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Two Problems in Metric Diophantine Approximation, II

PAUL A. CATLIN

Department of Mathematics, Ohio State University, Columbus, Ohio 43210

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For an arbitrary sequence $\{\alpha_n\}$ of nonnegative real numbers there is no known necessary and sufficient condition that for almost all x (in the sense of Lebesgue measure) there are infinitely many fractions p/q satisfying $|x - p/q| < \alpha_q/q$. With a restriction on $\{\alpha_n\}$ weaker than any previously used, except in a recent result of Erdös, we solve this problem and the analogous problem where p and q are required to be relatively prime.

Let $\{\alpha_n\}$ be a sequence of real numbers. We consider the question of when almost all real numbers x, in the sense of Lebesgue measure, can be approximated by infinitely many fractions p/q such that

$$|x - p/q| < \alpha_q/q, \tag{1}$$

and when almost all x can be so approximated when p/q is required to be reduced. We assume that $\alpha_q \leqslant 1/2$ for all q, but not necessarily that $\{\alpha_q\}$ is decreasing.

Let the sequence $\{\beta_n(\alpha)\}$ (also denoted $\{\beta_n\}$) be defined as follows.

$$\beta_n(\alpha) = \max(\alpha_n, \alpha_{2n}/2,..., \alpha_{in}/i,...). \tag{2}$$

Also, let the sequence $\beta_n(\alpha, m)$ be determined by

$$\beta_n(\alpha, m) = \max(\alpha_n, \alpha_{2n}/2,..., \alpha_{kn}/k), \tag{2'}$$

where $kn \leqslant m < (k+1)\,n$. Furthermore, we define $\{\gamma_n(\alpha)\}$ (also denoted $\{\gamma_n\}$ so that

$$\begin{aligned} \gamma_n(\alpha) &= 0 \qquad \text{if } \alpha_n/n \leqslant \alpha_{kn}/kn \text{ for some } k; \\ &= \alpha_n \qquad \text{otherwise.} \end{aligned} \tag{3}$$

Thus, by (2) and (3), we see that $\beta_n(\alpha) = \beta_n(\gamma)$.

It is our purpose in this paper to consider the following problems.

QUESTION 1. Is the divergence of $\sum \alpha_n \phi(n)/n$ a necessary and sufficient condition that for almost all x the relation (1) holds for infinitely many relatively prime p and q?

QUESTION 2. Is the divergence of $\sum \beta_n(\alpha)\, \phi(n)/n$ a necessary and sufficient condition that for almost all x the relation (1) holds for infinitely many p and q, not necessarily relatively prime?

The answer to each of these questions has been conjectured to be affirmative; in [2], we showed that these questions are equivalent. The first was conjectured by Duffin and Schaeffer [3], who showed that Question 1 is true when $\{\alpha_n\}$ satisfies the condition

$$\sum_{n=1}^{k} \alpha_n \phi(n)/n > c \sum_{n=1}^{k} \alpha_n$$

for some $c > 0$ and arbitrarily high k. The second question was first raised in [2]. Substantial progress toward affirmative answers to these questions was made by Erdös [5]. It is easy to show that divergence is a necessary condition in both problems. In this paper we shall prove the following result.

THEOREM. *For all sequences $\{\alpha_n\}$ such that*

$$\sum_{n=1}^{k} \beta_n(\alpha)\, \phi(n)/n > c \sum_{n=1}^{k} \gamma_n(\alpha) \tag{4}$$

for some constant $c > 0$ and certain arbitrarily high k, Questions 1 and 2 may be answered affirmatively.

An example of a sequence $\{\alpha_n\}$ for which (4) is not satisfied will be given following the proof of Theorem 1.

Work on the question of how many fractions satisfy (1) for a given x has been done by Erdös [4], LeVeque [7], Schmidt [8, 9], and others. Gallagher [6] has studied analogous problems in simultaneous approximation.

Since our proof of the equivalency of Questions 1 and 2 [2, Theorem 3] remains valid when condition (4) is imposed, we shall only consider Question 2 in our proof. Also, we only need to prove this theorem in the interval (0, 1).

The method of our proof is based on the proof of the result of Duffin and Schaeffer [3, Theorem I] cited above. Before we can give our proof, we must give several definitions and lemmas.

