Effects of Varying Nonlinearity and Their Singular Perturbation Flavour

U. Elias

Department of Mathematics, Technion-I.I.T., Haifa 32000, Israel
E-mail: elias@tx.technion.ac.il

and

H. Gingold

Department of Mathematics, West Virginia University, Morgantown, West Virginia 26506
E-mail: gingold@math.wvu.edu

Autonomous differential equations $y'' + f(y, p) = 0$ whose nonlinearity varies with a parameter p are studied. As a prototype, one may think of $y'' + f_0(y) + |y|^{p-1}g(y) = 0$. We discuss periodic solutions with initial values taken from various domains and their different types of convergence as $p \to \infty$. Equations $y'' - f(y, p) = 0$, $yf(y, p) > 0$ are similarly discussed, with the "period" of a solution replaced by its "maximal interval of existence." The study shows a natural link to singularly perturbed problems. It turns out that the family of ODEs under consideration are essentially a family of singularly perturbed problems. Solutions may develop "kinks" and higher order derivatives of solutions possess "boundary layers," namely sets of non-uniform convergence. Similarities and differences between this family and the more common singularly perturbed problems which abound in the literature emerge. © 2000 Academic Press

Key Words: varying nonlinearity; convergence of solutions; delta method; boundary layer; interval of existence; singular perturbations.

1. INTRODUCTION

In this paper we focus on the autonomous differential equations

$$y'' + f(y, p) = 0,$$
$$y(0) = \alpha, \qquad y'(0) = \beta, \tag{1}$$

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whose nonlinearity varies with a parameter p. The equation which moti-
vates our model is

$$y'' + f_0(y) + |y|^{p-1}g(y) = 0, \tag{2}$$

$$y(0) = \alpha, \qquad y'(0) = \beta, \tag{3}$$

where the nonlinearity varies with p due to the power y^{p-1}.

Differential equations with varying nonlinearities occur in various
branches of mathematical physics. An excellent source for these can be
found in recent papers [2, 3]. See also [1, Chap. 7; 5, Chaps 4, 7; 11]. The
Lane–Emden equation, $y'' + x^{-2}y' + y^p = 0$, is an example of a differen-
tial equation with an unrestricted nonlinearity generated by the physical
parameter p. It is of fundamental importance in astrophysics [5, Chap. 4].
The quantity $(p + 1)/p$ occurs in the assumed relation between the
pressure and density in a star configuration which is in equilibrium. The
index p is then an index of polytropy. See the many related references in
[5, Chap. 4]. The Emden–Fowler equation $y'' + y^p = 0$ is yet another
example. For the original discussion see [8], while for various mathematical
generalizations see [12, Chap. V; 14]. The equation $y'' + (\frac{2}{p-1})^2 y -
y|y|^{p-1} = 0$ occurs in the study of limiting behaviour of radially symmetric
solutions of $\Delta u = u|u|^{p-1}$ (see [6]). The last two ordinary differential
equations will turn out to be special cases of the more general differential
equations analyzed in this work. As a matter of fact our study with varying
and growing nonlinearities contains as a special case the equations of
motion in the setting of classical mechanics of the Hamiltonian $H = P^2 +
Q^{2p}$ which is studied in [4] in the setting of quantum mechanics.

Varying nonlinearities arise in a natural manner in the "delta method"
[2]. This technique is essentially a perturbation method which relies upon
the introduction of a parameter δ which quantifies in a certain manner
the nonlinearity in a given equation and expresses solutions as power
series of δ. The fact that in the δ-method there is a need to utilize
acceleration methods indicates that an asymptotic approach in which the
parameter p of nonlinearity tends to infinity could yield useful results. Our
analysis supplements the δ-method by studying the asymptotic-singularly
perturbed nature of solutions as $p \to \infty$.

These papers lead us in a natural way to ask among other questions the
following. Denote by $y_p(t)$ the solution of (1). Could we (should we) expect
the existence of $\lim_{p \to \infty} y_p(t)$, and if so, what are the qualitative properties
of $y_p(t)$ for large values of p? One should not be surprised at an attempt
to obtain qualitative information and robust approximations for large
families of solutions of differential equations depending on a finite range
of a parameter, like p, by taking the singular limit $p \to \infty$. This is a well

accepted approach in singularly perturbed problems which yield powerful results. Compare, e.g., with [13]. One of the many striking demonstrations of this is the excellent approximations obtained in [4] for moderate values of the parameter. Our own numerical calculations also support this approach. A rapid rate of convergence could make this conclusion valuable not only to equations where $p \to \infty$ but also to those equations with a finite variation of p. Although studies of nonlinear problems are abundant in the literature, we could not find works which deal with the variation of nonlinearities where a parameter measuring nonlinearity grows to infinity.

Let us consider first the case that the solutions of (1) are all periodic. It will turn out that under suitable assumptions, there are in (1) disjoint regions of the initial values in the phase plane which have dramatic effects on the evolution of the solution $y_p(t)$ of (1) as $p \to \infty$. For (α, β) in a certain open neighborhood of $(0, 0)$, we obtain simple results, namely that the solution $y_p(t)$ converges uniformly as $p \to \infty$, as well as its derivatives. For (α, β) on the boundary of this region non-uniform convergence of higher derivatives occurs as in boundary layer problems. Thus, it provides a source of intriguing behaviour. In the second region the solutions cease to converge uniformly. Finally, in the third region the solutions fail to converge at all. This is in analogy to the fact that $\lim_{p \to \infty} y^p$ is drastically different according to whether $|y|$ is less than, bigger than, or equal to 1.

