Oscillation and block-diagonalization

Uri Elias* and Harry Gingold

Dept of Mathematics, Technion - I.I.T., Haifa 32000, ISRAEL
Dept of Mathematics, West Virginia University, Morgantown, WV 26506

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Address:

U. Elias

Department of Mathematics

Technion

Haifa 32000, ISRAEL

Abstract. Asymptotic methods are used for the two term differential equation $y^{(n)} = p(x)y$, with $p(x)$ real analytic function at infinity. The differential equation is written as a system $Y' = A(x)Y$, and block diagonalization is utilized to study asymptotic solutions in the case of coalescing eigenvalues. It is shown that the system may be reduced to one standard system of dimension 2×2. We provide a necessary and sufficient condition for various types of oscillation of the two term differential equation near infinity. This complements's Hille-Kneser theory which gives a sufficient condition and separately a necessary condition for oscillation, while it leaves some cases unresolved.

1. Introduction. The aim of this work is to study some oscillatory properties of the equation

$$y^{(n)} = p(x)y, \tag{1.1}$$

where $p(x)$ is a sufficiently smooth real function on an interval $[a, \infty)$, through asymptotic methods. By standard techniques, the first step in handling the asymptotics of a scalar equation is to convert it into a vectorial differential system

$$Y' = A(x)Y \tag{1.2}$$

with the companion matrix A(x). Next, let $r(x) \equiv p(x)^{1/n}$ be one fixed branch of the nth root. The shearing transformation $Y(x) = V(x)Z_1, V = \text{diag}\,\{1, r, \ldots, r^{n-1}\}$ takes (1.2) into

$$Z_1' = A_1 Z_1 \tag{1.3}$$

with

$$A_1(x) = r \begin{pmatrix} 0 & 1 & 0 & \ldots & \\ & 0 & 1 & & \\ \vdots & & \ddots & & \\ & & & & 1 \\ 1 & & & & 0 \end{pmatrix} - r'/r \begin{pmatrix} 0 & & \ldots & \\ & 1 & & \\ \vdots & & \ddots & \\ & & & n-1 \end{pmatrix}. \tag{1.4}$$

The definition above assumes implicitly that $p(x)$ admits a single valued nth root. Such is the case when, for example, $p(x)$ is real valued and one signed.

It is necessary now to distinguish between three cases according to the relative magnitudes near infinity of the two terms in (1.4); that is, the ratio between $r(x)$

and $-r'(x)/r(x)$:

$$\lim_{x\to\infty}(-\frac{r^2}{r'}) = \lim_{x\to\infty}\left(\frac{p^{n+1}}{(-p'/n)^n}\right)^{1/n} \qquad \text{is zero, finite and nonzero or infinite.}$$

These three different cases and their application to oscillation had been discussed in [1]. In the present work we turn our attention only to the case when

$$\lim_{x\to\infty}\frac{p^{n+1}}{(-p'/n)^n} \qquad \text{is finite and nonzero.}$$

In this case we shall denote

$$\frac{p^{n+1}}{(-p'/n)^n} = \kappa + \Delta(x), \tag{1.5}$$

where $\kappa \neq 0$ is a constant and $\Delta(x) \to 0$ as $x \to \infty$. Such is, for example, the Euler equation $y^{(n)} + \kappa x^{-n} y = 0$.

Introducing $s(x) \equiv -r^2(x)/r'(x)$, (1.4) may be written as

$$A_1(x) = (-r'/r)\begin{pmatrix} 0 & s(x) & 0 & \dots & \\ & 1 & s(x) & & \\ & & 2 & & \\ \vdots & & & \ddots & s(x) \\ s(x) & & & & n-1 \end{pmatrix}. \tag{1.6}$$

The eigenvalues of $(-r/r')A_1(x)$ are the roots of the equation

$$\lambda(\lambda-1)\dots(\lambda-n+1) = \kappa + \Delta(x), \tag{1.7}$$

since $s^n(x) = \left(-r^2/r'\right)^n = p^{n+1}/(-p'/n)^n = \kappa + \Delta(x)$.

