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Oscillation of two-term differential equations through asymptotics

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§**1.** The aim of this paper is to study the oscillation theory of a two-term ordinary differential equation through its asymptotic solutions. The equation under consideration will be of the form

$$y^{(n)} = p(x)y, \tag{1.1}$$

where $p(x)$ is a sufficiently smooth, possibly complex valued function on an interval $[a, \infty)$. Asymptotic solutions for such equations are discussed in several works, e.g. [4-9]. The qualitative behaviour and the asymptotic representations of solutions of (1.1) are completely different according if $p(x)$ is "large", say $cx^{-(n-\epsilon)}; p(x)$ is "small", say $cx^{-(n+\epsilon)}$; and the intermediate case when $p(x)$ is close to cx^{-n}. When the coefficient $p(x)$ is "large", all the solutions of (1.1) which may potentially oscillate - do oscillate (For the rigorous meaning of this statement see Theorem 3). In the second case the equation is disconjugate and none of its solutions is oscillatory. And in the intermediate case some solutions may oscillate while the others don't. The asymptotics of the various cases were studied in $[4, 8, 5, 9].A$ summary of all three cases is found in [1, Chapter 2]. The domains of validity of the various asymptotic formulas in the literature do not overlap, and there are gaps among them waiting for further work. In this note we shall refine the asymptotic solutions for Eq. (1.1) in some cases when $p(x)$ is quite close to cx^{-n} and apply the results to study the oscillation of (1.1).

The main tool to obtain asymptotic solutions is the *Levinson theorem* [10] which we quote here as stated in [1]:

LEVINSON'S THEOREM. *Let $\Lambda(x)$ be an $n \times n$ diagonal matrix, $\Lambda(x) = diag$ $\{\lambda_1(x), \ldots, \lambda_n(x)\}$, and for each pair of integers $i \neq j$ and for all $t < x$,*

$$\int_t^x \Re\{\lambda_i(s) - \lambda_j(s)\} ds$$

is bounded either from above or from below.

Let the $n \times n$ matrix $R(x)$ satisfy

$$\int^\infty |R(x)| dx < \infty.$$

Then, as $x \to \infty$, the system $Y'(x) = \{\Lambda(x) + R(x)\} Y(x)$ has a matrix solution $Y(x)$ with the asymptotic form

$$Y(x) = \{I + o(1)\} \; diag \left\{ \exp \int^x \lambda_1(t) dt, \ldots \right\}.$$

This theorem will be used below.

The first step in handling the asymptotics of the scalar differential Eq. (1.1) is to convert it into a vectorial differential system $\bar{y}' = A(x)\bar{y}$ for the vector $\bar{y} = (y, y', \ldots, y^{(n-1)})$. The companion matrix A(x) is

$$A = \begin{pmatrix} 0 & 1 & 0 & & \\ & 0 & 1 & & \\ \vdots & & & \ddots & \\ & & & & 1 \\ p(x) & & & & 0 \end{pmatrix}.$$

From now on we consider the associated matrix differential system

$$Y' = A(x) Y. \tag{1.2}$$

The substitution $Y(x) = V(x) Z_1$ transforms Eq. (1.2) into

$$Z_1' = A_1 Z_1 \tag{1.3}$$

with $A_1 = V^{-1} A V - V^{-1} V'$.

Let $r(x) \equiv p(x)^{1/n}$ be one fixed branch of the nth root. This definition assumes implicitly that $p(x)$ admits a single valued nth root. Such is the case when, for example, $p(x)$ is real valued and one signed. In certain cases, say, when $p(x) < 0$, it may be more convenient to take $r = (-p)^{1/n}$. If we choose $V = \mathrm{diag}\ \{1, r, \dots, r^{n-1}\}$, then

$$V' = r' \ \mathrm{diag}\{0, 1, 2r, \dots, (n-1)r^{n-2}\},$$
$$V^{-1}V' = r'/r \ \mathrm{diag}\{0, 1, \dots, n-1\}$$

and

$$A_1(x) = r \begin{pmatrix} 0 & 1 & 0 & & \\ & 0 & 1 & & \\ \vdots & & & \ddots & 1 \\ p(x) & & & & 0 \end{pmatrix} - r'/r \begin{pmatrix} 0 & & \dots & \\ & 1 & & \\ \vdots & & \ddots & \\ & & & n-1 \end{pmatrix}. \tag{1.4}$$

From now on we have to separate between distinct cases according to the relative magnitudes near infinity of the two terms in (1.4); that is , the ratio between $r(x)$ and $r'(x)/r(x)$:

$$\lim_{x\to\infty} r^2(x)/r'(x) \quad \text{is zero, finite and nonzero or infinite.}$$

If the limit does not exist at all, the equation is not fit for our analysis.

