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SUBGRAPHS OF GRAPHS, I*

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Abstract. Let G and H be two simple graphs on p vertices. We give a sufficient condition, based on the minimum degree of the vertices of G and the maximum degree of the vertices of H, for H to be a subgraph of G.

1. Introduction

Throughout this paper, all graphs considered are finite and simple. The *degree* of a vertex v in the graph G is denoted $\deg_G(v)$. The vertex set of G is denoted $V(G)$. We say that a graph H can be embedded into the graph G if there is an injection $\pi : V(H) \to V(G)$ such that if v and w are adjacent in H, then $\pi(v)$ and $\pi(w)$ are adjacent in G. Given graphs G and H, each having p vertices, we give a sufficient condition for the existence of an embedding of H into G.

The notation $G(I, X)$ stands for a bipartite graph with an ordered bipartition of its vertex set. A bipartite graph $H(J, Y)$ is said to be *embedded* in $G(I, X)$ when a pair (π, θ) of injective mappings $\pi : J \to I$ and $\theta : Y \to X$ exists such that if $j \in J$ and $y \in Y$ are adjacent in $H(J, Y)$, then πj and θy are adjacent in $G(I, X)$. This embedding will be denoted by

$$(\pi, \theta) : H(J, Y) \to G(I, X).$$

First, we apply a theorem of Rado from transversal theory to ob-

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tain a sufficient condition that any bijection $\pi: J \to I$ can be extended to an embedding $(\pi, \theta): H(J, Y) \to G(I, X)$. An extensive family of bipartite graphs $G(I, X)$ and $H(J, Y)$ is constructed for which the above sufficient condition is also necessary. We then obtain a sufficient condition for embedding a graph H in a graph G, depending on the vertex independence number $\beta(H)$ of H, the maximum degree $\Delta(H)$ of the vertices of H, and the minimum degree $\delta(G)$ of the vertices of G. Finally, we give a constant [1] $c(H)$, depending only on $\Delta(H)$, such that if $\delta(G) \geqslant c(H)p-1$, then H can be embedded in G.

Given $X \subseteq V(G)$ and $a \in V(G)$, denote by X_a the set of vertices $x \in X$ adjacent to a (i.e., such that x and a are distinct and have a common edge). For $A \subseteq V(G)$, define the sets

$$X(A) = \bigcup_{a \in A} X_a, \qquad X^*(A) = \bigcap_{a \in A} X_a.$$

This notation is adopted throughout this paper and applies usually to bipartite graphs.

2. A theorem of Rado

We now give a variation of a theorem of Rado [2]. Our statement is more like the formulation in Mirsky's book [1, pp. 84–87].

Theorem 2.1 (Rado [2]). *Let $G(I, X)$ and $H(J, Y)$ be bipartite graphs with $|I| = |J|$ and $|X| \geqslant |Y|$. Suppose $\pi : J \to I$ is a bijective mapping. Then a necessary and sufficient condition that an embedding*

$$(\pi, \theta): H(J, Y) \to G(I, X)$$

exists is that for any nonempty subset $Y' \subseteq Y$,

$$\left| \bigcup_{y \in Y'} X^*(\pi J_y) \right| \geqslant \left| \bigcup_{y \in Y'} Y^*(J_y) \right|. \tag{2.1}$$

[1] We have recently improved Theorem 4.2 to the better, but probably not best possible, condition $\delta(G) \geqslant p(1-1/2\Delta(H))$.

In [1], condition (2.1) is taken to hold for any nonempty family $\mathcal{F}$ of subsets of J, not just the subfamilies of $(J_y : y \in Y)$. Suppose $J' \subseteq J$ and define

$$Y[J'] = Y^*(J') \setminus Y^*(J \setminus J').$$

We note that the nonempty $Y[J']$ are those of the form $Y[J_y]$ for $y \in Y$, and that the $Y[J_y]$ are the boolean atoms generated by $(Y_j : j \in J)$. Using these facts, one can easily see from the proof in [1] that (2.1) is equivalent to Mirsky's condition.