Let $s \equiv \lim_{x\to\infty} s(x) = \kappa^{1/n}$. (1.4) may be rewritten also as

$$A_1(x) = (-r'/r)\Big[\begin{pmatrix} 0 & s & 0 & \dots & \\ & 1 & s & & \\ & & 2 & & \\ \vdots & & & \ddots & s \\ s & & & & n-1 \end{pmatrix} + [s(x)-s]\begin{pmatrix} 0 & 1 & 0 & \dots & \\ & 0 & 1 & & \\ \vdots & & \ddots & & \\ & & & & 1 \\ 1 & & & & 0 \end{pmatrix}\Big] \tag{1.8}$$

Since $s(x) - s \to 0$, the last term of this expression is a small perturbation of the first one. The characteristic polynomial of the leading matrix

$$\begin{pmatrix} 0 & s & 0 & \dots & & \\ & 1 & s & & & \\ & & 2 & & & \\ \vdots & & & \ddots & s & \\ s & & & & n-1 & \end{pmatrix} \tag{1.9}$$

is $\lambda(\lambda-1)\dots(\lambda-n+1)-\kappa=0$. If its eigenvalues $\lambda_0,\dots,\lambda_{n-1}$ are all distinct, (1.8) may be diagonalized and under some additional assumptions, an asymptotic expression may be obtained for the solutions of system (1.2). Such result was proved in Theorem 4 of [1], and it was used to analyze the oscillatory behavior of the solutions of (1.1) and the asymptotic distribution of their zeros. However, this result definitely excludes the cases when some eigenvalues of (1.9) coincide and the leading matrix cannot be diagonalized. This excluded unresolved case is the subject of the present work and it will be discussed in Section 2. Section 3 is devoted to the implications of the asymptotics on oscillation theory.

For future applications, we mention an elementary lemma of [1] which summarizes some properties of the eigenvalues of (1.9):

Lemma 1. *The algebraic function $\lambda(\kappa)$ which is defined by*

$$\lambda(\lambda-1)\dots(\lambda-n+1)-\kappa=0 \tag{1.10}$$

has only real critical points κ, which are the local maxima and minima of the polynomial $x(x-1)\dots(x-n+1)$. Exactly 2 branches of $\lambda(\kappa)$ meet at each critical point. For each critical value κ_0, the corresponding double root $\lambda(\kappa_0)$ of (1.10) lies in the real interval $(k-1,k)$, where k is an admissible integer that satisfies the parity condition

$$(-1)^{n-k} = \ sgn[\kappa_0]. \tag{1.11}$$

From now on we enumerate the eigenvalues $\lambda_0,\dots,\lambda_{n-1}$ according to the following convention: For every admissible integer $k,\ 1\le k\le n-1$, which satisfies the parity condition (1.11), a pair of eigenvalues will be enumerated as λ_{k-1},λ_k if they lie on the two branches of (1.10) which intersect in the interval $(k-1,k)$. If $k=0$ is admissible, λ_0 will be the only root of (1.10) in $(-\infty,0)$, and if $k=n$ is admissible, then λ_{n-1} will be the root in $(n-1,\infty)$.

2. Block diagonalization. From now on it will be assumed that κ has one of the critical values such that the equation (1.10) has multiple zeros. This is the case which had been excluded from [1, Theorem 4].

Let the zeros of (1.7) be denoted by $\lambda_i(\Delta)$. Our assumption that (1.10) has multiple zeros means that eigenvalues of $(-r/r')A_1(x)$, coalesce as $x \to \infty$. As mentioned in Lemma 1, the branch points of (1.10) are exactly of multiplicity 2, thus two coalescing branches, say $\lambda_1(\Delta), \lambda_2(\Delta)$, of (1.7) are analytic functions of $\Delta^{1/2}$ near $\Delta = 0$. We also know by the special structure of (1.7) that for $\Delta > 0$ (or $\Delta < 0$) the branches $\lambda_1(\Delta)$, $\lambda_2(\Delta)$ are real valued, while for the other sign of Δ they are complex conjugates. Therefore

$$\lambda_1(\Delta) = \sum a_i\Delta^i + \Delta^{1/2}\sum b_i\Delta^i,$$
$$\lambda_2(\Delta) = \sum a_i\Delta^i - \Delta^{1/2}\sum b_i\Delta^i.$$

Anyway, $\lambda_1(\Delta)\lambda_2(\Delta)$ and $\lambda_1(\Delta) + \lambda_2(\Delta)$ are analytic functions of Δ near $\Delta = 0$.

