Math 17 - Review

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Review for Chapter 12

1. Given a parametric planar curve $x = f(t)$, $y = g(t)$, where $a \leq t \leq b$, how to eliminate the parameter? (Use substitutions, or use trigonometry identities, *etc*).

How to parameterize a curve $f(x, y) = 0$? (For polar form $r = f(\theta)$, one can set $x(t) = f(t)\cos t$ and $y(t) = f(t)\sin t$; for the special form $y = f(x)$, one can set $x(t) = t$ and $y(t) = f(t)$.)

2. Differentiation of parametric curves and the slope.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \ \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}.$$

For polar coordinates,

$$\frac{dy}{dx} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}, \ \cot\phi = \frac{1}{r}\frac{dr}{d\theta},$$

where ϕ denotes the angle between the tangent line at P and the radius OP from the origin.

3. Review of integration in Chapter 6 and how to apply the formulas to parametric curves.

Area under curve:	$A = \int y dx$
Vol. of rev. around x-axis:	$V_x = \int \pi y^2 dx$
Vol. of rev. around y-axis:	$V_y = \int \pi x^2 dy$
Arc length:	$s = \int ds = \int \sqrt{(dx)^2 + (dy)^2}$
Area of surf. of rev. around x-axis:	$A_x = \int 2\pi\|y\| ds$
Area of surf. of rev. around y-axis:	$A_y = \int 2\pi\|x\| ds$

4. Operation of vectors.

The addition and the multiplication of vectors behave pretty much the same way as the operations of numbers with some exceptions. The following are some reminders:

(i) The dot product of two vectors outputs a number.

(ii) The scalar product and the vector product output vectors.

(iii) The vector product of two vectors is not commutative.

5. Some important facts:

(i) If $\vec{a} \neq \vec{0}$, then $\vec{u} = \vec{a}/|\vec{a}|$ is the unit vector that has the same direction of $\vec{a}$;

(ii) Perpendicular test: $\vec{a}\vec{b} = 0$ iff $\vec{a}$ and $\vec{b}$ are perpendicular;
(iii) Parallel test: $\vec{a} \times \vec{b} = \vec{0}$ iff $\vec{a}$ and $\vec{b}$ are parallel.

6. Vector functions: The operations on vector functions can be done componentwise.

(Application to motions) With position vector $\vec{r}(t)$, velocity $\vec{v}(t)$ and acceleration $\vec{a}(t)$, one has $\vec{r}' = \vec{v}, \vec{v}' = \vec{a}$.

Review for Chapter 13

1. The distance formula and its relation to dot products.
If $\vec{r}$ has its head in (x_1, y_1, z_1) and tail at (x_2, y_2, z_2), then $\vec{r} =< x_1 - x_2, y_1 - y_2, z_1 - z_2 >$, and

$$|\vec{r}|^2 = \vec{r} \bullet \vec{r} = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2.$$

2. Important facts: (the angle θ below is the angle between the two vectors $\vec{a}$ and $\vec{b}$)
(i) $\vec{a} \bullet \vec{b} = |\vec{a}||\vec{b}| \cos\theta$, and so $\vec{a}$ and $\vec{b}$ are perpendicular if and only if $\vec{a} \bullet \vec{b} = 0$.
(ii) $|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 \sin^2\theta$, and so $\vec{a}$ and $\vec{b}$ are parallel to each other if and only if $\vec{a} \times \vec{b} = \vec{0}$, the zero vector.
(iii) $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.

3. Direction angles and numbers.
Let $\vec{r} =< x, y, z >$. Let the angles between $\vec{r}$ and the x, y and z axes be α, β and γ, respectively. Then these angles are the direction angles of $\vec{r}$ and the direction numbers are

$$\cos\alpha = \frac{\vec{r} \bullet \vec{i}}{|\vec{r}|}, \ \cos\beta = \frac{\vec{r} \bullet \vec{j}}{|\vec{r}|}, \ \cos\gamma = \frac{\vec{r} \bullet \vec{k}}{|\vec{r}|}.$$

4. The component of $\vec{a}$ along $\vec{b}$ is

$$\text{Comp}_{\vec{b}}\vec{a} = \frac{\vec{a} \bullet \vec{b}}{|\vec{b}|}.$$

5. Let L be a line in space that is parallel to $\vec{n} =< a, b, c >$ and passes through (x_0, y_0, z_0). Then the parametric equation of L is

$$r(\vec{t}) =< at + x_0, bt + y_0, ct + z_0 >,$$

and the symmetric equations of L is

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

6. The plane passes (x_0, y_0, z_0) with normal vector $\vec{n} =< a, b, c >$ has equation

$$< a, b, c > \bullet < x - x_0, y - y_0, z - z_0 >= 0.$$

The angle between two planes is the angle between the two normal vectors.