Since the solutions of Eq. (1) are periodic, we study the dependence of the periods on the initial values (α, β) and the parameter p. As $p \to \infty$, we show that the periods converge to strictly different limits, depending on the location of the point (α, β) in the various regions.

On the other hand, for the equation

$$
\begin{aligned}
& y'' - f(y, p) = 0, \\
& y(0) = \alpha, \qquad y'(0) = \beta,
\end{aligned}
\tag{4}
$$

none of the solutions is periodic under our assumptions and "period" is naturally replaced by another concept. We call $\tau_+ > 0$ the *forward escape time* if $[0, \tau_+)$ is the maximal interval of existence of the solution of (4) for $t > 0$. There is a similar notion of the *backward escape time* for $t < 0$, which is denoted by τ_-. The length of the maximal interval of existence is actually $\tau = \tau_+ + |\tau_-|$. If the solution of (4) exists only on a finite interval, a natural question to study is the dependence of the maximal interval of existence τ on the initial value (α, β) and on the parameter p, and its convergence as $p \to \infty$.

In order to get the meaning of the rich phenomena awaiting to be analyzed, consider the simpler setting of the first order Bernoulli equa-

tions

$$y' = ay + by^{p+1}, \qquad y(0) = y_0 > 0,$$

where p generates a varying nonlinearity and a, b are constants. The transformation $y = w^{-1/p}$ takes it into the linear "singularly perturbed" problem

$$-\frac{1}{p}w' = aw + b, \qquad w(0) = y_0^{-p}.$$

For $a \neq 0$, the solution of the Bernoulli equation is

$$y_p(t) = \left(y_0^{-p} e^{-pat} + \frac{b}{a} (e^{-pat} - 1) \right)^{-1/p}.$$

Let $a = i\beta$, β real. If y_0, b are real, then the complex valued solution y_p is defined for every real t and is periodic with period $T = 2\pi/|a|p$, which shrinks to zero as $p \to \infty$. If $a, b > 0$ and $y_0 > 1$, then the same holds for the maximal interval of existence τ of the solution.

If $a = 0$, the solution is given by

$$y_p(t) = (y_0^{-p} - pbt)^{-1/p}.$$

Let, for example, $b < 0$. If $y_0 > 1$, we have two boundary layers, namely subintervals of non-uniform convergence of y_p, at $t = 0$ and $t = \infty$, while y_p' does converge uniformly on every $[t_0, \infty)$, $t_0 > 0$. If $0 < y_0 \leq 1$, we have only one boundary layer at $t = \infty$. The backward escape time of the solution is $\tau_- = y_0^{-p}/pb < 0$. Notice that $\tau_- \to -\infty$ as $p \to \infty$ if $0 < y_0 < 1$, but $\tau_- \to 0$ as $p \to \infty$ for $y_0 \geq 1$.

Some of the features mentioned above reoccur in the analysis of the second order equations below. In [1, Chap. 7] we find a lengthy discussion of the nonautonomous ODEs

$$y'' + t^\sigma y^p = 0.$$

An additional glimpse into the nature of these singularly perturbed problems can be obtained by considering the special family of solutions in "closed form" $y = ct^w$ with $w = -(\sigma + 2)/(p - 1)$, $c = [\pm(\sigma + 2)(\sigma + p + 1)/(p - 1)]^{1/p-1}$. It is readily seen that on every fixed compact interval $t \in [\alpha, \beta]$, $\alpha > 0$, we have $\lim_{p \to \infty} y = 1$. Indeed, analogous conclusions are shown to hold for certain solutions of (1). This could lead to useful simplifications of solutions of important families of problems of mathematical physics. There are, however, additional difficulties associ-

ated with our singularly perturbed family of ODEs with varying nonlineari-
ties which make it all the more challenging. (These are also manifested in
[4] although the problems there are linear.) First, there is not always an
easily identifiable "reduced equation" for which an "outer solution" can
be calculated. Second, an outer solution cannot be expanded in an asymp-
totic series of powers of $1/p$. This raises a difficult question of what
alternative expansion could there be that could replace series expansions
in powers of $\varepsilon = 1/p$. For some thoughts in that direction see [10]. Third,
"boundary layers" may not show up at all in the solutions or their higher
derivatives on a finite interval. They need not show up in the solutions but
may become apparent in their higher derivatives or, of course, boundary
layers could develop in the solutions and their higher derivatives. It is easy
to show that solutions to the examples given above, e.g., the Bernoulli and
the second order equations, when considered as analytic functions of the
parameter p, possess an essential singularity at $p = \infty$. It is also possible to
show that if $f(y, p)$ in (4) is an analytic function of p, then the solutions of
(4) possess an essential singularity at $p = \infty$. It goes without saying that
the rich phenomena that characterize singularly perturbed problems in
numerous instances in mathematical physics are a direct consequence of
the analytic dependence of solutions on a physical parameter varying in a
neighbourhood of an essential singularity. The family of interesting prob-
lems, introduced by Bender *et al*. in recent years, provides a new source of
singularly perturbed nonlinear problems of a special flavour. Some of this
special flavour is presented here.

The study of specialized nonlinear problems with varying nonlinearities
is advocated in [9], where we find a study of ellipses with "growing
nonlinearity." It is noteworthy that even though the problems in [9] are
formulated in a geometrical algebraic setting, there are two underlying
similarities to the phenomena analyzed in this paper. Namely, non-uniform
convergence and portions of limiting trajectories turn out to be straight
line segments. It is now possible to put forward the following simplified
principle for equations with varying nonlinearities. Let $L_1(y, t) +
y^p L_2(y, t) = 0$ be an equation for an unknown $y(t)$ in a domain $t \in D$. Let
$L_1(y, t), L_2(y, t)$ be operators such that $0 < m \le |L_1(y, t)/L_2(y, t)| \le M$,
for certain constants m, M. Then, as $p \to \infty$, we have $|y(t)| \to 1$ for each
$t \in S$ and some $S \subset D$. This simplified argument is supported by [9] and by
this work.

