

## Two Problems in Metric Diophantine Approximation, I

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There is no known necessary and sufficient condition on a sequence  $\{\alpha_n\}$  of nonnegative real numbers such that for almost all  $x$  (in the sense of Lebesgue measure) there are infinitely many fractions  $p/q$  satisfying  $|x - p/q| < \alpha_q/q$ . Nor is any condition known when  $p/q$  is required to be reduced. We shall consider the relation of these problems to each other, and we shall discuss conjectured answers for these questions.

In this paper we shall consider the problem of finding a necessary and sufficient condition on a sequence  $\{\alpha_n\}$  so that for almost all  $x$  the relation

$$|x - p/q| < \alpha_q/q \quad (1)$$

holds for infinitely many pairs of integers  $p$  and  $q$ . By "almost all  $x$ " we mean "all  $x$  not in a set having Lebesgue measure zero." A characteristic of this type of problem is that (1) is satisfied for infinitely many  $p$  and  $q$  either for almost all  $x$  or for almost no  $x$ . A related problem also considered is the case where  $\gcd(p, q) = 1$  in (1). We shall study the relation between the problems of approximation by fractions and approximation by reduced fractions.

Duffin and Schaeffer [2] have proved the following theorem.

**DUFFIN-SCHAEFFER THEOREM.** *Let  $\{\alpha_n\}$  be a sequence of nonnegative numbers such that there is a constant  $c > 0$  with*

$$\sum_{i=1}^n \alpha_i \phi(i)/i > c \sum_{i=1}^n \alpha_i$$

*(where  $\phi$  is Euler's function) for arbitrarily many  $n$ . Then for almost all  $x$  or*

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for almost no  $x$ , respectively, there exist arbitrarily many relatively prime  $p$  and  $q$  such that (1) holds, according as  $\sum_{i=1}^{\infty} \alpha_i$  diverges or converges, respectively.

Related to this theorem is the following conjecture of Duffin and Schaeffer.

CONJECTURE 1. *The divergence of  $\sum \alpha_n \phi(n)/n$  is the necessary and sufficient condition that for almost all  $x$  relation (1) holds for infinitely many relatively prime  $p$  and  $q$ .*

We shall drop the requirement that  $p$  and  $q$  be relatively prime, and we shall give an analogous conjecture, which we shall prove to be equivalent to Conjecture 1.

Finally, we shall give an example of a sequence  $\{\alpha_n\}$  which has the properties that (1) holds infinitely often for almost all  $x$ , but if  $(p, q) = 1$  is also required, then (1) holds infinitely many times for almost no  $x$ .

Cassels [1] has investigated the problem without the restriction that  $p$  and  $q$  be relatively prime. Erdős [3] has recently proved a special case of Conjecture 1:

ERDŐS' THEOREM. *If  $\alpha_n = 0$  or  $\epsilon/n$  for all  $n$  and for some  $\epsilon > 0$ , then Conjecture 1 holds.*

Gallagher [4] has considered the more general problem of simultaneous approximation of a vector  $\mathbf{x} = (x_1, \dots, x_r)$  in  $\mathbf{R}^r$ , where each component  $x_i$  appears in an inequality analogous to (1). For the case  $r \geq 2$ , he has shown that the divergence of  $\sum \alpha_i^r$  is the desired criterion for simultaneous approximability.

Several authors have attempted to determine the number of solutions of (1) with  $q \leq h$ , for almost all  $x$ , or have studied the analogous problem in simultaneous approximation. For work on this and references to related results, see [4].

Let  $\mathcal{E}_n^\alpha$  denote the intersection of the interval  $(0, 1)$  with the union of open intervals of width  $2\alpha_n/n$ , each centered at the points  $m/n$ ,  $m = 0, 1, \dots, n$ . For convenience, we require that  $0 \leq \alpha_n \leq 1/2$ . The measure of  $\mathcal{E}_n^\alpha$ , denoted  $|\mathcal{E}_n^\alpha|$ , is  $2\alpha_n$ . Note that  $x \in \mathcal{E}_n^\alpha$  if and only if  $x$  satisfies relation (1).

Let  $\beta_n(\alpha)$  (denoted  $\beta_n$ ) be the sequence defined by

$$\beta_n(\alpha) = \max(\alpha_n, \alpha_{2n}/2, \alpha_{3n}/3, \dots). \quad (2)$$

We can augment terms of  $\{\alpha_n\}$  using (2) to get a sequence  $\{\beta_n\}$  for which the set of  $x$  satisfying

$$|x - p/q| < \beta_q/q \quad (3)$$

for infinitely many pairs of integers  $p$  and  $q$  has the same measure as the set of  $x$  for which (1) holds for infinitely many  $p$  and  $q$ . The following theorem describes this.

