Hamiltonian Claw-free Graphs and Line Graphs

Hong-Jian Lai

Department of Mathematics, West Virginia University

Joint work with Ye Chen, Keke Wang and Meng Zhang

Contents

Contents

Contents

- The Problem
- Line graphs and claw-free graphs
- 3-connected $\{K_{1,3}, P_k\}$-free graphs

Contents

- The Problem
- Line graphs and claw-free graphs
- 3-connected $\{K_{1,3}, P_k\}$-free graphs
- 3-connected $\{K_{1,3}, N_{s_1,s_2,s_3}\}$-free graphs

Contents

- The Problem
- Line graphs and claw-free graphs
- 3-connected $\{K_{1,3}, P_k\}$-free graphs
- 3-connected $\{K_{1,3}, N_{s_1,s_2,s_3}\}$-free graphs
- 4-connected line graphs

Hamiltonian Graphs

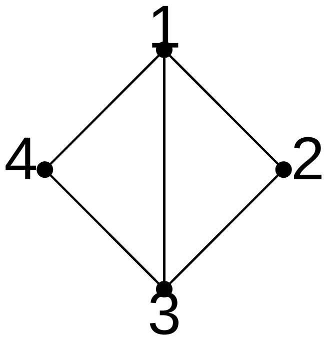
- A Hamilton cycle of a graph G: = a cycle containing all the vertices of G.
  A hamiltonian graph: = a graph with a Hamilton cycle.

Hamiltonian Graphs

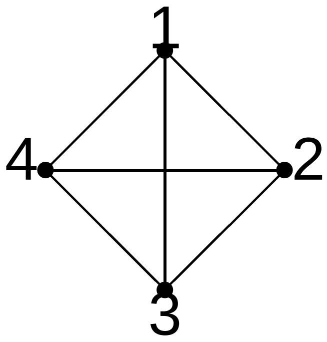
- A Hamilton cycle of a graph G: = a cycle containing all the vertices of G.
  A hamiltonian graph: = a graph with a Hamilton cycle.
- A Hamilton path: = a path containing all the vertices of G.
  A hamiltonian connected graph: = for any two vertices $u, v \in V(G)$, G has a (u, v)-Hamilton path.

Hamiltonian Graphs

- A Hamilton cycle of a graph G: = a cycle containing all the vertices of G.
  A hamiltonian graph: = a graph with a Hamilton cycle.
- A Hamilton path: = a path containing all the vertices of G.
  A hamiltonian connected graph: = for any two vertices $u, v \in V(G)$, G has a (u, v)-Hamilton path.
- Examples



Hamiltonian, not Hamiltonian-connected



Hamiltonian-connected

The Problem

- Observation: $K_{n+1,n} = n$, but $K_{n+1,n}$ cannot have a Hamilton cycle.

The Problem

- Observation: $K_{n+1,n} = n$, but $K_{n+1,n}$ cannot have a Hamilton cycle.
- Question: for which kind of graphs, high connectivity will warrant hamiltonicity?

The Problem

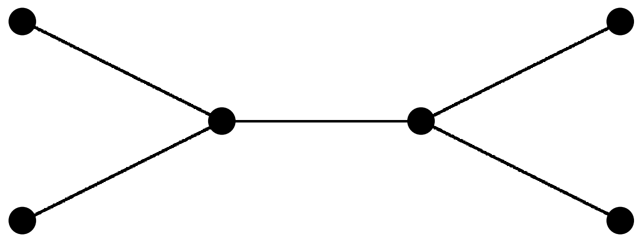
- Observation: $K_{n+1,n} = n$, but $K_{n+1,n}$ cannot have a Hamilton cycle.
- Question: for which kind of graphs, high connectivity will warrant hamiltonicity?
- Thomassen, Matthews and Sumner: conjectured such graphs exist.

Line Graphs

- $L(G)$: the line graph of a graph G, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are linked if and only if the corresponding edges in G share a vertex.

Line Graphs

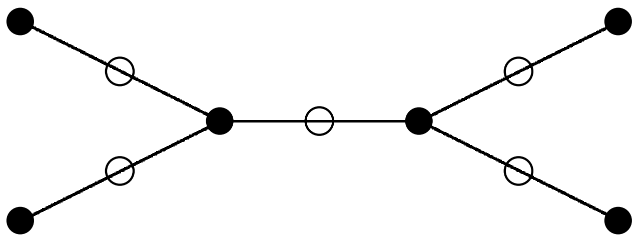
- $L(G)$: the line graph of a graph G, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are linked if and only if the corresponding edges in G share a vertex.
- $L(G)$ is a simple graph.