2. PERIODIC SOLUTIONS IN THE PHASE PLANE

We begin to study the autonomous differential equation (1). We do not
pretend to define which differential equation is "more nonlinear" than the

other as p varies. Instead, the following conventions will hold, motivated by Eq. (2), (3):

(a) For every fixed p, $f(y, p)$ is an odd, strictly increasing continuous function on $(-\infty, \infty)$.

The growing of nonlinearity is expressed by the following assumption:

(b) For every $y \in (0, a)$, $f(y, p)$ decreases (and on $(-a, 0)$ it increases) to a limit $f_0(y)$ as $p \to \infty$, where $f_0(y)$ is continuous on $(-a, a)$.

At $y = \pm a$, $f(\pm a, p)$ converges to finite limits which will be denoted, respectively, by $f_0(\pm a)$.

For $y > a$ and $y < -a$, $f(y, p)$ diverges to $\pm\infty$, respectively.

It follows that $f(y, p) \to f_0(y)$ uniformly for $|y| < a - \epsilon$ and $|f(y, p)| \to \infty$ uniformly for $|y| \geq a + \epsilon$ for every $\epsilon > 0$. Since we make no specific assumptions about $f_0(\pm a)$, $f_0(y)$ *may* have a discontinuous jump at $y = \pm a$, $\lim_{y \to a^-} f_0(y) \leq f_0(a)$.

Let us introduce the notations

$$F(y, p) = \int_0^y f(s, p)\, ds \qquad \text{for all } y,$$

$$F_0(y) = \int_0^y f_0(s)\, ds \qquad \text{for } y \in [-a, a].$$

Since $f_0(y)$ is continuous on $(-a, a)$ and bounded on $[-a, a]$, it follows under our assumptions that $\lim_{p \to \infty} F(y, p) = F_0(y)$ uniformly on $[-a, a]$. It is also clear that $F(y, p) \to +\infty$ uniformly for $|y| \geq a + \epsilon$.

In the study of Eq. (1) we keep in mind the equivalent Hamiltonian system

$$y' = z, \qquad z' = f(y, p),$$
$$y(0) = \alpha, \qquad z(0) = \beta.$$

The geometry of the integral curves in the (y, z) plane is trivial. As usual, we multiply (1) by y' and integrate to get the first integral

$$\tfrac{1}{2}y'^2 + F(y, p) = \tfrac{1}{2}\beta^2 + F(\alpha, p). \tag{5}$$

Since $F(y, p)$ is an even, nonnegative, convex function, these are convex, closed, and symmetric curves in the (y, y') plane. Consequently, all the solutions of (1) are periodic.

Since the topology of the trajectories is not interesting, we shall rather study their quantitative behaviour, their periods, and their dependence on

the initial values (α, β) and the varying parameter p. It follows by (5) that

$$\frac{dy}{dt} = \pm\sqrt{\beta^2 + 2F(\alpha, p) - 2F(y, p)}\,,$$

where the sign $\pm$ is determined in each quarter of the (y, y') plane. The period of the solution, $T = \int dt = \int (dt/dy)\, dy$, with the integration taken once around the closed trajectory, is given due to the twofold symmetry by

$$T_{\alpha, \beta}(p) = 4\int_0^{L(p)} \left(\beta^2 + 2F(\alpha, p) - 2F(y, p)\right)^{-1/2} dy, \qquad (6)$$

where $L = L(p)$ is the intersection point of the curve with the horizontal axis $y' = 0$. L is the unique positive solution of

$$F(y, p) = \tfrac{1}{2}\beta^2 + F(\alpha, p). \qquad (7)$$

(i) We begin our discussion in the region

$$\mathbf{A} = \left\{(y, z)\,|\, F_0(y) + \tfrac{1}{2}z^2 < F_0(a)\right\}$$

of the phase plane. This is an open, convex, symmetric region, centered at $(0, 0)$. Since F_0 is positive and increases on $(0, \infty)$, $\mathbf{A}$ is contained in the box $|y| < a$, $|z| < (2F_0(a))^{1/2}$. (If $f_0(y) \equiv 0$, $\mathbf{A}$ is empty.)

Let $(\alpha, \beta) \in \mathbf{A}$, that is, $\tfrac{1}{2}\beta^2 + F_0(\alpha) < F_0(a)$ and $|\alpha| < a$. Hence, for sufficiently large values of p,

$$\tfrac{1}{2}\beta^2 + F(\alpha, p) = \tfrac{1}{2}\beta^2 + F_0(\alpha) + o(1) < F_0(a)$$

and by (5)

$$\tfrac{1}{2}y'^2 + F(y, p) < F_0(a).$$

Thus the whole trajectory $(y_p(t), y_p'(t))$ through (α, β) lies in $\mathbf{A}$ for sufficiently large p. Since the closed trajectory is compact, we have in fact

$$\tfrac{1}{2}y'^2 + F(y, p) \leq F_0(a) - \delta, \qquad \delta > 0.$$

It follows that every trajectory which originates in a compact subset of the open region $\mathbf{A}$ never leaves $\mathbf{A}$ provided that p is sufficiently large.

