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ANOTHER BOUND ON THE CHROMATIC NUMBER OF A GRAPH

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Let G be a simple graph, let $\Delta(G)$ denote the maximum degree of its vertices, and let $\chi(G)$ denote its chromatic number. Brooks' Theorem asserts that $\chi(G) \leq \Delta(G)$, unless G has a component that is a complete graph $K_{\Delta(G)+1}$, or unless $\Delta(G)=2$ and G has an odd cycle. We show here that this bound can be improved if G does not contain certain types of subgraphs. For instance, if G has no 4-cycle, then $\chi(G) \leq \frac{2}{3}(\Delta(G)+3)$. A different result of a similar nature was recently obtained independently by us, by Borodin and Kostochka, and by Lawrence.

1. Introduction

Let G be a simple graph of maximum degree $\Delta(G)=h$ and chromatic number $\chi(G)$. The basic bound on $\chi(G)$ was given by Brooks [2]:

Theorem 1.1. *For any graph G,*

$$\chi(G) \leq \Delta(G)+1,$$

with equality if and only if either $\Delta(G)=2$ and G contains an odd cycle, or G contains a clique $K_{\Delta(G)+1}$.

The odd cycle or the clique $K_{\Delta(G)+1}$ of Brooks' Theorem is necessarily a component of G.

For graphs with no clique K_{r+1}, where r is not too large, Brooks' bound was improved independently by Borodin and Kostochka [1], by Catlin [3], and by Lawrence [5]:

Theorem 1.2. *For any graph G containing no K_{r+1}, where $3 \leq r \leq \Delta(G)$,*

$$\chi(G) \leq \frac{r}{r+1}(\Delta(G)+2).$$

Borodin and Kostochka's result was stated in more general terms, and Mitchem [7] generalized the proof in [3] to independently obtain the more general result of [1].

When we use the terms "clique" or "K_{r+1}" in G, we mean "complete subgraph", and not necessarily "maximal complete subgraph".

Let $K_{r+2}-e$ denote the complete subgraph on $r+2$ vertices, minus an edge. When we say that G contains no $K_{r+2}-e$ as a subgraph, we do not mean "induced subgraph". We mean that G contains no K_{r+2} also. A 4-cycle is a cycle on 4 edges.

The main result of this paper is:

Theorem 1.3. *Let G be a graph. If G has no $K_{r+2}-e$ as a subgraph, for some $r \geq 3$, then*

$$\chi(G) \leq \frac{r}{r+1}(\Delta(G)+3).$$

If also G has no 4-cycle, then

$$\chi(G) \leq \tfrac{2}{3}(\Delta(G)+3).$$

In the proofs of Theorem 1.2 (see [1] or [3]) a decomposition theorem of Lovász [6] was used. Here we develop a variation of it as part of the proof of Theorem 1.3. Other variations are in [4]. Lovász's result:

Theorem 1.4. *Let G be a graph and let n be a natural number. For any partition*

$$h_1+h_2+\cdots+h_n=\Delta(G)-(n-1)$$

of $\Delta(G)-(n-1)$, there is a decomposition of $V(G)$ into sets $X_1, X_2, \ldots, X_n$ such that $\Delta(G[X_i]) \leq h_i$, for $i=1, 2, \ldots, n$, where $G[X_i]$ is the subgraph of G induced by X_i.

One can obtain Theorem 1.2 from Theorems 1.1 and 1.4 by setting most of the h_i's equal to r.

Borodin and Kostochka stated in a recent communication that Kostochka has proved that if the girth of G is at least $2\Delta(G)^2$, then

$$\chi(G) \leq \tfrac{1}{2}(\Delta(G)+4).$$

The least value of $\Delta(G)$ for which Theorem 1.3 improves Theorems 1.1 and 1.2 is $\Delta(G)=10$: the theorem gives $\chi(G) \leq 8$ if G has no 4-cycle. The least value of $\Delta(G)$ for which Theorem 1.3 improves Theorems 1.1 and 1.2 when $r \geq 3$ is $\Delta(G)=18$: if G has a K_4 but has no K_5-e, then Theorem 1.3 (with $r=3$) gives $\chi(G) \leq 15$, while Theorem 1.2 gives only $\chi(G) \leq 16$.

We know of no examples that would show that Theorem 1.2 or 1.3 is best possible. Thus, we pose the following question: is there a constant $c_r > 0$ (depending only on r) such that for arbitrarily high values of h there are graphs G with $\Delta(G)=h$, with no K_{r+1}'s and with $\chi(G) > c_r\Delta(G)$?

