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# More on Asymptotic Approximation for Matrix Differential Equations

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An asymptotic approximation is obtained for solutions of a matrix differential equation with symmetric matrix coefficient, analytic on the real line. Our contribution consists of an asymptotic approximation that is valid even for various singularities near infinity. 

## 1. INTRODUCTION

This work considers the matrix differential equation

$$Y'' = A(t)Y, \tag{1.1}$$

where $A(t)$ is a hermitian, $n \times n$ matrix function, invertible and analytic on $(a, \infty)$.

For the analogous scalar equation,

$$y'' = a(t)y, \tag{1.2}$$

there exists under suitable assumptions the Liouville–Green (L-G, WKB) approximation $y \approx a^{-1/4}\exp(\pm\int a^{1/2})$. Improvements to the L-G approximations were sought in numerous works [1, 3, 15, 16]. An improvement of



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the L-G formula that holds in more general situations is

$$y \approx a^{-1/4} \exp\left( \pm \int \sqrt{a + \left(\frac{a'}{4a}\right)^2} \right). \tag{1.3}$$

The above approximations were derived by Hartman and Wintner [11], who proved their local validity as $t \to +\infty$, for the case where $\mathscr{R}e[a(t)]$ stays bounded away from 0. However, the validity of (1.3) in the case where $t = \infty$ is a singular regular point or a singular irregular point with $a(t)$ pure imaginary was not examined. Exceptional features of approximations for solutions and their derivatives, employing the unique combination $(a + (a'/4a)^2)^{1/2}$ were derived and proved in [6–8]. The approximations were shown to be independent of the type and location of singularities, namely valid in a half-neighborhood of a singular regular or a singular irregular point, whether finite or not. The only exclusion is the singular regular case, which involves a logarithmic solution. Moreover, it was shown in [7, Example 5.7, and 8, 10] that the approximations derived there, when interpreted appropriately, are also valid at a turning point. Consult, e.g., [1, 15, 16, 19] for the significance of turning points.

L-G approximations for matrix differential Eq. (1.1) were studied in [17] (see also references there). A similar problem was approached in [18] in the context of $C^*$-algebras. A paper by Edelstein [4] contains results concerning asymptotic approximations for solutions of (1.1) in a Banach space. In this case the exponential terms in the L-G approximation are replaced by evolutionary operators of the problem $du/dt = \pm A^{1/2}u$.

In [5], we developed an asymptotic formula for the matrix system (1.1) that is a matrix analog of the Liouville–Green approximation, under the assumption that the matrix $A(t)$ is invertible and analytic at the point $t = \infty$. However, it should come as no surprise that the range of validity of the asymptotic formula developed there does not include the cases where $t = \infty$ is a singular regular point or a so-called turning point, normally a point where an eigenvalue vanishes. This is so since the Liouville–Green approximation also fails in the analogous cases for the scalar (1.2).

In this work we intend to obtain approximate solutions of the matrix equation (1.1) in the spirit of (1.3). Our contribution consists of an asymptotic approximation that is valid even for various singularities at $t = \infty$. It will cover, for example, cases where $A(t) = t^r \tilde{A}(t)$, $r$ is any real number, and $\tilde{A}(t)$ is a hermitian, analytic matrix function on the real line near $t = \infty$, and $\tilde{A}(\infty)$ is invertible. Other cases that are covered are such that $A(t)$ is not invertible at $t = \infty$ and is possibly meromorphic there. Note that analyticity is a convenient working assumption in this work. However, it is used only to estimate functions and their derivatives near infinity, and it could have easily been replaced by other sets of assumptions. To keep our arguments short, this will not be done here.

Most of the results that are mentioned above are local approximations, i.e., they are valid on some neighborhood of $t = \infty$. In [7, 10] it was shown for the scalar equation (1.2) how to derive global approximations valid on an entire interval, one end of which is a turning point for the equation while the other end point could be a singular point. This is done without utilization of special or transcendental functions. In this paper we focus only on local approximations for (1.1). Nevertheless, a refined analysis should show, as in the scalar case, that global approximations could be derived for solutions of (1.1).

## 2. LINEAR TRANSFORMATIONS

We rewrite the second-order $n \times n$ matrix differential equation (1.1) as a first-order $2n \times 2n$ system,

$$Z' = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} Z, \quad Z = \begin{pmatrix} Y \\ Y' \end{pmatrix}. \tag{2.1}$$

The hermitian $A(t)$ may be diagonalized by a unitary matrix $U(t)$, $UU^* = I$,

$$U^{-1}AU = D(t) = \operatorname{diag}\{\lambda_1(t), \dots, \lambda_n(t)\}. \tag{2.2}$$

According to Rellich's theorem [13], the unitary $U(t)$ may be taken to be analytic in $t$, and $\lambda_1(t), \dots, \lambda_n(t)$ are the real, analytic eigenvalues of $A(t)$. By the invertibility of $A(t)$, $\lambda_j(t) \neq 0$ on $(a, \infty)$, $j = 1, \dots, n$.