Since $f(y, p) \to f_0(y)$ uniformly on every $[-a + \epsilon, a - \epsilon]$, this is the case along the whole trajectory through (α, β). Therefore, by standard results, the solution y_p of (1) converges uniformly, as well as y_p', y_p'', to the

corresponding solution of the limit equation

$$y'' + f_0(y) = 0, \qquad -a \le y \le a,$$
$$y(0) = \alpha, \qquad y'(0) = \beta. \tag{8}$$

Since also $F(y, p) \to F_0(y)$ uniformly, also the period (6) of the solution of (1) converges to that of (8), namely

$$T_{\alpha, \beta} = 2 \int_{-L}^{L} \left(\beta^2 + 2F_0(\alpha) - 2F_0(y) \right)^{-1/2} dy + o(1),$$
$$L = F_0^{-1} \left(F_0(\alpha) + \tfrac{1}{2}\beta^2 \right). \tag{9}$$

This is the simplest behaviour which Eq. (1) demonstrates. See Fig. 1.

(ii)   Let (α, β) be on the boundary of **A** but not one of the points $(\pm a, 0)$. In order to determine the location of the corresponding trajectory, let us estimate its intersections with the axis $y' = 0$. The intersection points are $(\pm L, 0)$, where L is the solution of (7). Since on ∂**A**, $\tfrac{1}{2}\beta^2 +$
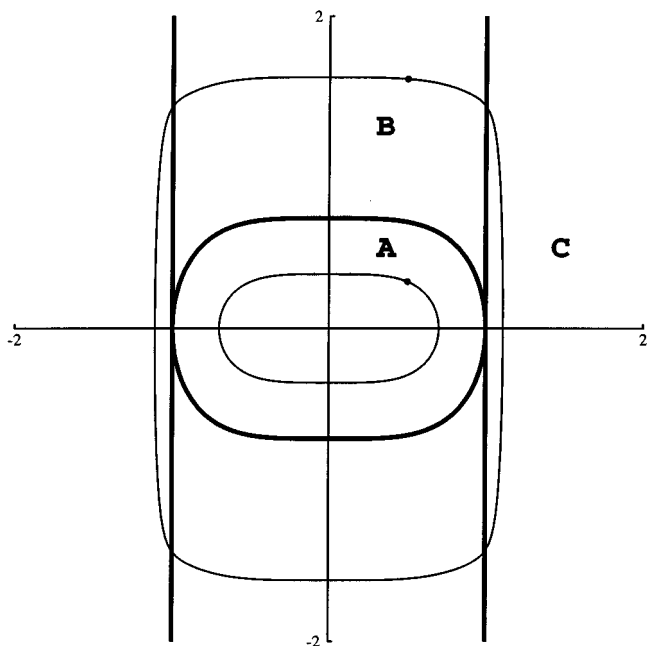

FIG. 1.   Trajectories of the equation $y'' + y^3 + y^p = 0$ with $p = 31$, through $(0.5, 0.3) \in$ **A**, $(0.5, 1.6) \in$ **B**.

$F_0(\alpha) = F_0(a)$, we rewrite (7) as

$$F(y, p) - F(\alpha, p) = F_0(a) - F_0(\alpha). \tag{10}$$

Here $|\alpha| < a$, since $(\alpha, \beta) \neq (\pm a, 0)$ according to our assumption. Due to the symmetry, let us assume that $0 \leq \alpha < a$. The left hand side of (10) is a strictly increasing function of y and at the point $y = \alpha$ it vanishes while the right hand side is positive. On the other hand, at $y = a$ the left hand side of (10), $F(a, p) - F(\alpha, p) = \int_\alpha^a f(y, p)\, dy$, is bigger than the right side, $F_0(a) - F_0(\alpha) = \int_\alpha^a f_0(y)\, dy$, since $f(y, p) \searrow f_0(y)$ on $(0, a)$. Therefore (10) has a unique positive solution $y = L(p)$ and it lies in (α, a). It is also clear by (10) that as $p \to \infty$, the intersection point cannot be bounded away from $y = a$ and it is $y = L(p) = a - o(1)$.

By convexity and symmetry, the whole closed trajectory through $(\alpha, \beta) \in \partial \mathbf{A}$ lies in $|y| < a - o(1)$. The same trajectory through $(\alpha, \beta) \in \partial \mathbf{A}$ intersects the axis $y = 0$ at $|y'| = (2(F_0(a) + F(\alpha, p) - F_0(\alpha)))^{1/2} = (2F_0(a))^{1/2} + o(1)$ which is outside of $\mathbf{A}$.

Since $F(y, p) \to F_0(y)$ uniformly on $[-a, a]$, the trajectory (5) of Eq. (1) and the trajectory

$$\tfrac{1}{2} y'^2 + F_0(y) = \tfrac{1}{2}\beta^2 + F_0(\alpha)$$

of the limit equation (8) are arbitrarily close for large values of p. This verifies the convergence of the derivatives $y_p(t)$, $y_p'(t)$ of (1) to those of (8). On the other hand, by our assumptions f_0 may be discontinuous at $y = a$, and the convergence of $f(y, p)$ to $f_0(y)$ may be non-uniform near our trajectory. If this is the case, it will be reflected in the behaviour of the second derivative, $y_p''(t)$, which may converge to a discontinuous limit. For Eq. (2), (3) we shall make a detailed statement about this question later on and show that $\lim y_p''(t)$ is actually discontinuous.

The trajectory which emerges from $(\alpha, \beta) = (\pm a, 0)$ stays in the box $|y| \leq a$, $|z| < (2F_0(a))^{1/2} + o(1)$.