Now we turn to calculate $\Delta(x)$. Suppose that $p(x)$ is analytic near infinity and close to the coefficient of the Euler equation:

$$p(x) = p_0 x^{-n}\left(1 + \theta_1 x^{-1} + \theta_2 x^{-2} + \ldots\right). \tag{2.1}$$

Then

$$\frac{-r'}{r} = -\frac{p'}{np} = \frac{1}{x}\left(1 + \frac{\theta_1}{n}x^{-1} + \ldots\right),$$
$$\kappa + \Delta(x) = \frac{p^{n+1}}{(-p'/n)^n} = p_0\left(1 + (\frac{n+1}{2n}\theta_1^2 - \theta_2)x^{-2} + \ldots\right).$$

Thus $\kappa = p_0$, and quite luckily,

$$\Delta(x) = \mathcal{O}(x^{-2}) \text{ as } x \to \infty. \tag{2.2}$$

Here $\Delta(x)$ tends to zero from one side as $x \to \infty$.

As long as $\lambda_i(\Delta) \neq \lambda_j(\Delta)$, $A_1(x)$ is diagonalized as

$$(\vec{v}_1 \,|\, \vec{v}_2 \,|\, \ldots)^{-1} A_1(x)\,(\vec{v}_1 \,|\, \vec{v}_2 \,|\, \ldots) = (-\frac{r'}{r})\mathrm{diag}\{\lambda_1(\Delta),\, \lambda_2(\Delta),\, \ldots\} \tag{2.3}$$

where $\vec{v}_i$ is an eigenvector that corresponds to the eigenvalue λ_i :

$$\vec{v}_i = \left(s^{n-1},\, s^{n-2}\lambda_i,\, s^{n-3}\lambda_i(\lambda_i - 1),\, \ldots,\, \lambda_i(\lambda_i - 1)\ldots(\lambda_i - n + 2)\right)^T$$
$$= \mathrm{diag}\{s^{n-1}, \ldots, s, 1\}\,(1,\, \lambda_i,\, \lambda_i(\lambda_i - 1),\, \ldots,\, \lambda_i(\lambda_i - 1)\ldots(\lambda_i - n + 2))^T.$$

When two eigenvalues coalesce, we rather reduce $A_1(x)$ to a block diagonal matrix. Replace $\vec{v}_1, \vec{v}_2$ by their linear combinations

$$\frac{1}{\lambda_2-\lambda_1}(\lambda_2\vec{v}_1-\lambda_1\vec{v}_2), \qquad \frac{1}{\lambda_2-\lambda_1}(-\vec{v}_1+\vec{v}_2)$$

and consider the matrix T whose columns are such combinations of pairs of eigenvectors,

$$\begin{aligned} T &\equiv (\vec{v}_1 \mid \vec{v}_2 \mid \ldots) \begin{pmatrix} \frac{\lambda_2}{\lambda_2-\lambda_1} & \frac{-1}{\lambda_2-\lambda_1} & 0 & \cdots \\ \frac{-\lambda_1}{\lambda_2-\lambda_1} & \frac{1}{\lambda_2-\lambda_1} & 0 & \cdots \\ 0 & 0 & 1 & \\ \vdots & \vdots & & \ddots \end{pmatrix} \\ &= \mathrm{diag}\{s^{n-1},\ldots,s,1\} \begin{pmatrix} 1 & 0 & \cdots \\ 0 & 1 & \\ -\lambda_1\lambda_2 & \lambda_1+\lambda_2-1 & \\ \lambda_1\lambda_2(\lambda_1+\lambda_2-3) & (\lambda_1+\lambda_2-1)(\lambda_1+\lambda_2-2)-\lambda_1\lambda_2 & \\ \vdots & \vdots & \end{pmatrix}. \end{aligned} \tag{2.4}$$

It is clear by the definitions of $\vec{v}_1, \vec{v}_2$ and T that the terms in its first two columns are symmetric polynomials of λ_1, λ_2. But we had already remarked that $\lambda_1(\Delta)\lambda_2(\Delta)$ and $\lambda_1(\Delta)+\lambda_2(\Delta)$ are analytic functions of Δ near $\Delta=0$, thus the matrix T, too, is analytic in Δ even when two eigenvalues coalesce as $x\to\infty$ and $\lambda_1(\Delta), \lambda_2(\Delta)$ are not analytic.