A natural initial step is to simplify (2.1) by some linear transformation $Z = TZ_1$. One convenient choice is to take

$$\begin{aligned} T &= \begin{pmatrix} A^{-1/4} & A^{-1/4} \\ A^{1/4} & -A^{1/4} \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \\ &= \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} D^{-1/4} & 0 \\ 0 & D^{1/4} \end{pmatrix} \begin{pmatrix} I & I \\ I & -I \end{pmatrix}, \\ T^{-1} &= \frac{1}{2} \begin{pmatrix} I & I \\ I & -I \end{pmatrix} \begin{pmatrix} D^{1/4} & 0 \\ 0 & D^{-1/4} \end{pmatrix} \begin{pmatrix} U^* & 0 \\ 0 & U^* \end{pmatrix}. \end{aligned} \tag{2.3}$$

It is easily verified by straightforward calculation that this takes (2.1) into

$$Z_1' = \left[ \begin{pmatrix} D^{1/2} & 0 \\ 0 & -D^{1/2} \end{pmatrix} - T^{-1}T' \right] Z_1. \tag{2.4}$$

In [5] this transformation was applied under the assumption that $A(t)$ is analytic and invertible at $t = \infty$, and so $T$ is analytic there. Consequently $T' = \mathscr{O}(t^{-2})$, and $T^{-1}T'$ was considered as a "small perturbation" of the dominant diagonal matrix. In the present work, on the other hand, we shall analyze (2.3) further to obtain an approximation that is also valid in singular cases. By (2.3),

$$T' = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} -\frac{1}{4}D^{-5/4}D' & 0 \\ 0 & \frac{1}{4}D^{-3/4}D' \end{pmatrix} \begin{pmatrix} I & I \\ I & -I \end{pmatrix}$$

$$+ \begin{pmatrix} U' & 0 \\ 0 & U' \end{pmatrix} \begin{pmatrix} D^{-1/4} & 0 \\ 0 & D^{1/4} \end{pmatrix} \begin{pmatrix} I & I \\ I & -I \end{pmatrix};$$

hence

$$-T^{-1}T' = \frac{1}{4} \begin{pmatrix} 0 & D^{-1}D' \\ D^{-1}D' & 0 \end{pmatrix} + L,$$

where

$$L = \frac{1}{2} \begin{pmatrix} I & I \\ I & -I \end{pmatrix} \begin{pmatrix} D^{1/4}U^*U'D^{-1/4} & 0 \\ 0 & D^{-1/4}U^*U'D^{1/4} \end{pmatrix} \begin{pmatrix} I & I \\ I & -I \end{pmatrix}. \tag{2.5}$$

So finally,

$$Z_1' = \left[ \begin{pmatrix} D^{1/2} & \frac{1}{4}D^{-1}D' \\ \frac{1}{4}D^{-1}D' & -D^{1/2} \end{pmatrix} + L \right] Z_1. \tag{2.6}$$

If $A(t)$ is not analytic at $t = \infty$, the diagonal terms are not necessarily the dominant ones. For example, at a singular-regular point at infinity, $A(t) = t^{-2} \sum A_i t^{-i}$, we have $\lambda_j \approx t^{-2}$ and $\lambda_j^{1/2}$, $\lambda_j'/4\lambda_j \approx t^{-1}$ are comparable. Similarly, any ratio $(\lambda_j'/4\lambda_j)/\lambda_j^{1/2}$ is available at a suitably chosen singularity. To handle a case as general as possible, we shall try to simplify the whole leading matrix by additional linear transformations. The first matrix coefficient consists of four diagonal blocks:

$$\left[\begin{array}{ccc:ccc} \lambda_1^{1/2} & & & \lambda_1'/4\lambda_1 & & \\ & \ddots & & & \ddots & \\ & & \lambda_n^{1/2} & & & \lambda_n'/4\lambda_n \\ \hdashline \lambda_1'/4\lambda_1 & & & -\lambda_1^{1/2} & & \\ & \ddots & & & \ddots & \\ & & \lambda_n'/4\lambda_n & & & -\lambda_n^{1/2} \end{array}\right].$$

Let $P$ be the $2n \times 2n$ permutation matrix that, by multiplication from the right, places the $i$th column at the $2i - 1$th place and the $n + i$th column at the $2i$th one, $i = 1, \ldots, n$. The transformation $Z_1 = PZ_2$ rearranges the four $n \times n$ blocks of (2.6) into a direct sum of $2 \times 2$ blocks and takes the whole Eq. (2.6) into

$$Z_2' = \left[ \bigoplus \begin{pmatrix} \lambda_j^{1/2} & \lambda_j'/4\lambda_j \\ \lambda_j'/4\lambda_j & -\lambda_j^{1/2} \end{pmatrix} + P^{-1}LP \right] Z_2. \tag{2.7}$$