    (iii) Region $\mathbf{B}$ is defined as the part of the strip $|y| \leq a$ outside the closure of $\mathbf{A}$,

$$\mathbf{B} = \left\{ (y, z) \mid |y| \leq a \text{ and } F_0(y) + \tfrac{1}{2} z^2 > F_0(a) \right\}.$$

For a solution through $(\alpha, \beta) \in \mathbf{B}$ we have

$$\tfrac{1}{2}\beta^2 + F(\alpha, p) > F_0(a)$$

by the definition of $\mathbf{B}$; on the other hand $\tfrac{1}{2}\beta^2 + F(\alpha, p) \leq \tfrac{1}{2}\beta^2 + F(a, p) \leq \tfrac{1}{2}\beta^2 + F_0(a) + o(1)$, so the trajectory through (α, β) is located in the ring

$$\mathbf{R} = \left\{ (y, z) \mid F_0(a) < \tfrac{1}{2} z^2 + F(y, p) \leq F_0(a) + \tfrac{1}{2}\beta^2 + o(1) \right\}$$

which surrounds region **A**. By the Poincare–Bendixson theorem the trajectory cannot stay forever in **R** ∩ **B** and must pass through the half planes $|y| > a$.

In **B**, $F(y, p) \to F_0(y)$ uniformly, thus the part of the trajectory which lies in **B** is close to a trajectory of (8). On $|y| \geq a + \epsilon$, $F(y, p) \to \infty$, therefore it follows from the definition of **R** that the parts of the trajectory which lie outside **B** must be located in fact in $a < |y| \leq a + o(1)$. These are two symmetric "steep" curves which connect the two segments of the trajectory which lie in **B**. See Fig. 1.

After the description of the geometric location of the curves (y_p, y_p'), we turn to its dynamic evolution. The time that takes a point $(y_p(t), y_p'(t))$ to pass through a subarc Γ of the trajectory is $\int_\Gamma dt = \int_\Gamma (dt/dy)\, dy$. In the present case our solution intersects **B** in two arcs during each period and the time it stays there during each crossing is

$$\int_{-a}^{a} \left(\beta^2 + 2F(\alpha, p) - 2F(y, p) \right)^{-1/2} dy$$

$$\to \int_{-a}^{a} \left(\beta^2 + 2F_0(\alpha) - 2F_0(y) \right)^{-1/2} dy.$$

The time it stays on each of the steep curves in $|y| \geq a$ is

$$2\int_{a}^{L} \left(2(\lambda - F(y, p)) \right)^{-1/2} dy,$$

where $\lambda = \lambda(p) = \frac{1}{2}\beta^2 + F(\alpha, p)$ and L is the intersection of the trajectory with the horizontal axis. As mentioned above, L is the solution of (7) and $a < L < a + o(1)$. Thus, L is a zero of $h(y) \equiv \lambda - F(y, p)$ on the right hand side of a. Now, $h'(y) = -f(y, p) < 0$, $h''(y) = -f'(y, p) < 0$, so the concave decreasing $h(y)$ is bounded from below by

$$h(y) \geq (\lambda/L)(L - y) \qquad \text{on } [0, L] \tag{11}$$

and $\lambda(p) > \frac{1}{2}\beta^2$. Thus

$$\int_{a}^{L} \left(2(\lambda - F(y, p)) \right)^{-1/2} dy \leq \int_{a}^{L} \left((2\lambda/L)(L - y) \right)^{-1/2} dy$$

$$= \lambda^{-1/2}\mathscr{O}\left((L - a)^{1/2} \right) = o(1)$$

as $p \to \infty$. That is, for large values of p, a solution through $(\alpha, \beta) \in$ **B** has a period

$$T_{\alpha, \beta} = 2\int_{-a}^{a} \left(\beta^2 + 2F_0(\alpha) - 2F_0(y) \right)^{-1/2} dy + o(1) \tag{12}$$

and it spends almost all of this period in **B**. Since the trajectory $(y_p(t), y_p'(t))$ crosses its arcs outside **B** in time $o(1)$, it is clear that $y_p''(t)$ cannot be bounded there as $p \to \infty$.

(iv)   The last region is

$$\mathbf{C} = \{(y, z) \mid |y| > a\}.$$

For $(\alpha, \beta) \in \mathbf{C}$, the intersection of the trajectory with $y' = 0$ is again the solution $y = L(p)$ of (7). Since (7) may be written as $\frac{1}{2}\beta^2 = F(y, p) - F(\alpha, p) = \int_\alpha^y f(s, p)\, ds$ and $f(y, p) \to \infty$ uniformly on $[\alpha, \infty)$, it is clear that $y = L$ cannot be bounded away from α and so $\alpha < L(p) < \alpha + o(1)$. Thus, for $(\alpha, \beta) \in \mathbf{C}$, $|y_p(t)| \le \alpha + o(1)$ for all t.

For every fixed y, $0 \le y < \alpha$, we have by (5) that $\frac{1}{2}y'^2 = \frac{1}{2}\beta^2 + F(\alpha, p) - F(y, p) = \int_y^\alpha f(s, p)\, ds \to \infty$ as $p \to \infty$, since $\alpha > a$. Therefore $|y_p'(t)| \to \infty$ whenever $|y_p(t)| \le \alpha - \epsilon$.

To estimate the period of the trajectory as $p \to \infty$, we use again the inequality (11). Thus, with $\lambda = \frac{1}{2}\beta^2 + F(\alpha, p)$,

$$T_{\alpha, \beta} = 4\int_0^L \left(2(\lambda - F(y, p))\right)^{-1/2} dy \le \int_0^L \left((2\lambda/L)(L - y)\right)^{-1/2} dy$$

$$= 8L/(2\lambda)^{1/2} \le 8(\alpha + o(1))/(2F(\alpha, p))^{1/2} \to 0.$$

The analysis up to this point may be summarized:

PROPOSITION.   (i) *For (α, β) in the interior of* **A**, *the solution y_p of Eq.* (1) *and its derivatives y_p', y_p'' converge, as $p \to \infty$, uniformly to the corresponding derivatives of the solution of the limit equation* (8). *The period $T_{\alpha, \beta}$ of y_p is given by* (9) *and it converges, as $p \to \infty$, to the period of* (8).