G

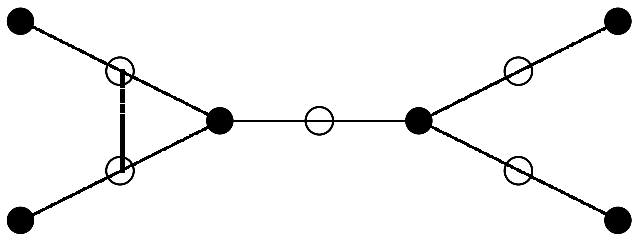
Line Graphs

- $L(G)$: the line graph of a graph G, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are linked if and only if the corresponding edges in G share a vertex.
- $L(G)$ is a simple graph.



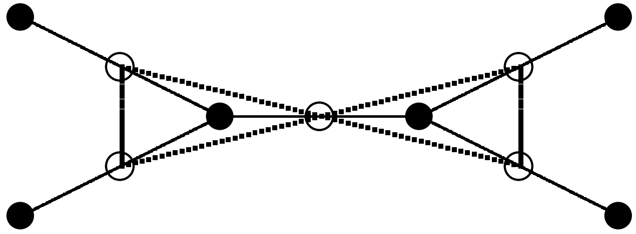
Line Graphs

- $L(G)$: the line graph of a graph G, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are linked if and only if the corresponding edges in G share a vertex.
- $L(G)$ is a simple graph.



Line Graphs

- $L(G)$: the line graph of a graph G, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are linked if and only if the corresponding edges in G share a vertex.
- $L(G)$ is a simple graph.



G: solid lines and closed circles

$L(G)$: dash lines and open circles

Claw-free Graphs

- a claw: an induced $K_{1,3}$

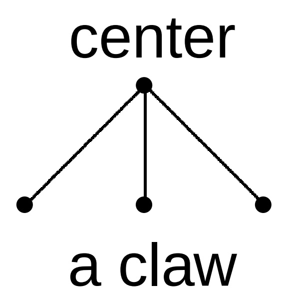


Figure 1.3

Claw-free Graphs

- a claw: an induced $K_{1,3}$

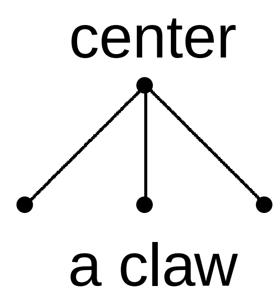
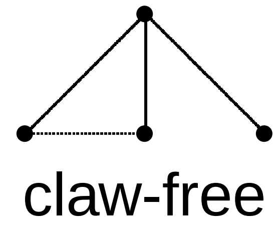


Figure 1.3

- claw free graph G: G does not contain an induced $K_{1,3}$



The Conjectures

- A claw-free graph G = a graph that does not have an induced $K_{1,3}$.

The Conjectures

- A claw-free graph G = a graph that does not have an induced $K_{1,3}$.
- Theorem: (Beineke and Robertson) Every line graph is also a claw-free graph.

The Conjectures

- A claw-free graph G = a graph that does not have an induced $K_{1,3}$.
- Theorem: (Beineke and Robertson) Every line graph is also a claw-free graph.
- Thomassen Conjecture: (JGT 1985) Every 4-connected line graph is hamiltonian.

The Conjectures

- A claw-free graph G = a graph that does not have an induced $K_{1,3}$.
- Theorem: (Beineke and Robertson) Every line graph is also a claw-free graph.
- Thomassen Conjecture: (JGT 1985) Every 4-connected line graph is hamiltonian.
- Matthews and Sumner Conjecture: (JGT 1984) Every 4-connected claw-free graph is hamiltonian.

Hamiltonian Line Graphs and Claw-free graphs

- Theorem (Zhan) Every 7-connected line graph is hamiltonian-connected.