The direct sum of blocks may be diagonalized again by a matrix $V$ of a similar block structure, and so the calculation is effectively reduced into one of a single $2 \times 2$ block. Its eigenvalues are

$$\pm\mu_j(t) = \pm\sqrt{\lambda_j + \left(\frac{\lambda_j'}{4\lambda_j}\right)^2}.$$

Note that while $\lambda_j$, $\lambda_j'$ are real valued, $\mu_j$ may be real valued or a pure imaginary function. The eigenvectors that correspond to $\pm\mu_j(t)$ may be taken, up to scalar factors, as

$$\begin{pmatrix} \lambda_j^{1/2} + \left(\lambda_j + \left(\lambda_j'/4\lambda_j\right)^2\right)^{1/2} \\ \lambda_j'/4\lambda_j \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -\lambda_j'/4\lambda_j \\ \lambda_j^{1/2} + \left(\lambda_j + \left(\lambda_j'/4\lambda_j\right)^2\right)^{1/2} \end{pmatrix},$$

respectively, where $(\,)^{1/2}$ always denotes the same branch. Now we introduce some convenient normalization. Let us put $\ell_j(t) = \lambda_j'/4\lambda_j^{3/2}$. Dividing by $\lambda_j^{1/2}$ takes the eigenvectors into

$$\begin{pmatrix} 1 + \sqrt{1 + \ell_j^2} \\ \ell_j \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -\ell_j \\ 1 + \sqrt{1 + \ell_j^2} \end{pmatrix}, \tag{2.8}$$

and if we normalize them to be unit vectors, some algebraic manipulations yield a diagonalizing block

$$\begin{pmatrix} g_j & -h_j \\ h_j & g_j \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \left(1 + \dfrac{1}{\sqrt{1 + \ell_j^2}}\right)^{1/2} & -\left(1 - \dfrac{1}{\sqrt{1 + \ell_j^2}}\right)^{1/2} \\ \left(1 - \dfrac{1}{\sqrt{1 + \ell_j^2}}\right)^{1/2} & \left(1 + \dfrac{1}{\sqrt{1 + \ell_j^2}}\right)^{1/2} \end{pmatrix},$$

where $g_j^2 + h_j^2 \equiv 1$ with complex valued $g_j, h_j, \ell_j(t)$. Since $\lambda_j$ is real valued, $\ell_j$ is restricted either to the real axis or to the imaginary one. To make the above formulas well defined, we shall choose $(1 - 1/\sqrt{1+\ell^2})^{1/2} = \ell/\sqrt{2} + \cdots +$ near $\ell = 0$ and $1 - 1/2\ell + \cdots$ near infinity, and a similar choice for the other terms. Thus no problem of multivalued functions is encountered. The corresponding diagonalizing matrix $V$ is of the form

$$V = \bigoplus \begin{pmatrix} g_j & -h_j \\ h_j & g_j \end{pmatrix}.$$

By the identity $\sin(\frac{1}{2}z) = (1/\sqrt{2})(1 - 1/(1 + \tan^2 z)^{1/2})^{1/2}$, the blocks are precisely

$$\begin{pmatrix} \cos\left(\frac{1}{2}\arctan \ell_j\right) & -\sin\left(\frac{1}{2}\arctan \ell_j\right) \\ \sin\left(\frac{1}{2}\arctan \ell_j\right) & \cos\left(\frac{1}{2}\arctan \ell_j\right) \end{pmatrix}, \tag{2.9}$$

a rotation by (the complex angle) $\frac{1}{2}\arctan \ell_j$.

The vectors in (2.8) are linearly independent, and the other considerations are valid, provided that $1 + \ell_j^2 \neq 0$, that is,

$$1 + \frac{\lambda_j'^2}{16\lambda_j^3} \neq 0 \tag{2.10}$$

on $(a, \infty)$. Since $1 + \ell_j^2$ is real valued and nonzero, it is either positive on all $(a, \infty)$ or negative there. As $\lambda_j(t)$ is also real and nonzero on $(a, \infty)$, (2.10) implies that $\mu_j = \sqrt{\lambda_j\left(1 + \ell_j^2\right)} \neq 0$ on $(a, \infty)$, and it is either real valued on the whole $(a, \infty)$ or pure imaginary there.

$V$ is analytic at every point where the $\ell_j(t)$'s are analytic or meromorphic and assumption (2.10) holds. At the singular end point $t = \infty$, $\lambda_j, \lambda_j'$ may vanish. However, if we assume that (2.10) also holds at $t = \infty$, then $V(t)$ will be bounded and analytic at $t = \infty$, too.