(ii)   *For $(\alpha, \beta) \in \partial\mathbf{A}$, y_p, y_p' converge to the solution of* (8) *and its derivative; however, y_p'' may approach a discontinuous function around the two extrema $y' = 0$.*

(iii)   *For large values of p, a solution through $(\alpha, \beta) \in \mathbf{B}$ spends most of its period* (12) *in* **B** *and only $o(1)$ in* **C**; *the restriction of the trajectory (y_p, y_p') to* **B** *approaches that of the limit equation* (8). *On its subarc in* **C**, *y_p, y_p' are bounded but y_p'' is unbounded as $p \to \infty$.*

(iv)   *For $(\alpha, \beta) \in \mathbf{C}$, y_p is bounded but y_p', y_p'' diverge to infinity almost everywhere as $p \to \infty$. The period of y_p converges to* 0.

3. AN EXAMPLE

In this section we discuss some specific features of Eq. (2), (3),

$$y'' + f_0(y) + |y|^{p-1}g(y) = 0,$$
$$y(0) = \alpha, \qquad y'(0) = \beta,$$

where $f_0(y), g(y)$ are odd, strictly increasing continuous functions on $(-\infty, \infty)$ and $p > 2$. This is Eq. (1) with $f(y, p) = f_0(y) + |y|^{p-1}g(y)$, $a = 1$,

$$F(y, p) = \int_0^y \left(f_0(s) + |s|^{p-1}g(s) \right) ds.$$

Let $F_0(y) = \int_0^y f_0(y)\, dy$ for $-1 \le y \le 1$ and define $G(y)$ by the mean value

$$G(y) = \frac{\int_0^y |y|^{p-1}g(y)\, dy}{\int_0^y |y|^{p-1}\, dy},$$

that is, $(|y|^p/p)G(y) = \int_0^y |y|^{p-1}g(y)\, dy$. Of course, G depends on p. According to this notation we can write $F(y, p)$ as

$$F(y, p) = F_0(y) + \frac{|y|^p}{p}G(y)$$

and the first integral (5) is

$$\frac{1}{2}y'^2 + F_0(y) + \frac{|y|^p}{p}G(y) = \frac{1}{2}\beta^2 + F_0(\alpha) + \frac{|\alpha|^p}{p}G(\alpha).$$

The following information about G will be needed:

LEMMA. *Let* $G(y) = \int_0^y s^{p-1}g(s)\, ds / \int_0^y s^{p-1}\, ds$. *If* $g, g' \ge 0$, *then* $G, G' \ge 0$ *and* $0 \le G(y) \le g(y)$ *for* $y \ge 0$.

Proof. By differentiation of $(y^p/p)G(y) = \int_0^y s^{p-1}g(s)\, ds$, one gets $y^{p-1}G(y) + (y^p/p)G'(y) = y^{p-1}g(y)$, that is,

$$y^{p-1}(g(y) - G(y)) = \frac{y^p}{p}G'(y). \tag{13}$$

On the other hand we integrate the same identity by parts,

$$\frac{y^p}{p}G(y) = \int_0^y s^{p-1}g(s) = \frac{y^p}{p}g(y) - \int_0^y \frac{s^p}{p}g'(s),$$

i.e.,

$$\frac{y^p}{p}(g(y) - G(y)) = \int_0^y \frac{s^p}{p}g'(s). \tag{14}$$

By comparing (13) and (14) we get

$$\frac{y^{p+1}}{p}G'(y) = \int_0^y \frac{s^p}{p}g'(s) \geq 0, \qquad y \geq 0,$$

and $G'(y) \geq 0$ follows by $g' \geq 0$. $G(y) \leq g(y)$ follows from (14).

Note that the last identity can be written also as

$$\frac{p+1}{p}G'(y) = \frac{\int_0^y s^p g'(s)}{\int_0^y s^p},$$

which may be generalized in various ways. ∎

According to the lemma, G, which is dependent on the parameter p, is nevertheless increasing and uniformly bounded by $g(y)$ which is independent of p.

Now we can add some details which are specific to Eq. (2), (3). Recall that for Eq. (2), (3) we have $a = 1$ and $\partial\mathbf{A}$ is the curve $F_0(y) + \frac{1}{2}z^2 = F_0(1)$. It was already seen that if (α, β) belongs to $\partial\mathbf{A}\backslash\{\pm a, 0)\}$, then the trajectory of (1) through this point intersects the axis $y' = 0$ of the phase plane at $(\pm y, 0)$ with $y = a - o(1)$ as $p \to \infty$. For Eq. (2), (3) the intersection point of the trajectory is the solution y of (10), which reads for Eq. (2), (3)

$$\left(F_0(y) + \frac{y^p}{p}G(y)\right) - \left(F_0(\alpha) + \frac{\alpha^p}{p}G(\alpha)\right) = F_0(1) - F_0(\alpha).$$