Hamiltonian Line Graphs and Claw-free graphs

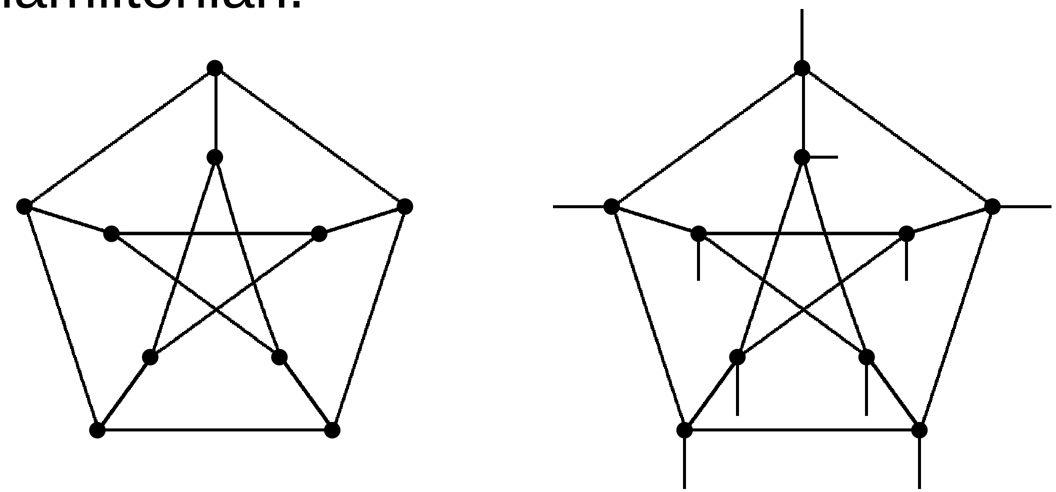
- Theorem (Zhan) Every 7-connected line graph is hamiltonian-connected.
- Theorem (Z. Ryjáček) These are equivalent:
  (A) Every 4-connected line graph is hamiltonian.
  (B) Every 4-connected claw-free graph is hamiltonian.

Hamiltonian Line Graphs and Claw-free graphs

- Theorem (Zhan) Every 7-connected line graph is hamiltonian-connected.
- Theorem (Z. Ryjáček) These are equivalent:
  (A) Every 4-connected line graph is hamiltonian.
  (B) Every 4-connected claw-free graph is hamiltonian.
- Theorem (Kaiser and P. Vrána) Every 5-connected claw-free graph with minimum degree at least 6 is hamiltonian.

3-connected Claw-free Graphs

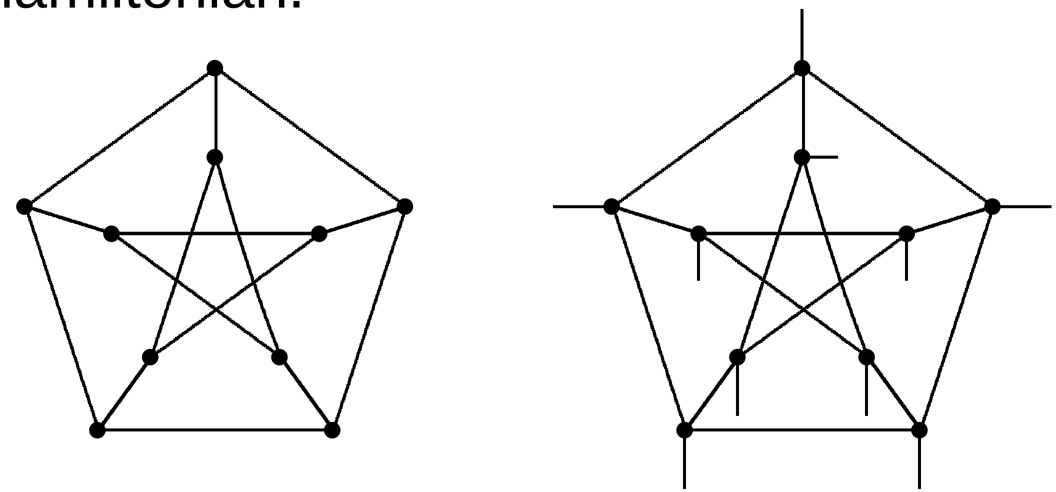
- In general, 3-connected claw-free graph may not be hamiltonian.



The Petersen graph P_{10} and P'_{10}

3-connected Claw-free Graphs

- In general, 3-connected claw-free graph may not be hamiltonian.



