesses no transversal, we have $y \in X_1$ such that

$$N(v_1, v_2)=y=N(v_1', v_2').$$

Thus, by (3.18) $b_1=1$ and $p_1=3$. Since $\delta(G_1) \geq \frac{2}{3}p_1$, we see that G_1 is a triangle, and

$$\delta(G_1)=2=\tfrac{2}{3}p_1,$$

which is condition (3.7) of Lemma 3.5. Define

$$Y_3=\{y\}, \tag{3.20}$$

and let each of the other two vertices of G_1 be put in separate singleton sets Y_1, Y_2. Then (3.5) of Lemma 3.5 is valid. We have

$$\delta(G_1)+\delta(G_2)=2+\tfrac{2}{3}(p_2-1)=\tfrac{2}{3}p_1+\tfrac{2}{3}(p_2-1)=\tfrac{2}{3}(p-1)=\delta(G),$$

which proves (3.3).

Finally, we must show that (3.8) of Lemma 3.5 holds in this subcase. Suppose that (3.8) is false, and that there is a vertex $y'' \in Y_1 \cup Y_2$ such that $G[v_1'', v_2'', y'']$ is a triangle in G, where $v_1'' \in V_1$ and $v_2'' \in V_2$. Then $\{v_1'', v_2''\}$ overlaps $\{v_1, v_2\}$ and $\{v_1', v_2'\}$, for otherwise, we would be in Subcase IA, because $\{N(v_1'', v_2''), N(v_1, v_2)\}$ or $\{N(v_1'', v_2''), N(v_1', v_2')\}$ would have a transversal. Without loss of generality, suppose that $v_1'' = v_1$ and $v_2'' = v_2'$. Then $N(v_1', v_2) = y''$, because otherwise $\{N(v_1, v_2'), N(v_1', v_2)\}$ would have a transversal, and we would be in Subcase IA. But then v_1, v_2, v_1', v_2' are not only adjacent to $y \in Y_3$ of Subcase IB; they are also adjacent to $y'' \in Y_1 \cup Y_2$, and so $\{N(v_1, v_2), N(v_1', v_2')\}$ possesses the transversal $\{y, y''\}$, and IA applies. Thus, (3.8) must hold if we are to avoid this contradiction. This completes Subcase IB.

Thus, all of the hypotheses of Lemma 3.5 hold. We conclude from Lemma 3.5 that $G-(Y_3 \cup V_3)$ is a bipartite graph. Thus, $\bar{G}-(Y_3 \cup V_3)$ consists of two cliques. By (3.14), (3.19), and (3.20), we have

$$|Y_3 \cup V_3| = b-1,$$

and so by Lemma 3.3, G is of type 1. This completes Case I.

Case II. Suppose that

$$p_1 \equiv p_2 \equiv 2 \pmod 3.$$

Since $\delta(G_i)$ is an integer, (3.12) implies

$$\delta(G_i) \geq \tfrac{2}{3}p_i - \tfrac{1}{3} \tag{3.22}$$

for $i = 1, 2$. Without loss of generality, assume

$$p_1 \leq p_2.$$

Write

$$b_1 = \tfrac{1}{3}p_1 - \tfrac{2}{3}, \qquad b_2 = \tfrac{1}{3}p_2 - \tfrac{2}{3},$$

and note that b_1 and b_2 are integers such that

$$b_1 + b_2 + 1 = b.$$

If we form a graph $G_i + z$, adding to $G_i (i = 1, 2)$ a new vertex z adjacent to every vertex of G_i, then by (3.22),

$$\delta(G_i + z) = \tfrac{2}{3}|X_i + z|,$$

and by Theorem 1.1, $b_i + 1$ pairwise disjoint triangles can be embedded in $G_i + z$. Therefore, b_i pairwise disjoint triangles and an edge disjoint from the b_i triangles, which we shall call the *free edge*, can be embedded in G_i, for $i = 1, 2$. We shall attempt to use the vertices of the two free edges to form an extra triangle, disjoint from the b_1 triangles in G_1 and the b_2 triangles in G_2, thus constituting $b_1 + b_2 + 1 = b$ pairwise disjoint triangles in G. By assuming that b pairwise disjoint triangles do not embed in G, we shall determine the structure of G in the attempt to find such an embedding.