Assuming that F_0, G are smooth enough, we try $y = 1 - cp^{-1} + o(p^{-1})$, i.e.,

$$F_0\left(1 - \frac{c}{p} + \ldots\right) + \frac{1}{p}\left(1 - \frac{c}{p} + \ldots\right)^p G\left(1 - \frac{c}{p} + \cdots\right)$$

$$= F_0(1) + \frac{\alpha^p}{p}G(\alpha).$$

Since the function G is dependent on p, $G(1 - c/p)$ must be estimated carefully. In

$$G(1 - c/p) = \int_0^{1-c/p} s^{p-1} g(s)\,ds \bigg/ \int_0^{1-c/p} s^{p-1}\,ds$$

$$\approx \int_0^{1-c/p} s^{p-1} g(s)\,ds / (e^{-c}/p)$$

we decompose the domain of integration into

$$[0, 1 - c/p] = \left[0, 1 - c/\sqrt{p}\,\right] \cup \left[1 - c/\sqrt{p}, 1 - c/p\right].$$

It is easily seen that most of the mass of integration lies on the second interval and that as $p \to \infty$, the limit is $g(1)$. Expanding now in negative powers of p,

$$F_0(1) + F_0'(1)(-cp^{-1}) + o(p^{-1}) + p^{-1}(e^{-c} + o(1))(g(1) + o(1))$$

$$= F_0(1) + o(p^{-1}).$$

Since $F_0' = f_0$, the comparison of p^{-1} terms yields

$$cf_0(1) = e^{-c} g(1)$$

which determines a unique positive c, independent of (α, β).

   This enables us to analyze the question of convergence of Eq. (2), (3) to the limit equation (8) as $p \to \infty$. While for every fixed interval $[-1 + \epsilon, 1 - \epsilon]$, $y^{p-1} g(y) \to 0$, at the rightmost point $(1 - c/p + \ldots, 0)$ of the trajectory one has

$$y^{p-1} g(y)|_{1-c/p} \to e^{-c} g(1) = cf_0(1) \neq 0 \qquad \text{as } p \to \infty.$$

Thus, Eq. (2), (3) is close to Eq. (8) on most of the trajectory through $(\alpha, \beta) \in \partial \mathbf{A}$ but not on the whole of it. This demonstrates that y_p'' fails to converge to the second derivative of the limit equation (8).

4. NON-PERIODIC SOLUTIONS IN THE PHASE PLANE

   Now we consider Eq. (4),

$$y'' - f(y, p) = 0,$$
$$y(0) = \alpha, \qquad y'(0) = \beta,$$

where $f(y, p)$ satisfies the same conditions as above. Here the first integral is

$$\tfrac{1}{2}y'^2 - F(y, p) = \tfrac{1}{2}\beta^2 - F(\alpha, p). \tag{15}$$

The first integrals (5) and (15) are the analogues of the families of ellipses and hyperbolas, respectively, studied in [9], hence the similarities in the non-uniform convergence and the straight segments in the limiting trajectories.

The topology of the trajectories (15) is completely different from those of (5), but is again very simple. The only critical point is the saddle point $(y, z) = (0, 0)$ with a global unstable manifold

$$W^u: y' = (2F(y, p))^{1/2}, \qquad \text{sgn } y = \text{sgn } y',$$

and a stable manifold

$$W^s: y' = -(2F(y, p))^{1/2}, \qquad \text{sgn } y = -\text{sgn } y'.$$

W^s naturally represents decreasing solutions which are defined for $t_0 \leq t < \infty$, and satisfy

$$y(t)y'(t) < 0,$$
$$y(t), y'(t) \to 0 \qquad \text{as } t \to +\infty.$$

While for Eq. (1) we are interested in the dependence of the periods on p for various initial values, here it is natural to study the *maximal interval of existence* τ, of each solution, and its behaviour as $p \to \infty$. In order that for every fixed, large p, all trajectories, except the stable one, will escape to infinity in both directions in finite times τ_+, τ_-, we shall add the assumption

(c)   For every fixed, sufficiently large p, $\int^\infty F(y, p)^{-1/2} \, dy < \infty$.

This assumption will be used below. Equation (2), (3) satisfies it for $p > 2$, while (4) does not necessarily satisfy it, unless $f_0(y)$ grows rapidly.

For Eq. (4), let us divide the phase plane into the regions

$$\mathbf{A} = \{(y, z) \mid |y| \leq a, \tfrac{1}{2}z^2 > F_0(y)\},$$
$$\mathbf{B} = \{(y, z) \mid |y| \leq a, \tfrac{1}{2}z^2 < F_0(y)\},$$
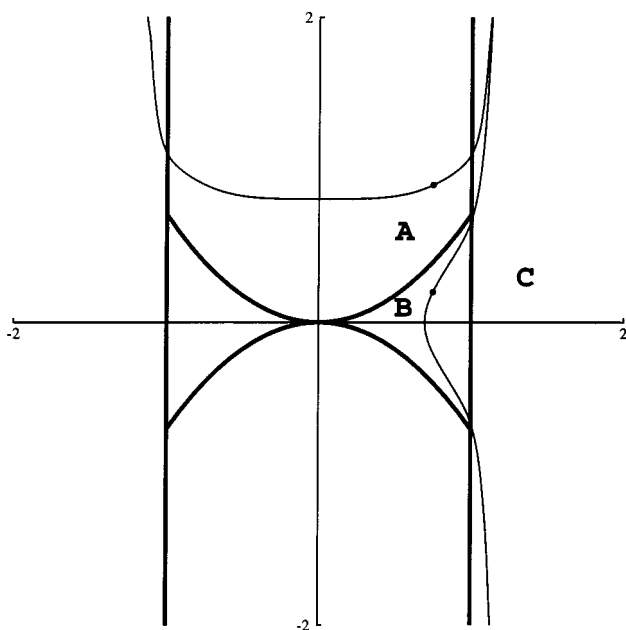$$\mathbf{C} = \{(y, z) \mid |y| > a\}.$$

FIG. 2.   Trajectories of the equation $y'' - y^3 - y^p = 0$ with $p = 31$, through $(0.75, 0.2) \in$ **B**, $(0.75, 0.9) \in$ **A**.