The Petersen graph P_{10} and P'_{10}

- The line graph $L(P'_{10})$ is not hamiltonian.

Forbidden Induced Subgraph Conditions

- Question: What should be forbidden in 3-connected graphs to warrant hamiltonicity?

Forbidden Induced Subgraph Conditions

- Question: What should be forbidden in 3-connected graphs to warrant hamiltonicity?
- Let $H_1, H_2, ..., H_c$ be graphs. A graph G is $\{H_1, H_2, ..., H_c\}$-free if G does not have an induced subgraph isomorphic to an H_i in the list.

Forbidden Induced Subgraph Conditions

- Question: What should be forbidden in 3-connected graphs to warrant hamiltonicity?
- Let $H_1, H_2, ..., H_c$ be graphs. A graph G is $\{H_1, H_2, ..., H_c\}$-free if G does not have an induced subgraph isomorphic to an H_i in the list.
- Theorem (Šoltés and HJL, JCTB 2001) Every 7-connected $\{K_{1,3}, K_5 - e, G_3\}$-free graph is hamiltonian-connected. (G_3 is the 3rd forbidden graph in Beineke-Robertson's characterization of a line graph).

Forbidden Induced Subgraph Conditions

- Question: What should be forbidden in 3-connected graphs to warrant hamiltonicity?
- Let $H_1, H_2, ..., H_c$ be graphs. A graph G is $\{H_1, H_2, ..., H_c\}$-free if G does not have an induced subgraph isomorphic to an H_i in the list.
- Theorem (Šoltés and HJL, JCTB 2001) Every 7-connected $\{K_{1,3}, K_5 - e, G_3\}$-free graph is hamiltonian-connected. (G_3 is the 3rd forbidden graph in Beineke-Robertson's characterization of a line graph).
- Theorem: (Proved by Ryjáček and Vrána in 2011 JGT, conjectured Šoltés and HJL by in JCTB 2001) Every 7-connected $K_{1,3}$-free graph is hamiltonian-connected.

Paths and Nets

- P_n = a path on n vertices.

Paths and Nets

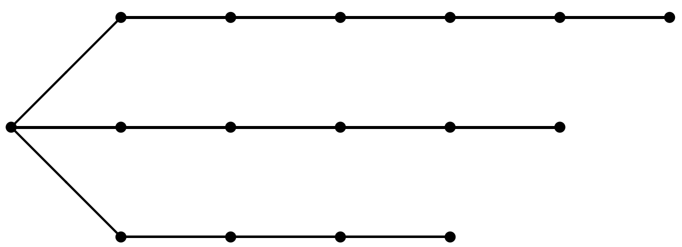
- P_n = a path on n vertices.
- Y_{s_1,s_2,s_3} = graph from disjoint paths P_{s_1+2}, P_{s_2+2} and P_{s_3+2} by identifying an end vertex of each of these three paths.

Paths and Nets

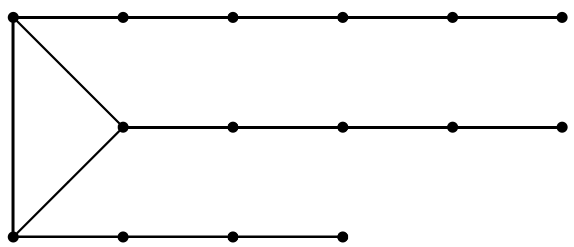
- P_n = a path on n vertices.
- Y_{s_1,s_2,s_3} = graph from disjoint paths P_{s_1+2}, P_{s_2+2} and P_{s_3+2} by identifying an end vertex of each of these three paths.
- $N_{s_1,s_2,s_3} = L(Y_{s_1,s_2,s_3})$

Paths and Nets

- P_n = a path on n vertices.
- Y_{s_1,s_2,s_3} = graph from disjoint paths P_{s_1+2}, P_{s_2+2} and P_{s_3+2} by identifying an end vertex of each of these three paths.
- $N_{s_1,s_2,s_3} = L(Y_{s_1,s_2,s_3})$
- Examples of Y_{s_1,s_2,s_3} and N_{s_1,s_2,s_3}



$Y_{5,4,3}$



$N_{5,4,3} = L(Y_{5,4,3})$

Conditions Forbidding Paths And Nets

- Theorem (Shepherd, PhD Thesis, U. of Waterloo, 1987) Every 3-connected $\{K_{1,3}, N_{1,1,1}\}$-free graph is hamiltonian.