§**2.** In this section we consider Eq. (1.1) when

$$\lim_{x\to\infty} r'(x)/r^2(x) = 0$$

This case had been already studied in [4,8,5,1]. It is known that if

$$|r'^2(x)/r^3(x)|, \quad |\left(r'(x)/r^2(x)\right)'| \in L_1(a, \infty),$$

then (1) has asymptotic solutions

$$y_\ell = (1 + o(1))p^{-\frac{n-1}{2n}} \exp \int^x \omega_\ell |p|^{1/n},$$

where ω_ℓ is the ℓth root of unity. The above assumptions hold if $p(x)$ is not too close to cx^{-n}, say, if $p(x) = cx^{-n+\epsilon}, \epsilon > 0$. However, they do not hold, for example, for $p(x) = x^{-n}\log^\beta x$, $0 < \beta \le n$, that is, $r(x) = x^{-1}\log^\alpha x$, $0 < \alpha \le 1$, since then r'^2/r^3 is not integrable. In the following theorem we prefer to assume that $r'^3/r^5 \in \mathbf{L}_1(a, \infty)$ and to include an additional term in the integral.

Theorem 1. *Let*

$$(i) \quad r'(x)/r^2(x) \to 0$$
$$(ii) \quad |r'^3(x)/r^5(x)|, \ |\left(r'(x)/r^2(x)\right)'| \in L_1(a,\infty).$$

Then system (1.2) has the asymptotic solution

$$Y = diag\{1, r, \ldots, r^{n-1}\} W(I + o(1)) \\ \times \ diag\left\{ r^{-\frac{n-1}{2}} \exp \int^x \left(p^{1/n} + \frac{n^2-1}{24n^2} \frac{p'^2}{p^{2+1/n}}, \right) \ldots \right\}, \tag{2.1}$$

where $W = \left(e^{\ell i \theta_k}\right)_{\ell,k=0}^{n-1}$ *and* $()^{1/n}$ *standing under the* exp *at the* (ℓ,ℓ)*th position of the second diagonal matrix is taken as the* ℓ*th branch of the root. The solution may be written equivalently as*

$$Y = diag\{1, r, \ldots, r^{n-1}\} W(I + o(1)) \\ \times \ diag\left\{ r^{-\frac{n-1}{2}} \exp \int^x \left(p + \frac{n^2-1}{24n} \frac{p'^2}{p^{1+1/n}}, \right)^{1/n} \ldots \right\}, \tag{2.2}$$

Remark. The last form has the desirable property that for $n = 2$ it reads

$$Y = \ \text{diag} \ \{1, p^{1/2}\} W(I + o(1)) \ \text{diag} \left\{ p^{-1/4} \exp \int^x \left(p + p'^2/16p^2\right)^{1/2}, \ldots \right\}$$

which is a known improvement of the WKB formula by Hartman and Wintner [7], [1, Section 2.2].

Proof. According to our assumption, the first term of (1.4) is the dominant one while the second may be considered as its small perturbation. The eigenvalues of the first matrix are $e^{i\theta_k}$, $k = 0, \ldots, n-1$, the nth roots of 1, and it may be diagonalized by a Fourier matrix of eigenvalues

$$W = \left(e^{\ell i \theta_k}\right)_{\ell,k=0}^{n-1} = \begin{pmatrix} 1 & 1 & \ldots. \\ e^{i\theta_0} & e^{i\theta_1} & \ldots. \\ e^{2i\theta_0} & e^{2i\theta_1} & \ldots. \\ \vdots & & \end{pmatrix}. \tag{2.3}$$

Thus, $Z_1 = W Z_2$ transforms (1.3) into $Z_2' = A_2 Z_2$, with

$$A_2(x) = r(x) \ \text{diag}\{e^{i\theta_1}, \ldots\} - r'/r W^{-1} \ \text{diag}\{0, 1, \ldots, n-1\} W.$$