**THEOREM 1.** *Let  $\{\alpha_n\}$  be a given sequence and let  $\{\beta_n\}$  be a sequence defined by (2). Then for any  $x$  the following are equivalent.*

- (i) *There are infinitely many integers  $p$  and  $q$  such that (1) holds.*
- (ii) *There are infinitely many integers  $p$  and  $q$  such that (3) holds. If  $x$  is irrational (as almost all  $x$  are), the following is equivalent to (i) and (ii).*
- (iii) *There are infinitely many integers  $p$  and  $q$  with  $(p, q) = 1$  such that (3) holds.*

*Proof.* First we show that (2) implies that for any  $Q$

$$\bigcup_{q>Q} \mathcal{E}_q^\beta = \bigcup_{q>Q} \mathcal{E}_q^\alpha. \quad (4)$$

For any  $q$  there is some  $i$  such that  $\beta_q = \alpha_{qi}/i$ . Therefore, since any interval of  $\mathcal{E}_q^\beta$  has width  $\beta_q/q$  and since any interval of  $\mathcal{E}_{qi}^\alpha$  has width  $\alpha_{qi}/qi$  and the same center as an interval of  $\mathcal{E}_q^\beta$ , we have  $\mathcal{E}_q^\beta \subseteq \mathcal{E}_{qi}^\alpha$  for some  $i$ . Thus, the left side of (4) is contained in the right side. Since  $\beta_q \geq \alpha_q$  by (2), we have  $\mathcal{E}_q^\alpha \subseteq \mathcal{E}_q^\beta$ , and the right side must be contained in the left side of (4), establishing equality.

Obviously, (i) implies (ii). Conversely, suppose that for some given  $x$  there are only finitely many  $p$  and  $q$  such that (1) holds. Hence there is some  $Q$  such that (1) is not satisfied for that  $x$  by any  $q > Q$ , whence  $x \notin \bigcup_{q>Q} \mathcal{E}_q^\alpha$ . Thus,  $x \notin \bigcup_{q>Q} \mathcal{E}_q^\beta$  by (4), and (3) is satisfied for only finitely many  $q$ . Hence, (i) and (ii) are equivalent.

Clearly, (iii) implies (ii). Suppose (ii) is true. Then there is an infinite set of fractions  $p/q$  such that (3) holds. Since  $\beta_{kq}/k \leq \beta_q$ , an inequality

$$|\alpha - kp/kq| < \beta_{kq}/kq$$

with  $(p, q) = 1$  implies that

$$|\alpha - p/q| = |\alpha - kp/kq| < \beta_q/q.$$

On the other hand, for any irrational  $x$  and a given reduced fraction  $p/q$ , there are only finitely many  $k$  such that

$$|x - kp/kq| < \beta_{kq}/kq, \quad (5)$$

since the left side of (5) is a positive constant and the right side tends to zero as  $k$  increases. Hence, there are infinitely many reduced fractions satisfying (3), and the proof is complete.

Next we consider the following conjecture.

CONJECTURE 2. *The divergence of*

$$\sum_{n=2}^{\infty} (\max_{i \in \mathbb{N}} (\alpha_{ni}/i)) \phi(n)/n$$

*is a necessary and sufficient condition that for almost all  $x$  there are infinitely many  $p$  and  $q$  such that (1) holds.*

THEOREM 2. *Divergence of the infinite sums in Conjectures 1 and 2 is a necessary condition that the conclusions hold.*

*Proof.* That the divergence of  $\sum \alpha_n \phi(n)/n$  is a necessary condition in Conjecture 1 is proved in [2, p. 251]. For Conjecture 2, let  $\{\beta_n\}$  be defined by (2), and suppose that  $\sum \beta_n \phi(n)/n$  converges. Hence, for almost no  $x$  are there infinitely many relatively prime  $p$  and  $q$  such that (3) holds, by the first statement of this proof. Thus, by the equivalence of conditions (i) and (iii) of Theorem 1, it follows that for almost all  $x$  there are only finitely many  $p$  and  $q$ , not necessarily relatively prime, such that (1) holds.

THEOREM 3. *Conjectures 1 and 2 are equivalent.*

*Proof.* Because we have Theorem 2, it is only necessary to show that the sufficiency conditions are equivalent.

Suppose that Conjecture 1 is true and suppose  $\sum \beta_n \phi(n)/n$  diverges. Then for almost all  $x$ , there are infinitely many  $p$  and  $q$  satisfying (3). Because of this and Theorem 2, Conjecture 1 implies Conjecture 2.

Suppose that Conjecture 1 is false; then because of Theorem 2, we have a sequence  $\{\alpha_n\}$  such that  $\sum \alpha_n \phi(n)/n$  diverges and such that for almost all  $x$  there are only finitely many relatively prime  $p$  and  $q$  such that (1) holds. We can assume that

$$\alpha_q = \max_{i \in \mathbb{N}} (\alpha_{qi}/i) = \beta_q \quad (6)$$

for all  $q$ , for if this were false, then we could increase terms for which

$\alpha_q < \max_{i \in \mathbb{N}} (\alpha_{qi}/i)$ , and by Theorem 1, we would not alter the desired properties of  $\{\alpha_n\}$ . For almost all  $x$  there are only finitely many solutions of (3) with coprime  $p$  and  $q$ . Hence, by Theorem 1, there are only finitely many solutions of (1). Thus, if Conjecture 1 is false, then so is Conjecture 2, and the theorem follows.