Conditions Forbidding Paths And Nets

- **Theorem** (Shepherd, PhD Thesis, U. of Waterloo, 1987) Every 3-connected $\{K_{1,3}, N_{1,1,1}\}$-free graph is hamiltonian.
- **Theorem** (Brousek, Ryj$\acute{a}\check{c}$ek and Favaron, DM 1999) Every 3-connected $\{K_{1,3}, N_{0,0,4}\}$-free graph is hamiltonian.

Conditions Forbidding Paths And Nets

- Theorem (Shepherd, PhD Thesis, U. of Waterloo, 1987) Every 3-connected $\{K_{1,3}, N_{1,1,1}\}$-free graph is hamiltonian.
- Theorem (Brousek, Ryj$\acute{a}\check{c}$ek and Favaron, DM 1999) Every 3-connected $\{K_{1,3}, N_{0,0,4}\}$-free graph is hamiltonian.
- Theorem (Luczak and Pfender, JGT 2004) Every 3-connected $\{K_{1,3}, P_{11}\}$-free graph is hamiltonian.

How Far Can We Go?

- Theorem (Xiong, Yan, Yan and HJL, JGT 2010) Every 3-connected $\{K_{1,3}, N_{0,0,8}\}$-free graph is hamiltonian.

How Far Can We Go?

- Theorem (Xiong, Yan, Yan and HJL, JGT 2010) Every 3-connected $\{K_{1,3}, N_{0,0,8}\}$-free graph is hamiltonian.
- $L(P(10)')$ is 3-connected, $\{K_{1,3}, N_{0,0,9}\}$-free, non-hamiltonian.

How Far Can We Go?

- Theorem (Xiong, Yan, Yan and HJL, JGT 2010) Every 3-connected $\{K_{1,3}, N_{0,0,8}\}$-free graph is hamiltonian.
- $L(P(10)')$ is 3-connected, $\{K_{1,3}, N_{0,0,9}\}$-free, non-hamiltonian.
- $L(P(10)')$ is 3-connected, $\{K_{1,3}, P_{12}\}$-free, non-hamiltonian.

How Far Can We Go?

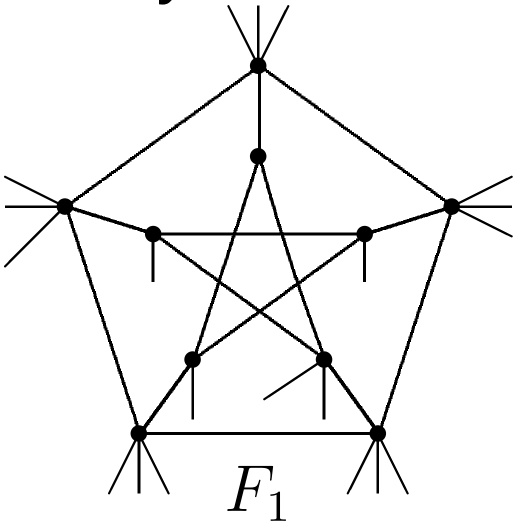
- Theorem (Xiong, Yan, Yan and HJL, JGT 2010) Every 3-connected $\{K_{1,3}, N_{0,0,8}\}$-free graph is hamiltonian.
- $L(P(10)')$ is 3-connected, $\{K_{1,3}, N_{0,0,9}\}$-free, non-hamiltonian.
- $L(P(10)')$ is 3-connected, $\{K_{1,3}, P_{12}\}$-free, non-hamiltonian.
- Open Problem (Xiong, Yan, Yan and HJL, JGT 2010) Is $L(P(10)')$ the only type of non-hamiltonian 3-connected $\{K_{1,3}, N_{0,0,9}\}$-free graph?

How Far Can We Go?

- Theorem (Xiong, Yan, Yan and HJL, JGT 2010) Every 3-connected $\{K_{1,3}, N_{0,0,8}\}$-free graph is hamiltonian.
- $L(P(10)')$ is 3-connected, $\{K_{1,3}, N_{0,0,9}\}$-free, non-hamiltonian.
- $L(P(10)')$ is 3-connected, $\{K_{1,3}, P_{12}\}$-free, non-hamiltonian.
- Open Problem (Xiong, Yan, Yan and HJL, JGT 2010) Is $L(P(10)')$ the only type of non-hamiltonian 3-connected $\{K_{1,3}, N_{0,0,9}\}$-free graph?
- Theorem YES.