Define a bipartite graph $B(X, Y)$ with vertex set the disjoint union of X and Y and edge set

$$\{\{x, y\} : x \in X,\ y \in Y \text{ and } \pi J_y \subseteq I_x\}.$$

Condition (2.1) is necessary and sufficient for a matching in $B(X, Y)$ which associates with each $y \in Y$ an element $\theta y \in X$. In this way the required embedding

$$(\pi, \theta) : H(J, Y) \to G(I, X)$$

is produced.

3. Subgraphs of bipartite graphs

It would be of interest to know a condition for a bipartite graph $H(J, Y)$ to be a subgraph of another bipartite graph $G(I, X)$. Theorem 2.1 provides a means of extending, if possible, a bijection $\pi : J \to I$ to an embedding of $H(J, Y)$ into $G(I, X)$. We shall let

(3.1) $$\Delta_J(H) = \max_{j \in J} \deg_{H(J,Y)}(j),$$

(3.2) $$\Delta_Y(H) = \max_{y \in Y} \deg_{H(J,Y)}(y).$$

Also, define

(3.3) $$\delta_I(G) = \min_{i \in I} \deg_{G(I,X)}(i),$$

(3.4) $\delta_X(G) = \min_{x \in X} \deg_{G(I,X)}(x).$

We first prove the following:

Theorem 3.1. *Let $G(I, X)$ and $H(J, Y)$ be bipartite graphs with $|I| = |J|$ and $|X| \geq |Y|$. If*

(3.5) $\Delta_Y(H)(|X| - \delta_I(G)) + \Delta_J(H)(|I| - \delta_X(G)) \leq |X|,$

then for any bijection $\pi : J \to I$, there is an injection $\theta : Y \to X$ such that

$$(\pi, \theta) : H(J, Y) \to G(I, X)$$

is an embedding.

Proof. By Theorem 2.1, it suffices to prove (2.1) for any subset Y' of Y. By (3.1) for all $j \in J$,

(3.6) $|Y_j| \leq \Delta_J(H),$

and by (3.2) for any $y \in Y$,

(3.7) $|J_y| \leq \Delta_Y(H).$

Let Y' be a fixed subset of Y. Let R be a minimum subset of J such that $R \cap J_y$ is nonempty for all sets J_y with $y \in Y'$. Every point in $\bigcup_{y \in Y'} Y^*(J_y)$ lies in Y_j for some $j \in R$, and so by (3.6),

(3.8) $\left|\bigcup_{y \in Y'} Y^*(J_y)\right| \leq \left|\bigcup_{j \in R} Y_j\right| \leq |R|\, \Delta_J(H).$

Case I. Suppose that

(3.9) $|R| > |I| - \delta_X(G).$

Then every subset of I with at most $|I| - \delta_X(G)$ vertices is disjoint from some member of $(\pi J_y : y \in Y')$. Every I_x for $x \in X$ contains at least $\delta_X(G)$ vertices and hence must include some $\pi J_{y'}$ with $y' \in Y'$. Since $x \in X^*(\pi J_{y'})$ and $x \in X$ is arbitrary,

$$X = \bigcup_{y \in Y'} X^*(\pi J_y).$$

We conclude that

$$\left|\bigcup_{y \in Y'} X^*(\pi J_y)\right| = |X| \geq |Y| \geq \left|\bigcup_{y \in Y'} Y^*(J_y)\right|,$$

and so (2.1) holds in this case.

Case II. Suppose that

$$(3.10) \qquad |R| \leq |I| - \delta_X(G).$$

Let y' be a fixed member of Y'. Then by (3.7) and (3.3), the set $X^*(\pi J_{y'})$ is the intersection of the

$$|\pi J_{y'}| = |J_{y'}| \leq \Delta_Y(H)$$

sets X_i, $i \in \pi J_{y'}$, each X_i containing all but at most $|X| - \delta_I(G)$ vertices of X. Hence $X^*(\pi J_{y'})$ contains all but at most $\Delta_Y(H)(|X| - \delta_I(G))$ vertices of X, and so,