We now construct an example of a sequence for which (1) is satisfied for almost all  $x$  infinitely often, but for almost no  $x$  by infinitely many coprime  $p$  and  $q$ . Let all terms of  $\{\alpha_n\}$  have the value zero, except for those with the following subscripts  $n$ :

$$n = 2, 3 \cdot 5 \cdot 7, 11 \cdot 13 \cdot 17 \cdot \cdots \cdot p_r, \dots, p_s p_{s+1} \cdots p_t, \dots,$$

where the  $i$ th value of  $n$  in this list has just enough prime factors so that  $\phi(n)/n < 2^{-i}$ , and the prime factors in the list run successively through all primes. Let all terms of  $\{\alpha_n\}$  with these subscripts have the value  $1/10$ . This construction is possible, since

$$\phi(n)/n = \prod_{i=1}^t (1 - 1/p_i),$$

and  $\prod_{i=1}^{\infty} (1 - 1/p_i)$  diverges. Using this sequence  $\{\alpha_n\}$ , let  $\{\beta_n\}$  be defined by (2).

Note that by (2) a particular positive term  $\alpha_m$  of  $\{\alpha_n\}$  causes  $\beta_d$  to be positive for all divisors  $d$  of  $m$ . Indeed, for any  $d$  there is some  $m$  such that

$$\beta_d = \alpha_m(d/m), \quad d \mid m.$$

Since 1 is the only common divisor of the subscripts of any two positive terms of  $\{\alpha_n\}$ , we require  $d > 1$  in what follows. Using the identity

$$\sum_{\substack{d \mid m \\ d > 1}} \phi(d)/m = (m - 1)/m$$

we get

$$\begin{aligned} \sum_{n=1}^{\infty} \beta_n \phi(n)/n &= \beta_1 + \sum_{m=2}^{\infty} \sum_{\substack{d \mid m \\ d > 1}} \alpha_m(d/m) \phi(d)/d \\ &= \alpha_2/2 + \sum_{m=2}^{\infty} \alpha_m(m - 1)/m \\ &> \sum \alpha_m/2 = \infty. \end{aligned}$$

Redefine  $\mathcal{E}_n^\alpha$  so that the intervals centered at 0 and 1 are disregarded.

The sets  $\mathcal{E}_n^\alpha$  (which equal  $\bigcup_{d|n} E_d^\alpha$ , where the union is over the divisors  $d$  of  $n$ , where  $d > 1$ , and where  $E_n^\alpha$  is defined as  $\mathcal{E}_n^\alpha$  was, with the restriction that  $(m, n) = 1$ ) have little overlap with each other, as we shall show. Suppose  $I'$  is an interval of  $\mathcal{E}_m^\alpha$  and  $I''$  an interval of  $\mathcal{E}_n^\alpha$  such that  $I', I''$  have nonempty intersection but distinct centers. Then, if  $i/m$  is the center of  $I'$  and  $j/n$  is the center of  $I''$ , we have

$$0 < |i/m - j/n| < \alpha_m/m + \alpha_n/n$$

or

$$0 < |in - jm| < n\alpha_m + m\alpha_n.$$

Without loss of generality, we may assume that

$$\alpha_m/m \leq \alpha_n/n,$$

whence,

$$0 < |in - jm| < 2m\alpha_n.$$

The number of solutions  $(i, j)$  of this inequality can be shown to be bounded by  $4m\alpha_n$ , by [2, Lemma I]. Hence, the overlap of  $\alpha_n$  and  $\alpha_m$  is bounded by  $(4m\alpha_n)(\alpha_m/m)$ , so that we have

$$|\mathcal{E}_n^\alpha \mathcal{E}_m^\alpha| < 8\alpha_n\alpha_m. \quad (7)$$

Now, the proof of the Duffin-Schaeffer theorem can be easily modified to show that almost all  $x$  are approximable infinitely often by (1). Specifically, using (7) in place of [2, Lemma II], we obtain in place of [2, Eq. (10)] the equation

$$|B| \geq \sum_{\nu=n}^m |\mathcal{E}_\nu^\alpha| - 4 \left( \sum_{\nu=n}^m \alpha_\nu \right)^2. \quad (10')$$

Also, the equation

$$\sum_{\nu=n}^m |\mathcal{E}_\nu^\alpha| = 2 \sum_{\nu=n}^m \alpha_\nu(\nu - 1)/\nu$$

can be used instead of using the analogous relation on  $\sum |E|$  and the second part of [2, Eq. (13)].

However, almost no  $x$  is approximable infinitely many times by reduced fractions satisfying (1), since

$$\sum \alpha_n \phi(n)/n < (1/10) \sum 2^{-i} < \infty.$$

This completes our example.

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