The transformation $Z_1 = TZ_2$ takes (1.3) into

$$Z_2' = A_2 Z_2, \qquad A_2 = T^{-1}A_1T - T^{-1}T' \tag{2.5}$$

Combining with (2.3),

$$\begin{aligned} &T^{-1}A_1T \\ &= \begin{pmatrix} \frac{\lambda_2}{\lambda_2-\lambda_1} & \frac{-1}{\lambda_2-\lambda_1} & 0 & \cdots \\ \frac{-\lambda_1}{\lambda_2-\lambda_1} & \frac{1}{\lambda_2-\lambda_1} & 0 & \cdots \\ 0 & 0 & 1 & \\ \vdots & \vdots & & \ddots \end{pmatrix}^{-1} (-\frac{r'}{r}) \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{pmatrix} \begin{pmatrix} \frac{\lambda_2}{\lambda_2-\lambda_1} & \frac{-1}{\lambda_2-\lambda_1} & 0 & \cdots \\ \frac{-\lambda_1}{\lambda_2-\lambda_1} & \frac{1}{\lambda_2-\lambda_1} & 0 & \cdots \\ 0 & 0 & 1 & \\ \vdots & \vdots & & \ddots \end{pmatrix} \\ &= (-\frac{r'}{r}) \begin{pmatrix} 0 & -1 & 0 & \ldots & 0 \\ \lambda_1\lambda_2 & \lambda_1+\lambda_2 & 0 & \ldots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & & \end{pmatrix}. \end{aligned} \tag{2.6}$$

Since $\lambda_1(\Delta)\lambda_2(\Delta)$, $\lambda_1(\Delta)+\lambda_2(\Delta)$ and T are analytic functions of Δ near $\Delta = 0$, and $\Delta(x) = \mathcal{O}(x^{-2})$ as $x \to \infty$,

$$\lambda_1\lambda_2 = \lambda_1^2(0) + \mathcal{O}(x^{-2}), \qquad \lambda_1 + \lambda_2 = 2\lambda_1(0) + \mathcal{O}(x^{-2}),$$

where $\lambda_1(0) = \lambda_1(\Delta(\infty))$; also $s^n(x) = \kappa + \Delta(x) = \kappa + \mathcal{O}(x^{-2})$ so

$$\mathrm{diag}\{s^{n-1}(x), \dots, s(x), 1\} = \ \mathrm{diag}\{s^{n-1}, \dots, s, 1\} + \mathcal{O}(x^{-2})$$

and

$$T(x) = T(\infty) + O(x^{-2}), \qquad T'(x) = \mathcal{O}(x^{-3}) \ \text{ as } x \to \infty.$$

In Equ. (2.5) the matrix coefficient will be

$$A_2 = T^{-1}A_1T - T^{-1}T'$$
$$= \frac{1}{x}\left(1 + \frac{\theta_1}{n}x^{-1} + \dots\right)\begin{pmatrix} 0 & -1 & 0 & \dots \\ \lambda_1^2(0) + \mathcal{O}(x^{-2}) & 2\lambda_1(0) + \mathcal{O}(x^{-2}) & 0 & \dots \\ 0 & 0 & & \\ \vdots & \vdots & & \end{pmatrix} + \mathcal{O}(x^{-3}). \tag{2.7}$$

The following lemma may be used to reduce (2.5) into a more simple system.