By the change of variables $Z_2 = VZ_3$, the first matrix in (2.7) is diagonalized, and our equation becomes

$$Z_3' = \left[\bigoplus \begin{pmatrix} \mu_j & 0 \\ 0 & -\mu_j \end{pmatrix} + V^{-1}P^{-1}LPV - V^{-1}V'\right] Z_3. \tag{2.11}$$

It follows by direct differentiation of (2.9) that

$$V^{-1}V' = \bigoplus \left(\tfrac{1}{2}\arctan \ell_j\right)' \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.12}$$

In this identity there is no essential difference whether the functions $\ell_j$ are analytic or meromorphic, since $(\arctan \ell)' = -(\arctan \ell^{-1})'$.

It is known that under suitable "smallness conditions" on the terms $V^{-1}P^{-1}LPV$, $V^{-1}V'$, the solutions of Eq. (2.11) have an asymptotic approximation

$$Z_3 = (I_{2n} + R(t)) \bigoplus \begin{pmatrix} \exp \int \mu_j & 0 \\ 0 & \exp \int -\mu_j \end{pmatrix} C,$$

where $R(t)$ is a $2n \times 2n$ continuous matrix function such that $R(\infty) = 0$. Several smallness conditions are summarized in [3]. The solutions of the corresponding Eq. (2.1) may be represented as

$$Z = T(t)PV(t)(I_{2n} + R(t)) \bigoplus \begin{pmatrix} \exp \int \mu_j & 0 \\ 0 & \exp \int -\mu_j \end{pmatrix} C.$$

Let us rewrite this as

$$Z = T(t)\big(PV(t)P^{-1}\big)\big(P(I_{2n} + R(t))P^{-1}\big) \times \left( P \bigoplus \begin{pmatrix} \exp \int \mu_j & 0 \\ 0 & \exp \int -\mu_j \end{pmatrix} P^{-1} \right) PC.$$

Each application of the inverse permutation matrix $P^{-1}$ rearranges the $n^2$ $2 \times 2$ blocks back into four $n \times n$ blocks. Thus

$$PVP^{-1} = \left( \begin{array}{c:c} \operatorname{diag}\left\{\cos\left(\frac{1}{2} \arctan \ell_j\right)\right\} & \operatorname{diag}\left\{-\sin\left(\frac{1}{2} \arctan \ell_j\right)\right\} \\ \hdashline \operatorname{diag}\left\{\sin\left(\frac{1}{2} \arctan \ell_j\right)\right\} & \operatorname{diag}\left\{\cos\left(\frac{1}{2} \arctan \ell_j\right)\right\} \end{array} \right),$$

$$P \bigoplus \begin{pmatrix} \exp \int \mu_j & 0 \\ 0 & \exp \int -\mu_j \end{pmatrix} P^{-1}$$

$$= \operatorname{diag}\left\{ \exp \int \mu_1, \exp \int \mu_2, \dots, \exp \int -\mu_1, \dots \right\},$$

$$P(I_{2n} + R)P^{-1} = I_{2n} + \tilde{R} = \left( \begin{array}{c:c} I_n + \tilde{R}_{11} & \tilde{R}_{12} \\ \hdashline \tilde{R}_{21} & I_n + \tilde{R}_{22} \end{array} \right).$$

If we put, for short, $\bar{\ell} = \operatorname{diag}\{\ell_j\}$, $\bar{\mu} = \operatorname{diag}\{\mu_j\}$, then

$$Z = \begin{pmatrix} UD^{-1/4} & UD^{-1/4} \\ UD^{1/4} & -UD^{1/4} \end{pmatrix} \left(\begin{array}{c:c} \cos\left(\frac{1}{2}\arctan \bar{\ell}\right) & -\sin\left(\frac{1}{2}\arctan \bar{\ell}\right) \\ \hdashline \sin\left(\frac{1}{2}\arctan \bar{\ell}\right) & \cos\left(\frac{1}{2}\arctan \bar{\ell}\right) \end{array}\right)$$

$$\times \left(\begin{array}{c:c} \left(I_n + \tilde{R}_{11}\right)\exp\int\bar{\mu} & \tilde{R}_{12}\exp\int -\bar{\mu} \\ \hdashline \tilde{R}_{21}\exp\int\bar{\mu} & \left(I_n + \tilde{R}_{22}\right)\exp\int -\bar{\mu} \end{array}\right) K. \tag{2.13}$$