Let $\mathscr{E}_n^\alpha$ denote the intersection of (0, 1) with the union of open intervals having width $2\alpha_n/n$ and centered at m/n, where $m = 0, 1,..., n$. Then the measure of $\mathscr{E}_n^\alpha$, denoted $| \mathscr{E}_n^\alpha |$, is $2\alpha_n$. Also, x satisfies (1) if and only if $x \in \mathscr{E}_q^\alpha$. Furthermore, let E_n^α be defined as $\mathscr{E}_n^\alpha$ is, except that m and n are required to be relatively prime.

LEMMA 1. *Let M and N be given positive integers. The number of positive integer pairs $\{x, y\}$ satisfying*

$$0 < | xN - yM | \leqslant A,$$

where $1 \leqslant x \leqslant M$, $1 \leqslant y \leqslant N$, is not greater than $2A$.

The proof of this is straightforward, and shall be omitted.

We shall use an asterisk (*) to denote the intersection of intervals of $\mathscr{E}_n^\gamma$ and $\mathscr{E}_m^\gamma$ that are not concentric.

LEMMA 2. *If $m \neq n$ then*

$$| \mathscr{E}_n^\gamma * \mathscr{E}_m^\gamma | < 8\gamma_n\gamma_m .$$

Proof. Suppose I' is an interval of $\mathscr{E}_m^\gamma$ and I'' an interval of $\mathscr{E}_n^\gamma$ such that I', I'' have nonempty intersection but distinct centers. Then if i/m is the center of I' and j/n is the center of I'', we have

$$0 < | i/m - j/n | < \gamma_m/m + \gamma_n/n$$

or

$$0 < | in - jm | < n\gamma_m + m\gamma_n .$$

Without loss of generality, we may assume that

$$\gamma_m/m \leqslant \gamma_n/n,$$

whence

$$0 < | in - jm | < 2m\gamma_n .$$

By Lemma 1, there are no more than $4m\gamma_n$ solutions of this inequality. Hence, the overlapping of these two sets $\mathscr{E}_n^\gamma$ and $\mathscr{E}_m^\gamma$, not counting the overlap of concentric intervals, is not greater than

$$(4m\gamma_n)(2\gamma_m/m) = 8\gamma_n\gamma_m .$$

LEMMA 3. *If A is a given set in* (0, 1) *consisting of a finite number of intervals and* $\{\alpha_n\}$ *is a given sequence, then*

$$| AE_n^\alpha | < | A || E_n^\alpha |(1 + cn^{-1/2}),$$

where c is a constant which depends only on the set A.

A proof of this appears in [3, pp. 247–248], and will not be repeated here.

LEMMA 4. *Suppose that $\beta(\alpha, m) = \beta(\gamma, m)$. Then*

$$\sum_{i=n+1}^{m} \sum_{j=n}^{i-1} | E_i^\beta E_j^\beta | \leqslant \sum_{i=n+1}^{m} \sum_{j=n}^{i-1} | \mathscr{E}_i^\gamma * \mathscr{E}_j^\gamma |,$$

where β denotes $\beta(\alpha, m)$, as defined in (2′).

Proof. Let $I_{i,k}^\beta$ denote the interval of E_i^β centered at k/i; let $\mathscr{I}_{p,s}^\gamma$ denote the interval of $\mathscr{E}_p^\gamma$ centered at s/p. It follows from the definition of $\{\beta_n(\gamma, m)\}$ (which can be substituted for $\{\beta_n(\alpha, m)\}$) that for every pair i, k there is a pair p, s (with $ti = p < m$, $tk = s$ for some integer t) such that $I_{i,k}^\beta = \mathscr{I}_{p,s}^\gamma$. Note that this correspondence of intervals of elements of $\{E_n^\beta\}$ to equal intervals of elements of $\{\mathscr{E}_n^\gamma\}$ is injective: specifically, if $\mathscr{I}_{p,s}^\gamma$ is equal to any interval $I_{i,k}^\beta$, then this latter interval is determined uniquely by setting $t = \gcd(p, s)$, $i = p/t$, $k = s/t$.

It follows that there is an injective correspondence from intervals of $E_i^\beta \cap E_j^\beta$ to equal intervals of $\mathscr{E}_p^\gamma * \mathscr{E}_q^\gamma$ (the asterisk is justified by the observation that E_i^β and E_j^β have no intervals with a common center, since $i \neq j$). From this injective relationship, the lemma follows.

Proof of Theorem. Let $\{\alpha_n\}$ satisfy the conditions of the theorem. Assume that $\sum_1^\infty \beta_i \phi(i)/i$ diverges, so that the condition of Question 2 is satisfied. Let

$$E = \sum_1^\infty E_q^\beta.$$

If the measure of E is 1, then for almost all x there is at least one pair of coprime p and q such that

$$| x - p/q | < \beta_q/q \tag{5}$$

holds. We suppose that $| E | < 1$ and show that this gives a contradiction.