Lemma 2. *Given two differential systems*

$$Z' = M(x)Z, \qquad M(x) = x^{-1}\sum_{i=0}^{\infty} M_i x^{-i}, \tag{2.8}$$

$$W' = N(x)W, \qquad N(x) = x^{-1}\sum_{i=0}^{\infty} N_i x^{-i} \tag{2.9}$$

near infinity, such that $M_0 = N_0$ and no two eigenvalues of M_0 differ by a positive integer. Then there exists a transformation

$$Z = (I + P(x))W, \qquad P(x) = \sum_{i=1}^{\infty} P_i x^{-i},$$

which reduces (2.8) to (2.9). The P_i's are determined by the recursion

$$P_\nu(M_0 - \nu I) - M_0P_\nu = M_\nu - N_\nu + \sum_{k=1}^{\nu-1}(M_{\nu-k}P_k - P_kN_{\nu-k}), \quad \nu = 1, 2, \dots \tag{2.10}$$

The proof is similar to the methods of [5, Chapter 5].

We are now quite free to choose the transformed equation. For example, we may take $M(x) = T^{-1}A_1T - T^{-1}T'$ as in (2.5) and $N(x) = T^{-1}A_1T$. According to this choice we disregard third order terms, since $M(x) - N(x) = -T^{-1}T' = \mathcal{O}(x^{-3})$. It follows that $P_1 = 0$, $Z = (I + P_2x^{-2} + \ldots)W$, P_2 is the solution of

$$P_2(M_0 - 2I) - M_0P_2 = N_2 - M_2, \tag{2.11}$$

and $N_2 - M_2$ is the coefficient of x^{-3} in $-T^{-1}T'$. Here W is a solution of the block diagonal system

$$W' = (-\frac{r'}{r})\begin{pmatrix} 0 & -1 & 0 & \ldots & 0 \\ \lambda_1\lambda_2 & \lambda_1+\lambda_2 & 0 & \ldots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & & \end{pmatrix} W. \tag{2.12}$$

and the solution of (1.2) is

$$Y(x) = \ \mathrm{diag}\{1, r, \ldots, r^{n-1}\}T(x)(I + \mathcal{O}(x^{-2}))W(x).$$

Thus, our problem is reduced essentially to the study of the 2×2 system

$$V' = (-\frac{r'}{r})\begin{pmatrix} 0 & -1 \\ \lambda_1\lambda_2 & \lambda_1+\lambda_2 \end{pmatrix} V. \tag{2.13}$$

This system has a singular-regular point and two coalescing eigenvalues at infinity.

3. Oscillation. In the literature of oscillation theory there are numerous "comparison theorems" of the following typical form:

If the equation $y^{(n)} = p_0(x)y$ is nonoscillatory/disconjugate/disfocal/etc, and $p(x)$ is "smaller" in a certain sense than $p_0(x)$, then the equation $y^{(n)} = p(x)y$ too has the same property.

A choice of $p_0(x)$ from a well known or easily solved equation leads to an explicit sufficient criterion for nonoscillation (disconjugacy, disfocality, ...). Interchanging the roles of $p_0(x)$ and $p(x)$ leads to a necessary criterion for the same property and as a rule there is a gap between the necessary condition and the sufficient one. A simple result of this type is the Hille-Kneser criteria [3]:

Let

$$\omega^* = \limsup x^2 p(x), \qquad \omega_* = \liminf x^2 p(x).$$

Then the equation $y'' + p(x)y = 0$ is nonoscillatory on (a, ∞) if $\omega^* < 1/4$ and oscillatory if $\omega_* > 1/4$, but it can be either oscillatory or nonoscillatory if $\lim x^2 p(x) = 1/4$.

Another result of the same flavor is the following [4, Corollary 1 of Section 4] which may be reformulated as following:

Let k be an integer of admissible parity, $M_k = \max_{[k-1,k]} |\lambda(\lambda-1)\dots(\lambda-n+1)|$ and

$$\omega^* = \limsup(-1)^{n-k} x^n p(x), \qquad \omega_* = \liminf(-1)^{n-k} x^n p(x).$$

Then equation (1.1) is eventually $(k, n-k)$-disfocal if $\omega^* < M_k$ and it is not $(k, n-k)$-disfocal on any ray (c, ∞) if $\omega_* > M_k$. If $\omega^* = \omega_* = \lim(-1)^{n-k} x^n p(x) = M_k$ the test is inconclusive.