2. The proof of Theorem 1.3

Consider r to be fixed by a hypothesis that G contains no $K_{r+2}-e$. If G has no 4-cycle, then set $r=2$. Let $\Delta(G)$ be denoted by h, and let

$$n=\left[\frac{h+2}{r+1}\right],$$

where the brackets denote the greatest integer function. Then we can write

$$h_1=h_2=\cdots=h_{n-1}=r,$$

and so for some integer h_n satisfying

$$r\leqslant h_n=h+2-n(r+1)+r\leqslant 2r$$

we have

$$h=h_1+h_2+\cdots+h_{n-1}+h_n+(n-2).$$

For a subset $X_i\subseteq V(G)$, we can write $G_i=G[X_i]$ (the subgraph of G induced by X_i). We also write $E(X_i)=E(G[X_i])$. Define the integer-valued function f by

$$f(X_1, X_2, \ldots, X_n)=h_1|X_1|+h_2|X_2|+\cdots+h_n|X_n| \\ -|E(X_1)|-|E(X_2)|-\cdots-|E(X_n)|,$$

where $(X_1, X_2, \ldots, X_n)$ is a decomposition of $V(G)$. In particular, assume that $(X_1, \ldots, X_n)$ is the decomposition of $V(G)$ that

(i) maximizes $f(X_1, \ldots, X_n)$;
(ii) if $r=2$, minimizes the total number of odd cycles in all G_i's for which $h_i=2$, such that (i) holds;
(iii) if $r>2$, minimizes the total number of cliques K_{r+1} in the G_i's for which $h_i=r$, again subject to (i).

Thus, if $h_n>r$, then G_n is not relevant to (ii) or (iii). Those odd cycles or cliques counted in (ii) or (iii) are clearly components of the respective subgraphs G_i, and we shall refer to them collectively as Brooks components, in order to use a common name.

By the maximality of f (i.e., by (i)),

$$0\leqslant f(X_1, \ldots, X_n)-f(X_1-x, X_2+x, X_3, \ldots, X_n)$$
$$0\leqslant f(X_1, \ldots, X_n)-f(X_1-x, X_2, X_3+x, \ldots, X_n)$$
$$\vdots$$
$$0\leqslant f(X_1, \ldots, X_n)-f(X_1-x, X_2, X_3, \ldots, X_n+x).$$

Then by the definition of f,

$$0 \leq h_1 |X_1| - h_1(|X_1|-1) + h_2 |X_2| - h_2(|X_2|+1) - |E(X_1)| + |E(X_1 - x)| - |E(X_2)| + |E(X_2 + x)|$$

$$\vdots$$

$$0 \leq h_1 |X_1| - h_1(|X_1|-1) + h_n |X_n| - h_n(|x_n|+1) - |E(X_1)| + |E(X_1 - x)| - |E(X_n)| + |E(X_n + x)|.$$

Hence, for $i = 2, 3, \ldots, n$,

$$0 \leq h_1 - h_i - \deg_{G_1} x + \deg_{G[X_i + x]} x,$$

and so for each $i \geq 2$ and for $i = 1$, we have

$$\deg_{G_1} x \leq h_1 - h_i + \deg_{G[X_i + x]} x.$$

We sum both sides of this system of inequalities, letting i run from 1 to n, and we obtain

$$\begin{aligned} n \deg_{G_1} x &\leq nh_1 - \sum_{i=1}^{n} h_i + \sum_{i=1}^{n} \deg_{G[X_i+x]} x \\ &= nh_1 - (h - n + 2) + \deg_G x \\ &\leq nh_1 + n - 2. \end{aligned}$$

Dividing both sides by n, we get

$$\deg_{G_1} x \leq h_1 + \frac{n-2}{n},$$

and since $\deg_{G_1} x$ and h_1 are integers,

$$\deg_{G_1} x \leq h_1.$$

In a similar manner we can obtain for $x \in X_i$ $(i \leq n)$

$$\deg_{G_i} x \leq h_i.$$

Hence, for $i = 1, 2, \ldots, n$,

$$\Delta(G_i) \leq h_i. \qquad (*)$$

We claim that it sufficies to show that $\chi(G_i) \leq h_i$ for each i. Were this so, then

$$\begin{aligned} \chi(G) &\leq \sum_{i=1}^{n} \chi(G_i) \leq \sum_{i=1}^{n} h_i = h - n + 2 \\ &= h - \left[\frac{h+2}{r+1}\right] + 2 \\ &\leq h - \frac{h+2}{r+1} + 1 - \frac{1}{r+1} + 2 \\ &= \frac{r}{r+1}(h+3), \end{aligned}$$

which is the conclusion of the theorem. Thus, we must show that $\chi(G_i) \leqslant h_i$ for each i. If $h_n > r$, then by $(*)$, Brooks' Theorem (Theorem 1.1) can be applied to G_n, for the hypotheses of Theorem 1.3 preclude Brooks components in G_n. Specifically, if $r=2$, then by hypothesis, G_n contains no 4-cycle, and hence no $K_{h_n+1}(h_n \geqslant 3)$; if $r>2$, then G_n contains no $K_{r+2}-e$ and hence, no $K_{r+2} \subseteq K_{h_n+1}(h_n \geqslant r+1)$. Therefore, we can restrict our attention to eliminating Brooks components from those subgraphs G_i for which $h_i = r$. Then when Brooks' Theorem is applied to these subgraphs, $\chi(G_i) \leqslant h_i$ follows.