Given a small positive number δ, let

$$A = E_1^\beta + E_2^\beta + \cdots + E_{q_1}^\beta,$$

and choose q_1 so large that

$$| A | > | E | - \delta.$$

Then A consists of a finite number of intervals; hence, if q_2 is sufficiently large, we have from Lemma 3 that

$$| AE_q^\beta | < | A || E_q^\beta |(1 + \delta), \qquad q \geqslant q_2(A, \delta). \tag{6}$$

Let n be greater than $q_1 + q_2$. Let

$$\alpha_i' = \alpha_i \quad \text{if } i \leqslant m; \qquad = 0 \quad \text{otherwise,}$$

where m is chosen large enough so that the following two relations are satisfied.

$$\sum_n^m \gamma_i \geqslant 1; \tag{7}$$

$$\sum_n^m | E_i^{\beta(\alpha')} | > c' \sum_n^m \gamma_i(\alpha') \tag{8}$$

for some constant c'. We show that these relations can be satisfied. By (4),

$$\sum_n^m | E_i^{\beta(\alpha')} | = \sum \beta_i(\alpha') \, \phi(i)/i > c \sum_1^m \gamma_i(\alpha') - \sum_1^{n-1} \beta_i \phi(i)/i. \tag{9}$$

By summing over k in both sides of

$$\sum_{d|k} | E_d | = | \mathscr{E}_k^\gamma | ,$$

we get an equality in which some terms $| E_d^\beta |$ may appear more than once on the left side. Hence,

$$\sum_{k=n}^m | E_k^\beta | \leqslant \sum_{k=n}^m | \mathscr{E}_k^\gamma | ,$$

so that

$$\sum_n^m \beta_i \phi(i)/i \leqslant \sum_n^m \gamma_i$$

Therefore, $\sum \gamma_i$ diverges, so that (7) follows and so that $c \sum_1^m \gamma_i(\alpha')$ in (9) can be made twice as large as $\sum_1^{n-1} \beta_i \phi(i)/i$. This gives

$$\sum_n^m | E_i^{\beta(\alpha')} | > \tfrac{1}{2} c \sum_1^m \gamma_i(\alpha'),$$

from which (8) follows.

Let

$$B = E_n{}^\beta + E_{n+1}^\beta + \cdots + E_m{}^\beta, \qquad m \geqslant n > q_1 + q_2,$$

where the superscript β denotes $\beta(\alpha', m)$. It follows from the definition of α' that $\beta(\alpha', m) = \beta(\gamma, m)$. Now,

$$\sum_n^m | E_i{}^\beta | \geqslant | B | \geqslant \sum_n^m | E_i{}^\beta | - \sum_{i=n+1}^m \sum_{j=n}^{i-1} | E_i{}^\beta E_j{}^\beta |,$$

so by Lemma 4,

$$| B | \geqslant \sum_n^m | E_i{}^\beta | - \sum_{i=n+1}^m \sum_{j=n}^{i-1} | \mathscr{E}_i{}^\gamma * \mathscr{E}_j{}^\gamma |,$$

whence, by Lemma 2

$$| B | \geqslant \sum_n^m | E_i{}^\beta | - 4 \left(\sum_n^m \gamma_i \right)^2. \tag{10}$$

It follows from (6) that

$$| AB | \leqslant \sum_n^m | AE_q{}^\beta | < | A | \left(\sum_n^m | E_q{}^\beta | \right) (1 + \delta). \tag{11}$$

We can see that

$$| E | \geqslant | A + B | = | A | + | B | - | AB |; \tag{12}$$

hence, using (10), (11), and (12), we have

$$| E | > | A | + \left(\sum_n^m | E_i{}^\beta | \right) [1 - | A | (1 + \delta)] - 4 \left(\sum_n^m \gamma_i \right)^2. \tag{13}$$

Choose δ so small that $| A | (1 + \delta) < 1$.

Substituting (8) into (13), we obtain

$$| E | > | A | + c' [1 - | A | (1 + \delta)] \, t - 4t^2, \tag{14}$$

where t represents $\sum_n^m \gamma_i(\alpha')$. The right side of (14) is an expression of the form $|A| + bt - 4t^2$, where

$$b = c'[1 - |A|(1 + \delta)] \qquad (0 < b < 1);$$

the maximum of this expression occurs when $t = b/8$.