We show in the two subcases below that *either G is of type 2, or there is a vertex $x_3 \in X_2$ such that the free edge in G_1 together with x_3 form a triangle in G.* It may be necessary to alter the embedding of b_1 triangles and the free edge into G_1 in order to accomplish this.

Let x_1, x_2 be the ends of the free edge in G_1. Without loss of generality, *choose the free edge from among all possible free edges so that*

$$\deg_{G_1}(x_1)+\deg_{G_1}(x_2)$$

is minimized. If x_1 and x_2 are adjacent in G to a vertex $x_3 \in X_2$, then x_1, x_2, x_3 is the desire triangle. Otherwise, x_1 and x_2 are adjacent to no common vertex in X_2. Then

$$\begin{aligned}\deg_{G_1}(x_1)+\deg_{G_1}(x_2) &\geq 2\delta(G)-p_2\\ &\geq \tfrac{4}{3}(p-1)-p_2=p_1+\tfrac{1}{3}p-\tfrac{4}{3}.\end{aligned} \tag{3.23}$$

Also, without loss of generality, assume that

$$\deg_{G_1}(x_1) \geq \deg_{G_1}(x_2).$$

These inequalities imply

$$2\deg_{G_1}(x_1) \geq \deg_{G_1}(x_1)+\deg_{G_1}(x_2) \geq p_1+\tfrac{1}{3}p-\tfrac{4}{3}. \tag{3.24}$$

We define

$$\pi : V(H_1) \to V(G_1)$$

to be an embedding of b_1 triangles K_3 and one edge-component K_2, constituting H_1, into G_1 such that the edge-component K_2 is mapped to the free edge x_1, x_2 that minimizes $\deg_{G_1}(x_1)+\deg_{G_1}(x_2)$. We shall alter π if necessary, and then either we shall extend π to an embedding of H into G, where H consists of b triangular components and one isolated vertex, or we shall show (Subcase IIA) that G is of type 2 or (following the subcases) that G is of type 1.

Define

$$M(x)=\{x' \in X_1 \colon \pi^{-1}(x) \text{ and } \pi^{-1}(x') \text{ are adjacent in } H_1\}.$$

For $i=1,2$, and $x \in V(G)$, define

$$N_i(x)=\{x' \in X_i : x \text{ and } x' \text{ are adjacent in } G\}.$$

We say that $x \in X_1$ is a *successor* of $x_1 \in X_1$ if each vertex of $M(x_1)$ is adjacent in G_1 to x. Denote the set of successors of x_1 by $S(x_1)$. We say that $x_1 \in X_1$ is a *predecessor* of $x \in X_1$ if x is a successor of x_1. Denote the set of predecessors of x by $P(x)$.

Subcase IIA. We adopt the following notation: $(x_1x_4)'$ denotes the transposition (x_1x_4) of x_1 and x_4 in X_1 if $x \neq x_4$; $(x_1x_4)'$ denotes the identity permutation if $x_1=x_4$.

Suppose that

$$\deg_{G_1}(x_2) \leq \tfrac{1}{3}(p-1).$$

First, we eliminate the possibility of strict inequality. If the inequality above is strict, then

$$|E(x_2, X_2)| = \deg_G(x_2) - \deg_{G_1}(x_2) > \tfrac{2}{3}(p-1) - \tfrac{1}{3}(p-1) = \tfrac{1}{3}(p-1).$$

Since x_1 is not adjacent to at most $\frac{1}{3}(p-1)$ vertices of G other than x_1, it is adjacent to one of the more than $\frac{1}{3}(p-1)$ vertices x_3 of X_2 incident with an edge of $E(x_2, X_2)$. Hence $G[x_1, x_2, x_3]$ is a triangle on the free edge in G_1 and a vertex of G_2.