For the solution $Y$ of (1.1) we get the approximation

$$\begin{aligned} Y(t) = {} & UD^{-1/4}\left[\sin\left(\tfrac{1}{2}\arctan \bar{\ell} + \frac{\pi}{4}I_n\right)\left(I_n + \tilde{R}_{11}\right)\right. \\ & \left. -\sin\left(\tfrac{1}{2}\arctan \bar{\ell} - \frac{\pi}{4}I_n\right)\tilde{R}_{21}\right] \\ & \times \operatorname{diag}\left\{\exp\int\mu_1, \ldots\right\}K_1 \\ & + UD^{-1/4}\left[-\sin\left(\tfrac{1}{2}\arctan \bar{\ell} - \frac{\pi}{4}I_n\right)\tilde{R}_{12}\right. \\ & \left. +\sin\left(\tfrac{1}{2}\arctan \bar{\ell} + \frac{\pi}{4}I_n\right)\left(I_n + \tilde{R}_{22}\right)\right] \\ & \times \operatorname{diag}\left\{\exp\int -\mu_1, \ldots\right\}K_2 \end{aligned} \tag{2.14}$$

and another approximation for $Y'$. It is worth mentioning that in the interior of $(a, \infty)$ where $\lambda_j \neq 0$, $\ell_j$ is finite. Consequently, $\frac{1}{2}\arctan \ell_j \neq \pm\pi/4$, and the corresponding diagonal matrices in (2.14) are invertible.

For $n = 1$, (2.14) reduces to the solutions of the scalar Eq. (1.2), which are equivalent to (6.1), (6.2) of [8]:

$$\begin{aligned} y_1 = {} & \left[(1 + r_{11})\left(\left(1 - \frac{1}{\sqrt{1+\ell^2}}\right)^{1/2} + \left(1 + \frac{1}{\sqrt{1+\ell^2}}\right)^{1/2}\right)\right. \\ & \left. +r_{21}\left(\left(1 - \frac{1}{\sqrt{1+\ell^2}}\right)^{1/2} - \left(1 + \frac{1}{\sqrt{1+\ell^2}}\right)^{1/2}\right)\right] \\ & \times a^{-1/4}\exp\left(\int\sqrt{a + (a'/4a)^2}\right), \end{aligned}$$

$$y_2 = \left[ r_{12} \left( \left( 1 - \frac{1}{\sqrt{1 + \ell^2}} \right)^{1/2} + \left( 1 + \frac{1}{\sqrt{1 + \ell^2}} \right)^{1/2} \right) + (1 + r_{22}) \right.$$
$$\left. \times \left( \left( 1 - \frac{1}{\sqrt{1 + \ell^2}} \right)^{1/2} - \left( 1 + \frac{1}{\sqrt{1 + \ell^2}} \right)^{1/2} \right) \right]$$
$$\times a^{-1/4} \exp\left( - \int \sqrt{a + (a'/4a)^2} \right),$$

with $\ell = a'/4a^{3/2}$.

## 3. ASYMPTOTIC APPROXIMATIONS

The formal asymptotic approximations that are suggested above will be justified if the perturbation terms in the reduced equation (2.11) are small in some sense. One simple result in this direction is the following modification of Levinson's theorem [3, Theorem 1.3.1]: Let the diagonal matrix $\Lambda(t) = \operatorname{diag}\{\mu_1(t), \ldots, \mu_n(t)\}$ satisfy the dichotomy condition that for each pair of integers $i, j$ and for all $x, t$ such that $a \le t \le x < \infty$, $\int_t^x \mathscr{R}e\{\mu_i(s) - \mu_j(s)\}\, ds$ is either bounded from above or is bounded from below. If the matrix $C(t)$ satisfies $\int_a^\infty |C(s)|\, ds < \infty$, then, as $t \to \infty$, the system $W'(t) = (\Lambda(t) + C(t))W(t)$ has a solution with the asymptotic form

$$W(t) = (I + o(1)) \operatorname{diag}\left\{ \exp \int_a^t \mu_1(s)\, ds, \ldots \right\}.$$

A simple result is mentioned in [2, p. 88], where the dichotomy condition is replaced by a stronger one, namely, that none of the differences $\mathscr{R}e\{\mu_i(s) - \mu_j(s)\}$ change sign.

We shall try to apply this version of Levinson's theorem for (2.11) and show that the perturbation terms $V^{-1}P^{-1}LPV$, $V^{-1}V'$ are integrable near infinity. This may be verified under various assumptions on the singular behavior of $A(t)$ at infinity.

Let us assume that

$$A(t) = t^h \tilde{A}(t),$$

where $h$ is an arbitrary real number, and the hermitian matrix function $\tilde{A}(t)$ is analytic and invertible on $(a, \infty)$ but possibly meromorphic and not necessarily invertible at $t = \infty$. Since the terms of $\tilde{A}(t)$ are meromorphic at $t = \infty$, we can write $A(t) = t^{h+p}\hat{A}(t)$, with a suitable positive integer $p$ and $\hat{A}(t)$ analytic at $t = \infty$. The eigenvalues of the hermitian $\hat{A}(t)$ are analytic at $t = \infty$ and are of the form $t^{p_j}\hat{\lambda}_j(t)$, where the $p_j$'s are negative integers, $\hat{\lambda}_j(\infty) \neq 0$, and the eigenvalues of $A(t)$ are