Our block-diagonalization technique yields some new information about the inconclusive case in the Hille-Kneser type criteria. While (2.13) is an intriguing system of its own sake, a more primitive choice of $N(x)$ will suffice in order to discuss the oscillatory properties of (1.1). We may take $M(x)$ as above, and $N(x)$ as its x^{-1} term,

$$N(x) = \frac{1}{x} \begin{pmatrix} 0 & -1 & 0 & \dots \\ \lambda_1^2(0) & 2\lambda_1(0) & 0 & \dots \\ 0 & 0 & & \\ \vdots & \vdots & & \end{pmatrix}.$$

The corresponding system $W' = N(x)W$ has a singular-regular point and two equal, real-valued eigenvalues $\lambda_1(0) = \lambda_2(0)$. After reducing it to Jordan form, its solution is found to be

$$W(x) = \begin{pmatrix} 1 & 0 & 0 & \dots \\ -\lambda_1(0) & -1 & 0 & \dots \\ 0 & 0 & & \\ \vdots & \vdots & & \end{pmatrix} \begin{pmatrix} x^{\lambda_1(0)} & x^{\lambda_1(0)} \log(x^{-1}) & 0 & \dots \\ 0 & x^{\lambda_1(0)} & 0 & \dots \\ 0 & 0 & & \\ \vdots & \vdots & & \end{pmatrix}.$$

Thus

$$Y(x) = \operatorname{diag}\{1, r, \dots, r^{n-1}\} T(x)(I + \mathcal{O}(x^{-2}))W(x).$$

From here it is obvious that to our 2×2 block there correspond two solutions of (1.1) of the shape

$$x^{\lambda_1(0)}(1 + \mathcal{O}(x^{-2})) \quad \text{and} \quad x^{\lambda_1(0)} \log(x^{-1})(1 + \mathcal{O}(x^{-2})). \tag{3.1}$$

Now, it is known that every nonoscillatory solution of (1.1) must satisfy

$$\begin{aligned} y^{(i)} &> 0, \qquad i = 0, \dots, k-1, \\ (-1)^{i-k} y^{(i)} &> 0, \qquad i = k, \dots, n \end{aligned} \tag{3.2}$$

for some integer k such that $(-1)^{n-k} p \geq 0$, and this system of inequalities is a necessary and sufficient condition for eventual $(k, n-k)$-disfocality. The nonoscillatory solutions (3.1) are compatible with (3.2) if and only if $k - 1 \leq \lambda_1(0) \leq k$. In our case $\lambda_1(0) = \lambda_2(0)$ is the double zero of the equation

$$\lambda(\lambda - 1) \dots (\lambda - n + 1) = \kappa$$

which is located in $[k-1, k]$, and this happens if and only if

$$(-1)^{n-k} \kappa = M_k = \max_{[k-1,k]} (-1)^{n-k} \lambda(\lambda - 1) \dots (\lambda - n + 1).$$

Thus we conclude that (1.1) is eventually $(k, n-k)$-disfocal even when

$$\lim_{x \to \infty} \frac{p^{n+1}}{(-p'/n)^n} = (-1)^{n-k} M_k.$$

On the other hand, when $p(x)$ is real analytic at $+\infty$, both limits

$$\lim_{x \to \infty} \frac{p^{n+1}}{(-p'/n)^n} = \kappa \qquad \text{and} \qquad \lim_{x \to \infty} x^n p(x) = \kappa$$

exist and are equivalent. When we combine with the known results about $\omega^* < M_k$ and $\omega_* > M_k$ with the above conclusion about $(-1)^{n-k}\kappa = M_k$, we may summarize

Theorem. *Let $p(x)$ be a real analytic function at $x = +\infty$ and k an integer such that $(-1)^{n-k} = sgn\,[p(x)]$. Then (1.1) is eventually $(k, n-k)$-disfocal if and only if*

$$0 \leq \lim (-1)^{n-k} x^n p(x) \leq M_k.$$

The new feature of this result is that it is both necessary and sufficient. This is achieved at the additional price of restricting $p(x)$ to the class of analytic functions. In the particular case $n = 2$ the necessary and sufficient condition is derived in [2].

If M_k is replaced by $M = \min\{M_k\}$, the minimum taken over the admissible values of $k, 1 \leq k \leq n-1$, we get a necessary and sufficient condition for nonoscillation.

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