Henceforth in this subcase, we shall suppose

$$\deg_{G_1}(x_2) = \tfrac{1}{3}(p-1).$$

By (3.23),

$$\deg_{G_1}(x_1) + \tfrac{1}{3}(p-1) \geq p_1 + \tfrac{1}{3}p - \tfrac{4}{3}.$$

Hence,

$$\deg_{G_1}(x_1) \geq p_1 - 1,$$

and so x_1 must be adjacent to each vertex of G_1. Therefore, $P(x_1) = G_1 - x_2$. Since $S(x_1) = N_1(x_2)$, we conclude that for any $x_4 \in N_1(x_2)$, $(x_1x_4)'\pi$ is an embedding of the b_1 triangles and free edge into G_1. Note that the embedding $(x_1x_4)'\pi$ makes $\{x_4, x_2\}$ the free edge. By the minimality of $\deg_{G_1}(x_1) + \deg_{G_1}(x_2)$,

$$\deg_{G_1}(x_4) + \deg_{G_1}(x_2) \geq \deg_{G_1}(x_1) + \deg_{G_1}(x_2),$$

whence,

$$\deg_{G_1}(x_4) = p_1 - 1.$$

Since x_4 may be any of the $\frac{1}{3}(p-1)$ vertices of $N_1(x_2)$, we know that the vertices of $X_1 - N_1(x_2)$ must be adjacent to each vertex of $N_1(x_2)$, a set of $\frac{1}{3}(p-1)$ vertices adjacent to all of G_1. Hence,

$$\delta(G_1) \geq \tfrac{1}{3}(p-1) = \deg_{G_1}(x_2).$$

Define the sets

$$T_1 = N_1(x_2), \qquad T_2 = N_2(x_2),$$
$$S_1 = X_1 - T_1, \qquad S_2 = X_2 - T_2.$$

We have already shown that $G[T_1]$ is a complete graph, and each vertex of S_1 is adjacent to every vertex of T_1. If there is an $x_4 \in T_1 = S(x_1)$ and a vertex $x_3 \in X_2$ such that $G[x_2, x_3, x_4]$ is a triangle in G, then we have accomplished the goal of this subcase, since $(x_1x_4)'\pi$ is an embedding of b_1 triangles and a disjoint edge mapped to $\{x_2, x_4\}$, which is the edge forming the triangle with x_3. Otherwise, no $x_4 \in T_1$ forms a triangle with x_2 and any vertex in X_2. Hence, no $x_4 \in T_1$ is adjacent to vertices of T_2. Now,

$$|T_2| = \deg_G(x_2) - \deg_{G_1}(x_2) \geq \tfrac{1}{3}(p-1),$$

and hence, any $x_4 \in T_1$, having degree at least $\tfrac{2}{3}(p-1)$ in G, must be adjacent to every vertex of $S_1 \cup T_1 \cup S_2 - x_4$. A similar argument shows that any vertex of T_2, not being adjacent to any vertex of T_1, a set of $\tfrac{1}{3}(p-1)$ vertices, is adjacent to any vertex of $S_1 \cup T_2 \cup S_2$ except itself. Note that this implies that $G[T_2]$ is like $G[T_1]$, a complete graph on $\tfrac{1}{3}(p-1)$ vertices. Also, note that any vertex of $S_1 \cup S_2$ is adjacent to every vertex of $T_1 \cup T_2$ in G.

Hence, $S_1 \cup S_2$ is a set of

$$|V(G) - (T_1 \cup T_2)| = p - \tfrac{2}{3}(p-1) = \tfrac{1}{3}(p-1) + 1 = b + 1$$

vertices whose removal from G leaves two components $G[T_i]$, $i = 1, 2$, each a complete graph on $\tfrac{1}{3}(p-1) = b$ vertices.