$$\lambda_j(t) = t^{r_j}\hat{\lambda}_j(t), \quad \hat{\lambda}_j(\infty) \neq 0, \qquad j = 1, \ldots, n,$$

with $r_j = h + p + p_j$. Thus $\lambda_j(t) = t^{r_j}[\hat{\lambda}_j(\infty) + \mathscr{O}(t^{-1})]$, $\lambda_j'/\lambda_j = r_j/t + \hat{\lambda}_j'/\hat{\lambda}_j = r_j/t + \mathscr{O}(t^{-2})$, and

$$\ell_j^2(t) = \frac{\lambda_j'^2}{16\lambda_j^3} = \frac{r_j^2}{16\hat{\lambda}_j(\infty)} t^{-r_j-2}\big(1 + \mathscr{O}(t^{-1})\big), \tag{3.1}$$

$$\mu_j(t) = \left(t^{r_j}\hat{\lambda}_j(t) + \frac{r_j^2}{16}t^{-2} + \mathscr{O}(t^{-3})\right)^{1/2}. \tag{3.2}$$

Assumption (2.10) is satisfied on some $[b, \infty]$ unless $r_j = -2$ and $\hat{\lambda}_j(\infty) = -1/4$ for some $j$, that is, when $\lambda_j(t) \approx -\frac{1}{4}t^{-2}$. This should not be surprising, since the equation $y'' + \frac{1}{4}t^{-2}y = 0$ is critically located between oscillatory and nonoscillatory equations. Except for this case, $V$ and $V^{-1}$ are bounded on some $[b, \infty]$.

Now we turn to the perturbation terms. By (3.1), $\ell_j(t) = ct^{-r_j/2-1}(1 + \mathscr{O}(t^{-1}))$, and

$$\big(\arctan \ell_j\big)' = \frac{\ell_j'}{1 + \ell_j^2} = \begin{cases} \mathscr{O}(t^{-r_j/2-2}) & \text{if } r_j > -2, \\ \mathscr{O}(t^{-2}) & \text{if } r_j = -2, \\ \mathscr{O}(t^{r_j/2}) & \text{if } r_j < -2, \end{cases} \tag{3.3}$$

that is, in every case it is $\mathscr{O}(t^{-1-\epsilon})$ for some $\epsilon > 0$. It follows by (2.12) that $V^{-1}V' \in \mathbf{L}[b, \infty)$.

Now we turn to the term $V^{-1}P^{-1}LPV$. Here we need a more detailed study of the matrix $L$, which is defined in (2.5). Since $U(t)$ is unitary, $UU^* = I$, it follows that $U^*U' = -U^{*\prime}U = -(U^*U')^*$ and $U^*U'$ is skew-hermitian. Put $U^*U' = (m_{jk})$, $m_{jk} = -\overline{m_{kj}}$, $j, k = 1, \ldots, n$. Then by (2.5),

$$L = \frac{1}{2}\left[\begin{array}{c:c} \left(\left[\left(\frac{\lambda_j}{\lambda_k}\right)^{1/4} + \left(\frac{\lambda_k}{\lambda_j}\right)^{1/4}\right]m_{jk}\right) & \left(\left[\left(\frac{\lambda_j}{\lambda_k}\right)^{1/4} - \left(\frac{\lambda_k}{\lambda_j}\right)^{1/4}\right]m_{jk}\right) \\ \hdashline \left(\left[\left(\frac{\lambda_j}{\lambda_k}\right)^{1/4} - \left(\frac{\lambda_k}{\lambda_j}\right)^{1/4}\right]m_{jk}\right) & \left(\left[\left(\frac{\lambda_j}{\lambda_k}\right)^{1/4} + \left(\frac{\lambda_k}{\lambda_j}\right)^{1/4}\right]m_{jk}\right) \end{array}\right].$$

The permutation $P$ rearranges $L$ into $n \times n$ blocks of size $2 \times 2$ such that the $(j,k)$th block of $P^{-1}LP$ is located in the $2j-1$ and the $2j$th row and the $2k-1$ and $2k$th column and consists of

$$m_{jk}\begin{pmatrix} \left(\frac{\lambda_j}{\lambda_k}\right)^{1/4} + \left(\frac{\lambda_k}{\lambda_j}\right)^{1/4} & \left(\frac{\lambda_j}{\lambda_k}\right)^{1/4} - \left(\frac{\lambda_k}{\lambda_j}\right)^{1/4} \\ \left(\frac{\lambda_j}{\lambda_k}\right)^{1/4} - \left(\frac{\lambda_k}{\lambda_j}\right)^{1/4} & \left(\frac{\lambda_j}{\lambda_k}\right)^{1/4} + \left(\frac{\lambda_k}{\lambda_j}\right)^{1/4} \end{pmatrix}.$$

By the way, if in addition, $A(t)$ is real valued symmetric and $U(t)$ is real orthogonal, then $U^T U'$ is skew-symmetric, and in particular, its diagonal elements, $m_{jj}$, vanish. Thus the diagonal blocks of $P^{-1}LP$ are all identically zero. By the block structure of $V$, the same holds for $V^{-1}P^{-1}LPV$ as well.