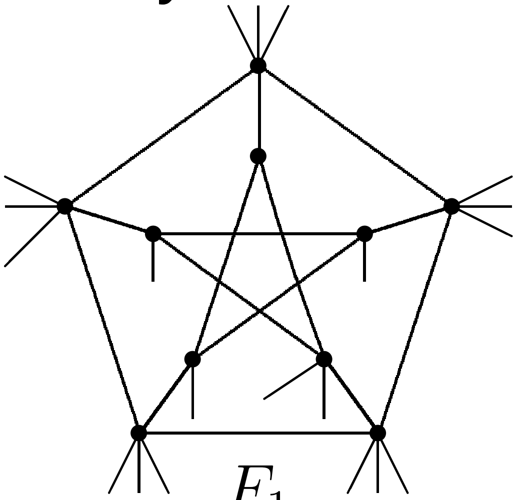
Forbidding P_{12}

- Example Let F_1 denote any graph in the following family.



Forbidding P_{12}

- Example Let F_1 denote any graph in the following family.



F_1

- $L(F_1)$ is 3-connected, P_{12}-free but non-hamiltonian.

Forbidding P_{12}

- **Theorem** Every 3-connected P_{12}-free line graph $L(G)$ if hamiltonian if and only if $G \notin F_1$.

Forbidding P_{12}

- **Theorem** Every 3-connected P_{12}-free line graph $L(G)$ if hamiltonian if and only if $G \notin F_1$.
- **Corollary** Let Γ be a 3-connected $\{K_{1,3}, P_{12}\}$-free graph. Then Γ is hamiltonian if and only if its closure $cl(\Gamma)$ is not the line graph $L(G)$, for any member G in F_1.

Forbidding P_{13}

- We have also determine 4 (classes of) graphs F_1, F_2, F_3, F_4 such that the following hold.

Forbidding P_{13}

- We have also determine 4 (classes of) graphs F_1, F_2, F_3, F_4 such that the following hold.
- Theorem Every 3-connected P_{13}-free line graph $L(G)$ if hamiltonian if and only if $G \notin F_1 \cup F_2 \cup F_3 \cup F_4$.

Forbidding P_{13}

- We have also determine 4 (classes of) graphs F_1, F_2, F_3, F_4 such that the following hold.
- Theorem Every 3-connected P_{13}-free line graph $L(G)$ if hamiltonian if and only if $G \notin F_1 \cup F_2 \cup F_3 \cup F_4$.
- Corollary Let Γ be a 3-connected $\{K_{1,3}, P_{13}\}$-free graph. Then Γ is hamiltonian if and only if its closure $cl(\Gamma)$ is not the line graph $L(G)$, for any member G in $F_1 \cup F_2 \cup F_3 \cup F_4$.

Forbidding general nets

- Theorem Let $s_1, s_2, s_3 > 0$ be integers such that $s_1 + s_2 + s_3 \leq 9$.
  (i) If $s_1 + s_2 + s_3 \leq 9$, every 3-connected $\{K_{1,3}, N_{s_1,s_2,s_3}\}$-free graph is hamiltonian.
  (ii) If $s_1 + s_2 \leq 8$, every 3-connected $\{K_{1,3}, N_{s_1,s_2,0}\}$-free graph is hamiltonian.

4-connected line graphs: Former Results

- Theorem Let $L(G)$ be a 4-connected line graph. Each of the following holds.
  (i) (Chen, Lai and Weng, 1994) If G is claw-free, then $L(G)$ is hamiltonian.
  (ii) (Kriesell, JCTB 2001) If G is claw-free, then $L(G)$ is hamiltonian-connected.
  (iii) (Y. Shao, M. Zhan and HJL, DM 2008) If G is quasi claw-free, then $L(G)$ is hamiltonian-connected.
  (iv) (Y. Shao, G. Yu, M. Zhan and HJL, DAM 2009) If G is almost claw-free, $L(G)$ is hamiltonian-connected.