By Lemma 3.4, either b pairwise disjoint triangles embed in G, or G is of type 2. The first possibility is contrary to hypothesis. The other possibility is a desired conclusion of Theorem 3.1. Hence, we can assume that there is a free edge in G_1, which together with some $x_3 \in X_2$, forms a triangle in G.

Subcase IIB. Suppose that

$$\deg_{G_1}(x_2) > \tfrac{1}{3}(p-1). \tag{3.25}$$

Let x_3 be a vertex of X_2 that is adjacent in G to x_2. Since $p_1 \leq p_2$,

$$\begin{aligned}\deg_G(x_2) &\geq \tfrac{2}{3}(p-1) \geq \tfrac{2}{3}(2p_1 - 1)\\ &= p_1 + \tfrac{1}{3}p_1 - \tfrac{2}{3} > p_1 - 1,\end{aligned}$$

and so x_3 exists. The successors $S(x_1)$ of x_1 in G_1 are the vertices of G_1 adjacent to x_2. We see that $S_1(x_1) = N_1(x_2)$. We have

$$\begin{aligned}|S(x_1) \cap N_1(x_3)| &\geq \deg_{G_1}(x_2) + \deg_G(x_3) - (p_2 - 1) - |S(x_1) \cup N_1(x_3)|\\ &\geq \deg_{G_1}(x_2) + \tfrac{2}{3}(p-1) - (p_2 - 1) - p_1\\ &= \deg_{G_1}(x_2) - \tfrac{1}{3}(p-1) > 0,\end{aligned}$$

by (3.25). Hence, there is a vertex $x_4 \in X_1$ that forms a triangle with x_2 and x_3 and is a successor of x_1.

If $x_1 \in S(x_4)$, then the embedding $(x_1 x_4)\pi$ maps the free edge in G_1 to $\{x_2, x_4\}$, which forms with $x_3 \in X_2$ a triangle in G as desired. Otherwise,

$$x_1 \notin S(x_4). \tag{3.26}$$

We shall find a vertex $x_5 \in X_1$ with $x_5 \in S(x_4) \cap P(x_1)$, whence $(x_1 x_4 x_5)\pi$ is the desired embedding of b_1 triangles and one edge into G_1.

In the image of the triangle embedded into G_1 having vertex x_4 are two other vertices, which we call x_6, x_7. The successors of x_4 are those vertices in G_1

adjacent to both x_6 and x_7. Hence, $x_1, x_6, x_7 \notin S(x_4)$, and

$$|S(x_4)| \geqslant \deg_{G_1}(x_6) + \deg_{G_1}(x_7) - p_1. \tag{3.27}$$

The predecessors $P(x_1)$ of x_1 in G_1 are those vertices $v \in X_1$ such that x_1 is adjacent to all vertices of $M(v)$. Now, x_1 is adjacent in $\bar{G}_1$ to $p_1 - \deg_{G_1}(x_1) - 1$ vertices $v' \in X_1$. Any such v' lies in exactly two sets $M(v)$, $v \in X_1$. Thus $x_1 \notin S(v)$ for at most

$$2p_1 - 2\deg_{G_1}(x_1) - 2$$

vertices v of $X_1 - M(x_1) = G_1 - x_2$. Since the remaining vertices of $G_1 - x_2$ are in $P(x_1)$, we have $x_2 \notin P(x_1)$, and

$$\begin{aligned} |P(x_1)| &\geqslant |X_1 - x_2| - (2p_1 - 2\deg_{G_1}(x_1) - 2) \\ &= 2\deg_{G_1}(x_1) - p_1 + 1 \geqslant \tfrac{1}{3}(p-1), \end{aligned} \tag{3.28}$$

by (3.24).