Next we simplify the second term of A_2. We have $W^*W = WW^* = nI$, since

$$(WW^*)_{k,\ell} = \sum_{j=0}^{n-1} e^{ki\theta_j}\overline{e^{\ell i\theta_j}} = \sum e^{(k-\ell)i\theta_j} = n\delta_{k,\ell}.$$

Thus, the matrix in the second term of $A_1(x)$ may be written as $1/nW^*$ diag$\{0,1,\ldots,n-1\}W$. The (k,k)th term of this Hermitian matrix is

$$1/n\sum_{j=0}^{n-1}\overline{e^{ji\theta_k}}e^{ji\theta_k} = 1/n\sum_{j=0}^{n-1} j = (n-1)/2,$$

hence by the scalar transformation $Z_2 = r(x)^{\frac{n-1}{2}}Z_3$ we get a similar differential system with coefficient $A_3(x) = A_2(x) - \frac{n-1}{2}I$. Hence

$$r^{-1}(x)A_3(x) = \text{diag}\{e^{i\theta_0},\ldots\} - r'/r^2 M,$$

where

$$M = 1/nW^* \ \text{diag}\{0,1,\ldots,n-1\}W - \frac{n-1}{2}I,$$

and the main diagonal of M consists of zeros. For short let us denote $\epsilon(x) \equiv -r'/r^2$.

Lemma 1. *The eigenvalues of diag* $\{e^{i\theta_0},\ldots\} + \epsilon M$ *are*

$$\mu_\ell = e^{i\theta_\ell} + \frac{n^2-1}{24}e^{-i\theta_\ell}\epsilon^2 + \mathcal{O}(\epsilon^3), \qquad \ell = 0,\ldots,n-1.$$

Since the proof of the lemma is not related to the main idea of the theorem, it is delayed to the end of the section.

Lemma 2. *Let* $d_0,\ldots,d_{n-1}$ *be distinct numbers and* $\epsilon(x)\to 0$. *There exists a matrix* $T(x) = I + Q(x)$, $Q(x) = O(\epsilon)$ *and diag*$Q(x)\equiv 0$, *such that*

$$T^{-1}\ \textit{diag}\{d_0,\ldots\} + \epsilon(x)MT = \ \textit{diag}\{\mu_1(x),\ldots\}.$$

Moreover,

$$T^{-1}T' = O(\epsilon').$$

This is a standard result. See [6, Lemma 1].

Returning to the proof of the theorem, we make the transformation $Z_3 = TZ_4$ which results in our case

$$
\begin{aligned}
A_4 &= T^{-1}A_3T - T^{-1}T' \\
&= r(x)\ \mathrm{diag}\ \{\mu_0(x), \dots, \} - O(\epsilon') \\
&= \mathrm{diag}\left\{ r\left(e^{i\theta_0} + \frac{n^2-1}{24} e^{-i\theta_0}(r'/r^2)^2 + O((r'/r^2)^3) \right), \dots \right\} + O((r'/r^2)') \\
&= \mathrm{diag}\left\{ \left(p^{1/n} + \frac{n^2-1}{24n^2}\frac{p'^2}{p^{2+1/n}} \right), \dots \right\} + O((r'/r^2)^3) + O((r'/r^2)'),
\end{aligned}
$$

where $p^{1/n}$ standing at the ℓth position of the diagonal matrix means the ℓth branch of the root. The last equality results from $re^{i\theta} = p^{1/n}$, $r'e^{i\theta} = p'p^{1/n-1}/n$ for the corresponding branch of $p^{1/n}$. According to our assumptions the conditions of the Levinson Theorem are satisfied and

$$
Z_4(x) = (I + o(1))\ \mathrm{diag}\left\{ \exp \int^x \left(p^{1/n} + \frac{n^2-1}{24n^2}\frac{p'^2}{p^{2+1/n}} \right), \dots \right\}.
$$

When we substitute successively Z_3, Z_2, Z_1 and Y, formula (2.1) of Theorem 1 follows. The integrand in the diagonal term,

$$
p^{1/n}\left(1 + \frac{n^2-1}{24n^2}\frac{p'^2}{p^{2+2/n}} \right),
$$

differs from

$$
p^{1/n}\left(1 + \frac{n^2-1}{24n}\frac{p'^2}{p^{2+2/n}} \right)^{1/n}
$$

by

$$
\mathcal{O}\left(p^{1/n}\frac{p'^4}{p^{4+4/n}} \right) = \mathcal{O}\left(r'^4/r^7 \right),
$$

which is integrable according to our assumptions. Thus the required formula (2.2) is proved.