7. Vector functions and their limits, derivatives, and antiderivatives.

One key thing to remember: do it componentwise. Note that the product rules for derivatives are very similar to the product rule for ordinary functions, with exception in its vector product form.

8. Given $r(\vec{t})$, how do you find the velocity, the speed, the acceleration, the unit tangent vector, the principal normal vector, the tangent component and the normal component of the acceleration, and the curvature.

9. Given the velocity and the acceleration together with some initial conditions, how do you find the position vector $r(\vec{t})$?

10. Given the equation of a surface and the cutting planes, how do you describe the traces?

11. Given the equation of a plane curve C and an axis L, how do you find an equation of the surface generated by revolving C about L?

12. The Cylindrical and the spherical coordinates.

The formulae about cylindrical coordinates:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \quad \text{and } x^2 + y^2 = r^2,\ \tan\theta = \frac{y}{x}.$$

The formulae about spherical coordinates:

$$\begin{cases} x = \rho\cos\theta\sin\phi \\ y = \rho\sin\theta\sin\phi \\ z = \rho\cos\phi \end{cases} \quad \text{and } x^2 + y^2 + z^2 = \rho^2.$$

Review for Chapter 14

1. Limits of multivariable functions. All the limit laws we learnt in MATH 15 remain valid and the techniques are similar (substitution for continuous functions, canceling zero-factors in both the numerator and the denominator, making use of a known limit, etc).

2. How to show nonexistence of some limits? (Use different paths to approach the limit.)

3. How to find partial derivatives? (Regard it as a derivative and use everything valid for derivatives.)

4. How to find tangent objects? (For surface with equation $z = f(x, y)$, use normal vector $\vec{n}(x, y) =< f_x(x, y), f_y(x, y), -1 >$. For surface with equation $F(x, y, z) = 0$, then the gradient is a normal. **Important: You CANNOT use the vector functions $\vec{n}(x, y)$ or $\nabla F(x, y, z)$ as normals. You HAVE TO EVALUATE them at a point in the tangent object**.

5. How to find extrema of a function with a bounded domain region? (Step 1: Find critical points INSIDE the region. Step 2: Find extrema on each curve of the boundary of the region by Math 15 techniques. Step 3: By comparison to pick up the absolute extrema).

6. How to classify the critical points (if they are local extrema and what kind)? (Use $\triangle(x, y) = f_{xx} f_{yy} - (f_{xy})^2$.)

7. Differentials and its applications.

$$df(x, y, z) = \nabla f(x, y, z) \bullet < dx, dy, dz > .$$

One can use df as an approximation to $f(x + dx, y + dy, z + dz) - f(x, y, z)$.

8. Chain rules: (Go check them on pages 736 and 739).

9. Implicit partial differentiation. (This time the equation defines one variable (z, say) as a function of the other variables (x, y, say). Hence when you partially differentiate both sides of the equation with respect to x or y, you cannot regard z as a constant.)

10. Application of implicit partial differentiation: when a surface is given by an equation, one can use implicit partial differentiation to find the tangent plane by using $\vec{n} =< z_x, z_y, -1 >$ as a normal.

11. The gradient and the directional derivatives:

$$\nabla f(x, y, z) =< f_x, f_y, f_z > \text{ and } D_{\vec{n}} f(x, y, z) = \nabla f(x, y, z) \bullet \vec{n}, \text{ if } |\vec{n}| = 1.$$

Review for Chapter 15

1. Evaluation of double integrals: (x, y coordinates)

$$\int\int_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx, \text{ if } R \text{ is } a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x).$$

$$\int\int_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dydx, \text{ if } R \text{ is } c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y).$$

When you can evaluate the integral by either way, you may want to choose a simpler way.

2. Evaluation of double integrals: (Polar coordinates)

$$\int\int_R f(x,y)dA = \int_a^b \int_{g_1(r)}^{g_2(r)} f(r\cos\theta, r\sin\theta)rd\theta dr, \text{ if } R \text{ is } a \leq r \leq b,\ g_1(r) \leq \theta \leq g_2(r).$$

$$\int\int_R f(x,y)dA = \int_a^b \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)rdrd\theta, \text{ if } R \text{ is } a \leq \theta \leq b,\ h_1(\theta) \leq r \leq h_2(\theta).$$

A useful fact: $dA = rdrd\theta = dxdy$.