See Fig. 2. For $|\alpha| < |a|$,

$$\tfrac{1}{2}y'^2 - F(y, p) = \tfrac{1}{2}\beta^2 - F(\alpha, p)$$
$$= \left(\tfrac{1}{2}\beta^2 - F_0(\alpha)\right) - \left(F(\alpha, p) - F_0(\alpha)\right) \quad (16)$$
$$= \left(\tfrac{1}{2}\beta^2 - F_0(\alpha)\right) + o(1).$$

(i)   Let $(\alpha, \beta) \in$ **A**. It is clear that the trajectory through (α, β) extends for all $|y| \le a$, and since $\tfrac{1}{2}\beta^2 - F_0(\alpha) > 0$, we have in this strip

$$\tfrac{1}{2}y'^2 = F(y, p) + \left(\tfrac{1}{2}\beta^2 - F_0(\alpha)\right) + o(1) > F_0(y).$$

Thus, the trajectory stays in **A** until it intersects $|y| = a$ and crosses into **C**. On **A**, y_p, y_p' converge uniformly to the corresponding solution of the limit equation and its derivative, respectively, but near $y = \pm a$, y_p'' does not necessarily converge.

By (13), $dt = (\beta^2 + 2F(y, p) - 2F(\alpha, p))^{-1/2} \, dy$; therefore the solution stays in **A** for a time of

$$\int_{-a}^{a} \left(\beta^2 + 2F(y, p) - 2F(\alpha, p) \right)^{-1/2} dy$$

$$\rightarrow \int_{-a}^{a} \left(\beta^2 + 2F_0(y) - 2F_0(\alpha) \right)^{-1/2} dy,$$

since $F(y, p) \rightarrow F_0(y)$ uniformly on $[-a, a]$. The time the trajectory stays in **C** before it escapes to infinity is

$$\int_{a}^{\infty} \left(\beta^2 + 2F(y, p) - 2F(\alpha, p) \right)^{-1/2} dy.$$

By assumption (c), this quantity is finite for large, fixed values of p. Since $F(y, p) \rightarrow \infty$ uniformly for $y \geq a + \epsilon$ as $p \rightarrow \infty$, it is even $o(1)$. It follows that the maximal time interval of existence of the solution through $(\alpha, \beta) \in \mathbf{A}$ is

$$\tau_{\alpha, \beta} = \int_{-a}^{a} \left(\beta^2 + 2F_0(y) - 2F_0(\alpha) \right)^{-1/2} dy + o(1). \qquad (17)$$

Note the similarity between the maximal interval of existence $\tau_{\alpha, \beta}$ of Eq. (4) and the period $T_{\alpha, \beta}$ of Eq. (1).

   (ii)   For $(\alpha, \beta) \in \mathbf{B}$, we have again (16), however, now $\frac{1}{2}\beta^2 - F_0(\alpha) < 0$. The trajectory stays in **B** until it intersects either $y = a$ (if $\alpha > 0$) or $y = -a$ (if $\alpha < 0$) and crosses into **C**. It intersects $z = 0$ at $y = L(p) = F^{-1}(F(\alpha, p) - \frac{1}{2}\beta^2)$, and due to symmetry, the total time it stays in **B** is $2\int_{L(p)}^{a}(2F(y, p) - 2F(\alpha, p) + \beta^2)^{-1/2}$. Since

$$\left(\beta^2 + 2F_0(y) - 2F_0(\alpha) \right)^{-1/2} \approx (2f_0(L)(y - L))^{-1/2} \qquad \text{near } L,$$

it is easy to see that for large values of p, the maximal interval of existence is now

$$\tau_{\alpha, \beta} = 2\int_{L}^{a} \left(\beta^2 + 2F_0(y) - 2F_0(\alpha) \right)^{-1/2} dy + o(1),$$

$$L = F_0^{-1}\left(F_0(\alpha) - \tfrac{1}{2}\beta^2 \right). \quad (18)$$

   (iii)   Next, take a point $(\alpha, \beta) \in \partial\mathbf{A} \cap \partial\mathbf{B}$, i.e., the curves $\frac{1}{2}z^2 = F_0(y)$, $-a < y < a$. These curves are the stable and unstable manifolds of

the limit equation $y'' - f_0 y = 0$, $-a < y < a$. Then

$$\tfrac{1}{2}y'^2 - F(y, p) = \tfrac{1}{2}\beta^2 - F(\alpha, p) = F_0(\alpha) - F(\alpha, p) < 0,$$

and the corresponding trajectory of (4) lies in **B**. Since $\tfrac{1}{2}\beta^2 = F_0(\alpha)$, $\tau_{\alpha, \beta}$ becomes

$$2^{3/2} \int_{L(p)}^a \left(F(y, p) - F(\alpha, p) + F_0(\alpha)\right)^{-1/2} dy$$

$$\to 2^{3/2} \int_0^a F(y, p)^{-1/2} dy$$

which may be finite or infinite. In this case the limit of the solution of Eq. (4) and the solution of the limit equation are different.

(iv) If $(\alpha, \beta) \in \mathbf{C}$, then $\tfrac{1}{2}y'^2 - F(y, p) = \tfrac{1}{2}\beta^2 - F(\alpha, p) \to -\infty$ as $p \to \infty$ and for sufficiently large p the whole trajectory lies in **C** and

$$\tau_{\alpha, \beta} = o(1) \qquad \text{as } p \to \infty.$$

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