Recall that $A(t) = t^{h+p}\hat{A}(t)$, where $\hat{A}(t)$ is analytic at $t = \infty$. Therefore the matrix $U(t)$ that diagonalizes $A(t)$ may be chosen to be analytic at $t = \infty$ and $U^*U' = \mathscr{O}(t^{-2})$. It follows from the above calculations that each term of $L$ and $P^{-1}LP$ consists of expressions of the type

$$\mathscr{O}\left(t^{-2+(r_j - r_k)/4}\right),$$

which is integrable on $[a,\infty)$ if $r_j - r_k < 4$. The differences $r_j - r_k$ are independent of $h, p$ and are necessarily integers. Thus the matrix $P^{-1}LP$ is integrable if

$$\max_{j,k} |r_j - r_k| \leq 3.$$

Since $V, V^{-1}$ are bounded near $t = \infty$, them $V^{-1}P^{-1}LPV = VP^{-1}LPV \in \mathbf{L}[b,\infty)$. By this we show that the perturbation terms in (2.11) are integrable and the following is proved.

THEOREM 1. *Consider Eq.* (1.1), *where* $A(t) = t^h\tilde{A}(t)$, *$h$ is a real number, and $\tilde{A}(t)$ is a $n \times n$ hermitian matrix function and is analytic and invertible on* $(a,\infty)$, *but possibly meromorphic and not necessarily invertible at $t = \infty$. Let* $\lambda_1(t), \dots, \lambda_n(t)$ *be the eigenvalues of* $A(t)$, $\lambda_j(t) = t^{r_j}\hat{\lambda}_j(t)$, $\hat{\lambda}_j(\infty) \neq 0$. *If* $\max_{j,k}|r_j - r_k| \leq 3$ *and*

$$1 + \frac{\lambda_j'^2}{16\lambda_j^3} \neq 0 \qquad on\ (a,\infty],$$

*then the solutions of Eq.* (1.1) *and those of the corresponding first-order system* (2.1) *may be represented by* (2.14) *and* (2.13), *respectively*.

Let us consider a special case when $A(t) = t^h\tilde{A}(t)$ and $\tilde{A}(t)$ is analytic and invertible, not only on the open interval $(a, \infty)$, but even at $t = \infty$. Then all $r_j$'s are equal to the same $h$, $\lambda_j(t) = t^h\tilde{\lambda}_j(t)$, and all we have to check is condition (2.10).

THEOREM 2. *Consider the second-order differential system* (1.1) *with*

$$A(t) = t^h\tilde{A}(t),$$

*where $h$ is an arbitrary real number, and $\tilde{A}(t)$ is a $n \times n$ hermitian matrix function on the real line and analytic on $(a, \infty]$ and invertible there. Let $\lambda_1(t), \ldots, \lambda_n(t)$ denote the real eigenvalues of $A(t)$, $\lambda_j(t) = t^h\tilde{\lambda}_j(t)$ and assume that*

$$1 + \frac{\lambda_j'^2}{16\lambda_j^3} \neq 0 \qquad on\ (a, \infty].$$

*Then the solutions of Eq.* (1.1) *and those of the corresponding first-order system* (2.1) *may be represented by* (2.14) *and* (2.13), *respectively*.

EXAMPLE. Let

$$A(t) = \begin{pmatrix} t^\alpha & t^\gamma \\ t^\gamma & t^\beta \end{pmatrix},$$

where $\alpha$, $\beta$, and $\gamma$ arbitrary real numbers and consider, for example, the case $\gamma > \alpha, \beta$. $A(t)$ can be written as

$$t^\gamma \begin{pmatrix} t^{\alpha-\gamma} & 1 \\ 1 & t^{\beta-\gamma} \end{pmatrix},$$

which does not satisfy the analyticity assumptions of Theorem 1 if $\alpha - \gamma$, $\beta - \gamma < 0$ are not integers. Nevertheless the arguments of Theorem 1 hold with minor modifications. For, near $t = \infty$, $\lambda_{1,2} = \pm t^\gamma(1 + \frac{1}{2}t^{\alpha-\gamma} + \frac{1}{2}t^{\beta-\gamma} + \cdots)$, and we take (the nonunitary)

$$U(t) = \begin{pmatrix} 1 & -1 - \frac{1}{2}(t^{\beta-\gamma} - t^{\alpha-\gamma}) + \cdots \\ 1 + \frac{1}{2}(t^{\beta-\gamma} - t^{\alpha-\gamma}) + \cdots & 1 \end{pmatrix}.$$