Suppose first that x_4 is not adjacent to x_1. Then

$$x_2, x_6, x_7 \notin P(x_1),$$

and we combine (3.27), (3.28), (3.22), and $2p_1 \leqslant p$ to get

$$x_1, x_2, x_6, x_7 \notin S(x_4) \cap P(x_1), \qquad x_6, x_7 \notin S(x_4) \cup P(x_1),$$

and

$$\begin{aligned} |S(x_4) \cap P(x_1)| &\geqslant |S(x_4)| + |P(x_1)| - |X_1 - \{x_6, x_7\}| \\ &\geqslant \deg_{G_1}(x_6) + \deg_{G_1}(x_7) - p_1 + \tfrac{1}{3}p - \tfrac{1}{3} - p_1 + 2 \\ &\geqslant 2\delta(G_1) - 2p_1 + \tfrac{1}{3}p + \tfrac{5}{3} \\ &\geqslant 2(\tfrac{2}{3}p_1 - \tfrac{1}{3}) - 2p_1 + \tfrac{1}{3}p + \tfrac{5}{3} \\ &= \tfrac{1}{3}p - \tfrac{2}{3}p_1 + 1 \geqslant 1. \end{aligned} \tag{3.29}$$

Suppose, otherwise, that x_4 is adjacent to x_1. Then $G[x_1, x_2, x_4]$ is a triangle, and $\{x_6, x_7\}$ is a free edge. Thus, by choice of $\{x_1, x_2\}$ and (3.23),

$$\begin{aligned} \deg_{G_1}(x_6) + \deg_{G_1}(x_7) &\geqslant \deg_{G_1}(x_1) + \deg_{G_1}(x_2) \\ &\geqslant p_1 + \tfrac{1}{3}p - \tfrac{4}{3}. \end{aligned} \tag{3.30}$$

We combine (3.27), (3.28), (3.30) and

$$p_1 + p_2 = p$$

to obtain

$$\begin{aligned} |S(x_4) \cap P(x_1)| &\geqslant |S(x_4)| + |P(x_1)| - p_1 \\ &\geqslant \deg_{G_1}(x_6) + \deg_{G_1}(x_7) - p_1 + \tfrac{1}{3}p - \tfrac{1}{3} - p_1 \\ &\geqslant p_1 + \tfrac{1}{3}p - \tfrac{4}{3} - 2p_1 + \tfrac{1}{3}p - \tfrac{1}{3} \\ &= \tfrac{2}{3}p - \tfrac{2}{3}p_1 - \tfrac{1}{3}p_1 - \tfrac{5}{3} \\ &\geqslant \tfrac{2}{3}p_2 - \tfrac{1}{3}p_1 - \tfrac{5}{3} \\ &= \tfrac{1}{3}(p_2 - p_1) + (\tfrac{1}{3}p_2 - \tfrac{5}{3}). \end{aligned} \tag{3.31}$$

Note that both of the terms in the last line of (3.31) are nonnegative if $p_2 \geq 5$, and if $p_2 > 5$, then the last line is positive. If $p_2 \leq 5$, then $p_1 \leq p_2$ and $p_i \equiv 2 \pmod 3$ imply one of the following three cases:

$$p_2 = p_1 = 5;$$

$$p_2 = 5, \qquad p_1 = 2;$$

or

$$p_2 = p_1 = 2.$$

If $p_2 = p_1 = 5$, then (3.23) gives

$$\deg_{G_1}(x_1) + \deg_{G_1}(x_2) \geq 7,$$

whence $\deg_{G_1}(x_1) \geq \deg_{G_1}(x_2)$ implies that x_1 is adjacent to every vertex of G_1 except itself, whence $x_4 \in P(x_1)$, in violation of (3.26). If $p_2 = 5$, $p_1 = 2$, then the last line of (3.31) is 1, which is as desired. If $p_1 = p_2 = 2$, then $p = 4$ and $\delta(G) \geq \frac{2}{3}(p-1)$ imply G is K_4, $K_4 - e$ (e an edge), or a quadrilateral, all of which satisfy the theorem. Hence, under our hypotheses, the last line of (3.31) and the last line of (3.29) may be assumed to be positive.