4-connected line graphs: Former Results

- Theorem Let $L(G)$ be a 4-connected line graph. Each of the following holds.
  (i) (Chen, Lai and Weng, 1994) If G is claw-free, then $L(G)$ is hamiltonian.
  (ii) (Kriesell, JCTB 2001) If G is claw-free, then $L(G)$ is hamiltonian-connected.
  (iii) (Y. Shao, M. Zhan and HJL, DM 2008) If G is quasi claw-free, then $L(G)$ is hamiltonian-connected.
  (iv) (Y. Shao, G. Yu, M. Zhan and HJL, DAM 2009) If G is almost claw-free, $L(G)$ is hamiltonian-connected.
- Both QCF and ACF contain CF properly.

4-connected line graphs: DCT and P3D Graphs

- For nonadjacent $x, y \in V(G)$ with $N(x) \cap N(y) \neq \emptyset$, define

4-connected line graphs: DCT and P3D Graphs

- For nonadjacent $x, y \in V(G)$ with $N(x) \cap N(y) \neq \emptyset$, define
- $J_G(x, y) = \{u \in N_G(x) \cap N_G(y) : N_G[u] \subseteq N_G[x] \cup N_G[y]\}$, and

4-connected line graphs: DCT and P3D Graphs

- For nonadjacent $x, y \in V(G)$ with $N(x) \cap N(y) \neq \emptyset$, define
- $J_G(x,y) = \{u \in N_G(x) \cap N_G(y) : N_G[u] \subseteq N_G[x] \cup N_G[y]\}$, and
- $J'_G(x,y) = \{u \in N_G(x) \cap N_G(y) : v \in N_G(u) - (N_G[x] \cup N_G[y]) \Longrightarrow N_G(x) \cup N_G(y) \cup N_G(u) - \{x, y, v\} \subseteq N_G(v)\}$.

4-connected line graphs: DCT and P3D Graphs

- Definition: (Ainouche, Favaron and Li, DM 2008) G is **dominated claw toed** (DCT) if for any claw $[a, a_1, a_2, a_3]$ centered as a, $J_G(a_1, a_2) \cup J_G(a_2, a_3) \cup J_G(a_1, a_3) \neq \emptyset$..

4-connected line graphs: DCT and P3D Graphs

- Definition: (Ainouche, Favaron and Li, DM 2008) G is **dominated claw toed** (DCT) if for any claw $[a, a_1, a_2, a_3]$ centered as a, $J_G(a_1, a_2) \cup J_G(a_2, a_3) \cup J_G(a_1, a_3) \neq \emptyset$..
- Definition: (Broersma and Vumar, Math. Methods Oper. Res. 2009) G is P_3**-dominated** (P3D) if for any non adjacent $x, y \in V(G)$ with $N(x) \cap N(y) \neq \emptyset$, $J_G(x, y) \cup J'_G(x, y) \neq \emptyset$.

4-connected line graphs: DCT and P3D Graphs

- Definition: (Ainouche, Favaron and Li, DM 2008) G is **dominated claw toed** (DCT) if for any claw $[a, a_1, a_2, a_3]$ centered as a, $J_G(a_1, a_2) \cup J_G(a_2, a_3) \cup J_G(a_1, a_3) \neq \emptyset$..
- Definition: (Broersma and Vumar, Math. Methods Oper. Res. 2009) G is P_3**-dominated** (P3D) if for any non adjacent $x, y \in V(G)$ with $N(x) \cap N(y) \neq \emptyset$, $J_G(x, y) \cup J'_G(x, y) \neq \emptyset$.
- Each of DCT and P3D graphs properly contain QCF and ACF.

4-connected line graphs: New Results

- Theorem Suppose $\kappa(L(G)) \geq 3$ and $L(G)$ does not have an independent 3-vertex cut. Then
  (i) If G is a DCT graph, then $L(G)$ is hamiltonian.
  (ii) If G is a P3D graph, then $L(G)$ is hamiltonian.

4-connected line graphs: New Results

- **Theorem** Suppose $\kappa(L(G)) \geq 3$ and $L(G)$ does not have an independent 3-vertex cut. Then
  (i) If G is a DCT graph, then $L(G)$ is hamiltonian.
  (ii) If G is a P3D graph, then $L(G)$ is hamiltonian.
- **Corollary**:
  (i) Every 4-connected line graph of a DCT graph is hamiltonian.
  (ii) Every 4-connected line graph of a P3D graph is hamiltonian.

Thank You