$U' = \mathscr{O}(t^{\alpha-\gamma-1}) + \mathscr{O}(t^{\beta-\gamma-1})$, and so $U^{-1}U' \in \mathbf{L}[a, \infty)$. Equation (3.1) is replaced by

$$\ell^2(t) = ct^\gamma\big(1 + \mathscr{O}(t^{\alpha-\gamma}) + \mathscr{O}(t^{\beta-\gamma})\big),$$

and (3.3) is modified depending on whether $\gamma < -2$, $\gamma = -2$, or $\gamma > -2$. For $\gamma = -2$, for example, we have $(\arctan \ell(t))' = \mathscr{O}(t^{\alpha-\gamma-1}) + \mathscr{O}(t^{\beta-\gamma-1}) \in \mathbf{L}[a,\infty)$.

Theorems 1 and 2 are easily extended to more general cases, say, when

$$A(t) = t^r \tilde{A}(t^q),$$

where $r, q$ are real numbers and $q > 0$, and $\tilde{A}(s)$ is real analytic on $(a,\infty)$ but possibly meromorphic at $s = \infty$. Indeed, if $U(s) = U(\infty) + \mathscr{O}(s^{-1})$ is the unitary matrix that diagonalizes $\tilde{A}(s)$, we apply in (2.3) the matrix $U(t^q)$. Then $U^{-1}(t^q)\, d/dt\, U(t^q) = \mathscr{O}(t^{-2q})qt^{q-1} = \mathscr{O}(t^{-q-1}) \in \mathbf{L}[a,\infty)$. The eigenvalues of $A(t)$ are of the shape $\lambda_j(t) = t^{p_j}\sum c_{k_j} t^{-kq}$ and

$$\ell_j^2(t) = \frac{\lambda_j'^2}{16\lambda_j^3} = \frac{\mathscr{O}(t^{p_j-1})^2(1 + \mathrm{O}(t^{-q}))}{\mathscr{O}(t^{p_j})^3} = \mathscr{O}(t^{-p_j-2})(1 + \mathscr{O}(t^{-q})).$$

Consequently,

$$\frac{\ell_j'}{1 + \ell_J^2} = \begin{cases} \mathscr{O}(t^{-p_j/2-2}) & \text{if } p_j > -2, \\ \mathscr{O}(t^{-q-1}) & \text{if } p_j = -2, \\ \mathscr{O}(t^{p_j/2}) & \text{if } p_j < -2, \end{cases}$$

are all integrable near $t = \infty$. The rest of the discussion follows as in the previous theorems.

## 4. AN APPLICATION TO OSCILLATION

One of the main questions of oscillation theory is the existence of *conjugate points*. Recall that for a vector differential equation,

$$\mathbf{y}'' = A(t)\mathbf{y}, \tag{4.1}$$

$\eta(t)$ is called a *conjugate point of* $t$ if the equation has a nontrivial solution $\mathbf{y}$ such that

$$\mathbf{y}(t) = \mathbf{y}(\eta(t)) = 0.$$

If a conjugate point exists for all sufficiently large values of $t$, the equation is called *oscillatory*.

As in the detailed calculations of [5], it can be shown that a conjugate point $\eta(t)$ exists for large values of $t$ if $\mu_j = (\lambda_j + (\lambda_j'/4\lambda_j)^2)^{1/2}$ is pure imaginary for some $j$ (and the corresponding $\exp\int\mu_j$ is a trigonometric function), and that

$$\mathcal{I}m \int_t^{\eta(t)} \sqrt{\lambda_j + \left(\frac{\lambda_j'}{4\lambda_j}\right)^2}\, ds = m\pi + o(1) \tag{4.2}$$

for some $m = 1, 2, \ldots$. Thus, Eq. (4.1) is oscillatory if

$$\mathcal{I}m \int^{\infty} \sqrt{\lambda_j + \left(\frac{\lambda_j'}{4\lambda_j}\right)^2}\, ds = \infty. \tag{4.3}$$

For the case $n = 1$, we thus recover under the assumptions of Theorems 1 and 2 the necessary and sufficient condition for the oscillation of the scalar equation (1.2), as obtained in [9].

It was conjectured in [12] that for a symmetric matrix $A(t)$, Eq. (4.1) is oscillatory if

$$\lim_{t\to\infty} \lambda_{\min}\left\{\int_0^t A(s)\, ds\right\} = -\infty,$$

where $\lambda_{\min}$ denotes the minimal eigenvalue of the corresponding matrix. In [14] this is verified with the additional assumption that

$$\liminf_{t\to\infty} t^{-1} \operatorname{trace}\left\{\int_0^t A(s)\, ds\right\} < \infty.$$

For Eq. (4.1) with $A(t) = t^h A$, where $A$ is a constant symmetric matrix, (4.3) is a weaker assumption. This is, of course, the result of the additional assumptions about the analytic behavior of the coefficient matrix.

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