Therefore, whether or not x_4 and x_1 are adjacent, there is a vertex $x_5 \neq x_1$ or x_2, such that

$$x_5 \in S(x_4) \cap P(x_1),$$

and so we have a closed alternating chain in G_1 represented by the permutation

$$\alpha = (x_1 x_4 x_5).$$

Hence, $\alpha\pi$ is an embedding of the b_1 triangles and one edge into G_1. The free edge is determined by $\alpha\pi$ to be $\{x_2, x_4\}$, since x_1 is permuted to x_4 and since $x_2 \neq x_5$ guarantees that x_2 is fixed. Thus, the free edge is part of a triangle $G[x_2, x_3, x_4]$, as desired. *This concludes Subcase IIB.*

To complete Case II and the proof of the theorem, we verify that all the hypotheses, and hence the final conclusion, of Lemma 3.5 apply to G_1 and G_2, and then we show that G is of type 1.

Since we have assumed that

$$b = b_1 + 1 + b_2$$

triangles do not embed in G, and since $b_1 + 1$ triangles embed in $G_1 + x_3 = G[X_1 + x_3]$, we know that we cannot embed b_2 triangles in $G_2 - x_3$. Now,

$$|V(G_2) - x_3| = 3b_2 + 1,$$

and by (3.22),

$$\begin{aligned}\delta(G_2 - x_3) &\geq \delta(G_2) - 1 \geq \tfrac{2}{3}p_2 - \tfrac{1}{3} - 1\\ &= \tfrac{2}{3}(|X_2 - x_3| - 1) = 2b_2.\end{aligned}$$

Therefore, by the induction hypothesis, $G_2 - x_3$ is of type 1 or type 2. Hence,

$$\delta(G_2 - x_3) = \tfrac{2}{3}(p_2 - 2) = \tfrac{2}{3}p_2 - \tfrac{4}{3},$$

whence, by (3.22), x_3 is adjacent to every vertex of $G_2 - x_3$ having degree $\tfrac{2}{3}p_2 - \tfrac{4}{3} = \delta(G_2 - x_3)$ in $G_2 - x_3$.

By Lemmas 3.7 and 3.8, we know that $b_2 - 1$ triangles embed in $G_2 - x_3$, and that such an embedding uses all but 4 vertices of $G_2 - x_3$. Moreover, these 4 vertices all have degree $\delta(G_2 - x_3)$ in $G_2 - x_3$, and they induce a 4-cycle. Now, x_3 is adjacent to all four of these vertices, and hence forms a triangle with 2 of them. Let v_1 and v_2 denote the other 2 vertices on this 4-cycle. Note that v_1 and v_2 are adjacent. We shall show that there are b_1 choices of a vertex $y_3 \in X_1$ such that $G[v_1, v_2, y_3]$ is a triangle. If b_1 disjoint triangles can be embedded in $G_1 - y_3$, then, counting the triangle containing x_3, the triangle $G[v_1, v_2, y_3]$, and the $b_2 - 1$ triangles of $G_2 - x_3$, we have b pairwise disjoint triangles in G, contrary to assumption. Hence, b_1 triangles do not embed in $G_1 - y_3$. For this to happen,

$$b_1 \geqslant 1.$$

Thus, by (3.22), we may apply Lemma 3.6, with

$$\{z, z'\} = \{x_3, y_3\},$$

and conclude that both $G_1 - y_3$ and $G_2 - x_3$ are of type 1. Since

$$\delta(G) = 2b = 2b_1 + 2 + 2b_2,$$

and since

$$\deg_{G_2}(v_j) = 2b_2 + 1 \qquad (j = 1, 2),$$

each v_j $(j = 1, 2)$ is adjacent to at least $2b_1 + 1$ vertices of G_1. Hence, there are at least

$$|E(v_1, X_1)| + |E(v_2, X_1)| - p_1 \geqslant 2(2b_1 + 1) - (3b_1 + 2) = b_1 \geqslant 1$$

choices $y_3 \in X_1$ such that $G[v_1, v_2, y_3]$ is a triangle. Therefore, as we already remarked, we may apply Lemma 3.6 and conclude that both $G_1 - y_3$ and $G_2 - x_3$ are of type 1.