Suppose by way of contradiction that G_i contains a Brooks component C_0 (an odd cycle if $\Delta(G_i)=r=2$, a clique K_{r+1} if $\Delta(G_i)=r>2$), for some i. Let $x_0 \in V(C_0)$.

If x_0 is adjacent to at least h_j+1 vertices in each set X_j for $j \neq i$, then

$$\deg_G x_0 \geqslant h_i + \sum_{j \neq i} (h_j+1) = h+1 > h,$$

contrary to hypothesis. Hence, there is some j such that x_0 is adjacent to h_j or fewer vertices in X_j.

Notice that as x_0 is moved from X_i to X_j, the maximality of f is preserved:

$$f(X_1, \ldots, X_n) = f(X_1, \ldots, X_i - x, \ldots, X_j + x, \ldots, X_n).$$

If $h_j > r$, then $j=n$, and since $(X_1, \ldots, X_n)$ was chosen to minimize the number of Brooks components in $G_1, \ldots, G_{n-1}$ (condition (ii) or (iii)), x_0 must lie in a Brooks component in G_n. But, as we have already seen, Brooks' Theorem and the hypotheses preclude Brooks components in G_n, if $h_n > r$. Therefore, $h_j = r$, although j may still equal n.

To avoid violating condition (ii) or (iii), that the number of Brooks components is minimized, subject to (i), the destruction of C_0 in G_i must be accompanied by the formation of a Brooks component C_1 containing x_0, as x_0 is moved from G_i to X_j to form $G[X_j + x_0]$. An odd arc $C_0 - x_0$ is left behind in $G_i - x_0$ if $r=2$, and if $r \geqslant 3$, then a clique $K_r = C_0 - x_0$ remains.

We repeat this process by picking a vertex $x_1 \neq x_0$ in $V(C_1)$ and moving it out of $X_j + x_0$. Another Brooks component C_2 is formed, leaving behind an odd arc $(r=2)$ or a clique $K_r(r>2)$.

Since G is finite, this sequence $C_0, C_1, C_2, \ldots$ of Brooks components in the subgraphs will eventually double back on itself for the first time. A vertex x_{m-1} will be moved into a set X^* of the altered decomposition, where it will be part of a Brooks component C_m in $G[X^* + x_{m-1}]$, and where C_m overlaps C_k for some $k < m$. Then either $C_m - x_{m-1}$ is the odd arc $C_k - x_k$ left behind as some vertex x_k in the sequence

$$x_0, x_1, \ldots, x_k, \ldots, x_{m-1}$$

was moved out $(r=2)$, or $C_m - x_{m-1} = C_k - x_k$ is a clique $K_r(r>2)$ in $G[X^*]$ formed under similar circumstances. In the first case $(r=2)$, x_{m-1}, x_k, and the two endpoints of the odd arc

$$C_m - x_{m-1} = C_k - x_k$$

form a 4-cycle, contrary to hypothesis; in the second case ($r > 2$), x_{m-1}, x_k, and the r-clique

$$C_m - x_{m-1} = C_k - x_k$$

form a $K_{r+2} - e$ in G, again contrary to the hypothesis of the theorem.

Thus, there are no Brooks components in the G_i's, and so, as already demonstrated, the conclusions of the theorem follow.

References

[1] O.V. Borodin and A.V. Kostochka, On an upper bound of a graph's chromatic number, depending on the graph's degree and density, J. Combinatorial Theory Ser. B 23 (1977) 247–250.

[2] R.L. Brooks, On colouring the nodes of a network, Proc. Cambridge Philos. Soc. 27 (1941) 194–197.

[3] P.A. Catlin, A bound on the chromatic number of a graph, Discrete Math. 22 (1978) 81–83.

[4] P.A. Catlin, Graph decompositions satisfying extremal degree constraints, J. Graph Theory 2 (1978) 165–170.

[5] J. Lawrence, Covering the vertex set of a graph with sub-graphs of smaller degree, Discrete Math. 21 (1978) 61–68.

[6] L. Lovász, On decomposition of graphs, Studia Sci. Math. Hungar. 1 (1966) 237–238.

[7] J. Mitchem, On the genus of graphs with Lick–White number k, preprint.