In order to satisfy the condition $t = b/8$, we proceed as follows: obviously, a decrease in the length of some of the intervals will not cause an increase in E. Let z be a real number in (0, 1). Let $E_q'^{\beta}$ be the set in (0, 1) consisting of open intervals, each of length $2z\beta_q/q$, with centers at p/q, where p and q are coprime and $0 < p < q$. Since $z < 1$, $E_q'^{\beta} \subseteq E_q{}^{\beta}$. Keeping A the same as before, we use in place of B the set

$$B_z = E_n'^{\beta} + \cdots + E_m'^{\beta}.$$

Proceeding as before, (14) becomes

$$|E| > |A| + b\left(\sum_n^m z\gamma_i\right) - 4\left(\sum_n^m z\gamma_i\right)^2.$$

Choose z such that $\sum_n^m z\gamma_i = b/8$. By (7) and the definition of b, z is in (0, 1), as it should be. Thus,

$$|E| > |A| + (c'^2/16)[1 - |A|(1 + \delta)]^2.$$

Letting δ approach 0 we must have $|A|$ approach $|E|$, so that

$$|E| \geqslant |E| + (c'^2/16)(1 - |E|)^2,$$

implying that $|E| = 1$.

We have thus shown that for almost all x in (0, 1), (5) is satisfied for at least one pair of coprime p and q.

To show that (5) is satisfied for arbitrarily many coprime p and q for almost all x, let m be some natural number, and let $\{\alpha_q{}^*\}$ be a new sequence defined by

$$\begin{aligned} \alpha_q{}^* &= 0 \quad \text{if} \quad q \leqslant m \\ &= \alpha_q \quad \text{if} \quad q > m. \end{aligned}$$

Then the sequence $\{\alpha_q{}^*\}$ satisfies the divergence condition if $\{\alpha_q\}$ does, so for almost all x there is at least one set of p and q such that

$$|x - p/q| < \alpha_q{}^*/q.$$

Hence, (5) is true for some $q > m$. Let D_n ($n = 1, 2, 3, \ldots$) be the sets in

(0, 1) for which (5) is true for at least one pair of p, q with $q > m$. Let D be the set common to all D_m. Then the measure of each D_m is 1, so the measure of D is 1. It follows that (5) is true for infinitely many coprime p and q for almost all x. By [2, Theorem 1], this is equivalent to the assertion that for almost all x, (1) holds for infinitely many p and q. This proves the theorem.

Next, we give an example to show that (4) does not hold for all sequences $\{\alpha_n\}$. Let $p_1 = 2$, $p_2 = 3$, $p_3 = 5$,... denote the prime numbers in ascending order. Let $M_1 = \prod_{i=1}^{k} p_i$, where k is chosen large enough so that $\sum_{i=1}^{k} 1/p_i > 2$. Let $M_2 = \prod_{i=k+1}^{m} p_i$, where m is chosen large enough so that $\prod_{k+1}^{m} 1/p_i > 2^2$, and for any natural number j, let M_j be the product of enough of the next few successive primes so that $\sum 1/p_i > 2^j$, where the sum is over these next few primes. Let

$$\begin{aligned} \alpha_n &= n/M_j \qquad \text{if } n = M_j/p_i \text{ for some } i \text{ and } j; \\ &= 0 \qquad \text{otherwise.} \end{aligned} \tag{15}$$

This implies that

$$\begin{aligned} \beta_n(\alpha) = \max_{i \in N} (\alpha_{ni}/i) &= n/M_j \qquad \text{if } n \mid M_j,\ n < M_j \text{ for some } m; \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Then we have

$$\begin{aligned} \sum_{2}^{M_t} \beta_n \phi(n)/n &= \sum_{j=1}^{t} M_j^{-1} \sum_{\substack{d \mid M_j \\ 1<d<M_j}} \phi(d) \\ &= \sum_{1}^{t} [1 - M_j^{-1} - \phi(M_j)/M_j] \\ &< \sum_{1}^{t} 1 = t, \end{aligned} \tag{16}$$

and

$$\sum_{2}^{M_t} \gamma_n = \sum_{j=1}^{t} \left(\sum 1/p_i\right) > \sum_{1}^{t} 2^j = 2^{t+1} - 1, \tag{17}$$

where the third summation in (17) is over the reciprocals of primes which divide M_j. Hence, by (16) and (17), we see that for the sequence $\{\alpha_n\}$ defined by (15), the condition

$$\sum \beta_n \phi(n)/n > c \sum \gamma_n$$

does not hold for any constant $c > 0$.

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