Next, we establish the hypotheses of Lemma 3.5.

Since $G_1 - y_3$ and $G_2 - v_3$ are of type 1, where

$$v_3 = x_3,$$

and since they have $3b_1 + 1$, $3b_2 + 1$ vertices, respectively, there are sets $Y_3' \subseteq X_1 - y_3$ and $V_3' \subseteq X_2 - v_3$ with

$$|Y_3'| = b_1 - 1, \qquad |V_3'| = b_2 - 1,$$

such that $G_1 - y_3 - Y_3'$ and $G_2 - v_3 - V_3'$ are complete bipartite graphs $Y_1 \cup Y_2$ and $V_1 \cup V_2$, respectively. Define

$$Y_3 = Y_3' + y_3, \qquad V_3 = V_3' + v_3.$$

Thus, (3.4) and (3.5) of Lemma 3.5 hold, and also

$$|Y_3 \cup V_3| = b_1 + b_2 = b - 1. \tag{3.32}$$

Since $G_1 - y_3$ is of type 1, if $y \in Y_1 \cup Y_2$, then

$$\deg_{G_1 - y_3}(y) = 2b_1 = \tfrac{2}{3}(p_1 - 2).$$

Now $\delta(G_i) \geq \frac{2}{3}p_1 - \frac{1}{3}$, and hence y is adjacent to $y_3 \in X_1$. Therefore, for any $y \in Y_1 \cup Y_2$,

$$\deg_{G_1}(y) = \tfrac{2}{3}(p_1 - 2) + 1 = \delta(G_1),$$

and (3.7) of Lemma 3.5 is established. Similarly, since $G_2 - v_3$ is of type 1, (3.6) may be established, and also for any $v \in V_1 \cup V_2$,

$$\deg_{G_2}(v) = \delta(G_2).$$

By (3.22),

$$\delta(G_1) + \delta(G_2) \geq \tfrac{2}{3}p_1 - \tfrac{1}{3} + \tfrac{2}{3}p_2 - \tfrac{1}{3} = \tfrac{2}{3}(p - 1) = \delta(G),$$

ınd (3.3) is established. Thus, having proved (3.3) through (3.7) of Lemma 3.5, ve conclude from Lemma 3.5 that any vertex $y \in Y_1 \cup Y_2$ is adjacent to every ertex in V_j for some $j \in \{1, 2\}$.

Suppose by way of contradiction that some $y \in Y_1 \cup Y_2$ is adjacent in G to ertices $v_1 \in V_1$ and $v_2 \in V_2$ (i.e., suppose that (3.8) is false). Thus, $G[y, v_1, v_2]$ is a riangle. By Lemma 3.7, for any vertices $v_5 \in V_1 - v_1$ and $v_4 \in V_2 - v_2$, there is an mbedding of $b_2 - 1$ triangles into $G_2 - \{v_1, v_2, v_3, v_4, v_5\}$, since $G_2 - v_3$ is of type . Note that $G[v_3, v_4, v_5]$ is also a triangle. We conclude from Lemma 3.7 that for ny vertices $y_1 \in Y_1$, $y_2 \in Y_2$, there is an embedding of $b_1 - 1$ pairwise disjoint riangles into $G_1 - \{y, y_1, y_2, y_3\}$, since $G_1 - y_3$ is of type 1. Including the $b_2 - 1$ riangles of $G_2 - \{v_1, v_2, v_3, v_4, v_5\}$ and the 3 triangles $G[y_1, y_2, y_3]$, $G[y, v_1, v_2]$, nd $G[v_3, v_4, v_5]$, we have

$$(b_1 - 1) + (b_2 - 1) + 3 = b$$

airwise disjoint triangles embedded in G, contrary to assumption. Hence, (3.8) ıolds, and by Lemma 3.5, $G - (Y_3 \cup V_3)$ is a complete bipartite graph. By (3.32) nd Lemma 3.3, G is of type 1. This completes the proof of Theorem 3.2.

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