$$(3.11) \qquad |X^*(\pi J_{y'})| \geq |X| - \Delta_Y(H)(|X| - \delta_I(G)).$$

Therefore,

$$\begin{aligned} \left|\bigcup_{y \in Y'} X^*(\pi J_y)\right| &\geq |X^*(\pi J_{y'})| \\ &\geq |X| - \Delta_Y(H)(|X| - \delta_I(G)) && \text{(by (3.11))}, \\ &\geq \Delta_J(H)(|I| - \delta_X(G)) && \text{(by (3.5))}, \\ &\geq \Delta_J(H)\,|R| && \text{(by (3.10))}, \\ &\geq \left|\bigcup_{y \in Y'} Y^*(J_y)\right| && \text{(by (3.8))}, \end{aligned}$$

whence (2.1) follows. Since (2.1) holds in either case, this theorem follows from Rado's theorem.

The minimum degree sequence required to obtain the conclusions of Theorem 3.1 may be relaxed a bit from the requirements of our result. However, in a certain sense, Theorem 3.1 is best possible, as we now show.

For positive integers a, b, m, n, choose disjoint sets I, X, J, Y such that

$$|I| = |J| \quad \text{and} \quad |X| = |Y|,$$
$$am \leqslant |X| \quad \text{and} \quad bn \leqslant |Y|,$$
$$mn \leqslant |I|.$$

Let $\pi : J \to I$ be a fixed bijective mapping. Then partitions

$$J = J^0 \cup J^1 \cup \ldots \cup J^n,$$
$$Y = Y^0 \cup Y^1 \cup \ldots \cup Y^n,$$
$$I = I^0 \cup I^1 \cup \ldots \cup I^m,$$
$$X = X^0 \cup X^1 \cup \ldots \cup X^m$$

exist with

$$|J^j| = m \quad \text{and} \quad |Y^j| = b \quad \text{for } 1 \leqslant j \leqslant n,$$
$$|I^i| = n \quad \text{and} \quad |X^i| = a \quad \text{for } 1 \leqslant i \leqslant m,$$
$$|\pi J^i \cap I^i| = 1 \quad \text{for } 1 \leqslant j \leqslant n,\ 1 \leqslant i \leqslant m.$$

The sets J^0, Y^0, I^0, X^0 may be empty.

We construct bipartite graphs $H(J, Y)$ and $G(I, X)$, with the evident vertex sets, and edge sets defined by

$$E(H(J, Y)) = \{\{j, y\} : j \in J^i \text{ and } y \in Y^i \text{ for } 1 \leqslant i \leqslant n\},$$
$$E(G(I, X)) = \{\{i, x\} : i \in I \text{ and } x \in X\} \setminus \{\{i, x\} : i \in I^j \text{ and } x \in X^j \text{ for } 1 \leqslant j \leqslant m\}.$$

We readily see that any embedding

$$(\pi, \theta) : H(J, Y) \to G(I, X)$$

forces $\theta Y^i \subseteq X^0$ for $1 \leqslant i \leqslant n$. Such embeddings then exist if and only if

$$\left| \bigcup_{i=1}^{n} Y^i \right| \leqslant |X| - \left| \bigcup_{i=1}^{m} X^i \right|,$$

or

$$bn \leqslant |X| - am,$$

or

$$am + bn \leqslant |X|.$$

But this is exactly condition (3.5) of Theorem 3.1.

4. Subgraphs of simple graphs

In this section we consider the problem of giving a nontrivial sufficient condition for a graph to be embedded in another graph. We shall use Theorem 3.1.

Recall that $\Delta(H)$ is the maximum degree of the vertices of H, $\delta(G)$ is the minimum degree of the vertices of G, and $\beta(H)$ is the cardinality of a maximum set of independent vertices of H.

Theorem 4.1. *Let G and H be graphs on p vertices. If*

$$(4.1)\qquad \delta(G) \geqslant p - \frac{\beta(H)}{2\Delta(H)} - 1,$$

then H can be embedded in G.

Proof. Let G and H be graphs satisfying the hypothesis. There exists a chain

$$H_m \subseteq H_{m+1} \subseteq \dots \subseteq H_\Delta = H$$

of subgraphs of H satisfying the following conditions:

(i) The graph H_m is edgeless.

(ii) $H_{\Delta-1}$ has $\beta(H)$ fewer vertices than H.