Some applications of double integrals.

2a. Area of region R is $\int\int_R dA$.

2b. The volume between $z = f(x,y)$ and $z = g(x,y)$ when (x,y) are in R is $\int\int_R (f(x,y) - g(x,y))dA$.

2c. (Applications in Physics) Let $\rho(x,y)$ be the density of lamina whose region is R. Then the mass and the centroid of the lamina $(\overline{x}, \overline{y})$ is

$$\begin{aligned} \text{mass } m = &\ \int\int_R \rho(x,y)dA \\ \overline{x} = &\ \frac{1}{m}\int\int_R x\rho(x,y)dA \\ \overline{y} = &\ \frac{1}{m}\int\int_R y\rho(x,y)dA \end{aligned}$$

3. Evaluation of triple integrals: (rectangular coordinates) The main idea is the same as the cross section idea. The following gives a way to reduce a triple integral to a double integral.

$$\int\int\int_T f(x,y,z)dV = \int\int_R (\int_{h_2(x,y)}^{h_1(x,y)} f(x,y,z)dz)dA \text{ if } T \text{ is } h_1(x,y) \leq z \leq h_2(x,y),\ (x,y) \text{ in } R.$$

4. Evaluation of triple integrals: (cylindrical coordinates and spherical coordinates) The main relationship among rectangular, cylindrical and spherical coordinates is

$$dV = dxdydz = rdrd\theta dz = \rho^2 \sin\phi d\phi d\theta d\rho.$$

5. Some applications of triple integrals.

5a. The volume of the solid T is $\int\int\int_T dV$.

5b. The mass of a solid T with density $\rho(x,y,z)$ is $\int\int\int_T \rho(x,y,z)dV$.
5c. The centroid of T with density $\rho(x,y,z)$ is $(\overline{x},\overline{y},\overline{z})$, where

$$\overline{x}=\int\int\int_T x\rho(x,y,z)dV, \overline{y}=\int\int\int_T y\rho(x,y,z)dV, \overline{z}=\int\int\int_T z\rho(x,y,z)dV$$

6. Use double integral to find the surface area. If a surface is given by $\mathbf{r}(u,v) =< x(u,v), y(u,v), z(u,v) >$, where (u,v) is in R, then the area of the surface is

$$\int\int_R |\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}|dA.$$

Review for Chapter 16

1. A vector field is a vector function $\mathbf{F} =< P(x,y,z), Q(x,y,z), R(x,y,z) >$ or $\mathbf{F} = < P(x,y), Q(x,y) >$.

2. The gradient operator $\nabla =< \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} >$ is a linear operator.

3. div $\mathbf{F} = \nabla \bullet \mathbf{F}$.

4. curl $\mathbf{F} = \nabla \times \mathbf{F}$.

5. Line integrals: If C is a curve with parametric equations $< x(t), y(t), z(t) >$ with $a \le t \le b$, then

$$\int_C f(x,y,z)dx = \int_a^b f(x(t),y(t),z(t))x'(t)dt.$$

$$\int_C f(x,y,z)dy = \int_a^b f(x(t),y(t),z(t))y'(t)dt.$$

$$\int_C f(x,y,z)dz = \int_a^b f(x(t),y(t),z(t))z'(t)dt.$$

$$\int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}dt.$$

6. Conservative fields: A vector field $\mathbf{F} = < P, Q >$ is conservative if and only if $P_y = Q_x$. In this case, one can find its potential function f such that $\nabla f = \mathbf{F}$ by the following steps: (1) Find $f(x,y) = \int P dx + \phi(y)$.
(2) Set $Q = f_y$ to get a differential equation of $\phi'(y)$.
(3) Solve the equation to find $\phi(y)$ and thereby getting f.

7. Important fact: If $\mathbf{F} = < P, Q, R >$ is a conservative field, and if $\mathbf{T}$ is the unit tangent vector of the curve C, then $\int_C \mathbf{F} \bullet \mathbf{T} ds$ is independent of path. (Therefore, you can make

use of it to simplify the integral).

8. Green's Theorem: Let C be the closed curve bounding the region R and suppose that P, Q are both continuous and have continuous first order of derivatives, then

$$\oint_C P dx + Q dy = \int\int_R (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dA.$$