(iii) For any k, the removal of the edges of H_k from H_{k+1} leaves a bipartite graph with bipartition (J_k, Y_k) for $J_k = V(H_k)$ and $Y_k = V(H_{k+1}) \setminus V(H_k)$.

Such a sequence of graphs is constructed by removing from H_k a maximum set of independent vertices to obtain H_{k-1}. The three conditions are then trivially satisfied.

The graph H_m, having no edges, can be embedded in G. Let this be a basis for induction. We shall let π_k denote the embedding of H_k into G, and by using Theorem 3.1, we shall extend π_k to an embedding of H_{k+1} into G.

Let

$$(4.2)\qquad I_k = \pi_k[V(H_k)],$$

$$(4.3)\qquad X_k = V(G) \setminus I_k.$$

Let $G(I_k, X_k)$ denote the bipartite graph obtained from G by removing all edges not joining vertices of I_k and X_k. The desired conclusion of Theorem 3.1 is that there is an injection $\theta_k : Y_k \to X_k$ so that π_k and θ_k define an embedding of the bipartite graph of (iii) into $G(I_k, X_k)$. Since π_k embeds H_k also, π_k and θ_k define an embedding π_{k+1} of H_{k+1} into G. Thus, we must derive the hypothesis of Theorem 3.1.

The condition $|I| = |J|$ of Theorem 3.1 is a consequence of (4.2). To obtain $|Y| \leqslant |X|$, note that

$$\begin{aligned} |Y_k| &= |V(H_{k+1})| - |V(H_k)| \\ &= |V(H_{k+1})| - |I_k| \\ &\leqslant |V(G)| - |I_k| = |X_k| \qquad \text{(by (4.3))}. \end{aligned}$$

Finally, we must prove (3.5). In $G(I_k, X_k)$,

$$\begin{aligned} \delta_X(G(I_k, X_k)) &= \min_{x \in X_k} \deg_{G(I_k, X_k)}(x) \\ &\geqslant \delta(G) - |X_k| + 1 \\ &\geqslant p - \frac{\beta(H)}{2\Delta(H)} - |X_k| \qquad \text{(by (4.1))}, \\ &= |I_k| - \frac{\beta(H)}{2\Delta(H)} \qquad \text{(by (4.3))}, \end{aligned} \tag{4.4}$$

and similarly,

$$\begin{aligned} \delta_I(G(I_k, X_k)) &\geqslant \delta(G) - |I_k| + 1 \\ &\geqslant |X_k| - \frac{\beta(H)}{2\Delta(H)} \qquad \text{(by (4.1) and (4.3))}. \end{aligned} \tag{4.5}$$

Observe that the degree of the vertices of the bipartite graph described in (iii) is at most $\Delta(H)$. For $k = m, \ldots, \Delta-1$, we have

$$\begin{aligned} \Delta(H)(|X_k| - \delta_I) + \Delta(H)(|I_k| - \delta_X) &\leqslant \Delta(H)\frac{\beta(H)}{2\Delta(H)} + \Delta(H)\frac{\beta(H)}{2\Delta(H)} \\ &\qquad \text{(by (4.5) and (4.4))}, \\ &= \beta(H) = |X_{\Delta-1}| \\ &\qquad \text{(by (ii) and (4.3))}, \\ &\leqslant |X_k| \qquad (\text{since } X_{\Delta-1} \subseteq X_k), \end{aligned}$$

from which (3.5) follows. This completes the proof.

Theorem 4.2. *Let G and H be graphs on p vertices. If*

$$(4.6) \qquad \delta(G) \geqslant p\left(1 - \frac{1}{2\Delta(H)(\Delta(H)+1)}\right) - 1,$$

then H can be embedded in G.

Proof. In view of Theorem 4.1, it suffices to prove that H has $p/(\Delta(H)+1)$ independent vertices. Let S be a maximal set of independent vertices of H. Since S is maximal, every vertex of H either lies in S or is adjacent to a member of S. But at most $\Delta(H)+1$ vertices are adjacent to or equal to a given vertex of S. Hence S must have at least $p/(\Delta(H)+1)$ members, in order that all p vertices of H are adjacent to or equal to a member of S.

References

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