Proof of Lemma 1. The characteristic equation of $\mathrm{diag}\{d_0, \dots, d_{n-1}\} + \epsilon M$ is

$$
\det \begin{pmatrix} d_0 - \lambda & \epsilon m_{01} & \dots & \epsilon m_{0,n-1} \\ \epsilon m_{10} & d_1 - \lambda & & \\ \vdots & & \ddots & \\ \epsilon m_{0,n-1} & \dots & & d_{n-1} - \lambda \end{pmatrix}
$$

$$
= \prod (d_i - \lambda) + \epsilon^2 \sum_{j \neq k} m_{jk} m_{kj} \left[\prod_{i \neq j,k} (d_i - \lambda) \right] + O(\epsilon^3) = 0
$$

and its ℓth root is obviously $\lambda_\ell = d_\ell + c\epsilon^2 + O(\epsilon^3)$. When one puts this back into the equation and compares powers of ϵ, the result is

$$\lambda_\ell = d_\ell + \epsilon \sum_{k \neq \ell} m_{\ell k} m_{k\ell}/(d_\ell - d_k) + O(\epsilon^3).$$

In our case $m_{k\ell} = 1/n \sum_{j=0}^{n-1} W_{kj}^* j W_{j\ell} = 1/n \sum_{j=0}^{n-1} j e^{-ji\theta_k} e^{ji\theta_\ell}$, and utilizing that those are nth roots of unity, a short calculation results in $m_{k\ell} = \left(e^{i(\theta_\ell - \theta_k)} - 1\right)^{-1}$ and $m_{k\ell} m_{\ell k} = (2 - 2\cos(\theta_\ell - \theta_k))^{-1}$. Thus $\sum_{k\neq\ell} m_{\ell k} m_{k\ell}/(d_\ell - d_k) = 1/2 e^{-i\theta_{ell}} \sum_{k\neq\ell}(1 - \cos(\theta_\ell - \theta_k))^{-1}(1 - e^{i(\theta_k - \theta_\ell)})^{-1}$. As $\theta_\ell - \theta_k = 2\pi(\ell - k)/n$, the last expression equals $1/4 e^{-i\theta_\ell} \sum_{k=1}^{n-1}(1 - \cos(2\pi k/n))^{-1}$. Now $\sum_{k=1}^{n} 1/(1 - x_k) = P_{n-1}'(1)/P_{n-1}(1)$, where $P_{n-1}(x) = \prod_{k=1}^{n-1}(x - x_k)$. So all we need is a polynomial whose roots are $\theta_k = \cos(2\pi k/n)$, $k = 1, \ldots, n-1$. Now

$$\begin{aligned}&(\sin(n\theta/2)/\sin(\theta/2))^2 \\ &\quad = n + 2(n-1)\cos\theta + 2(n-2)\cos(2\theta) + \ldots + 2\cos(n-1)\theta\end{aligned}$$

with $x = \cos(\theta)$ is such $P_{n-1}(x)$. [12, Vol. II, p. 74]. A direct calculation gives $P_{n-1}(1) = n^2$, $P_{n-1}(1) = n^2(n^2 - 1)$, and the lemma follows.

§**3.** When $p(x)$ is real valued, Theorem 1 reveals information about the oscillation of Eq. (1.1) and the distribution of the zeros if its solutions. In this case, $r'/r^2 \to 0$ implies that $p(x)$ does not vanish eventually and is of a fixed sign. Thus, for real $p(x)$ it is more convenient to take $r \equiv |p|^{1/n}$ and $p^{1/n} = |p|^{1/n} e^{i\theta_k}$, $k = 0, \ldots, n-1$, where $e^{i\theta_k}$ are the nth roots of sgn$[p]$. The solution of system (1.2),

$$\begin{aligned} Y = &\ \operatorname{diag}\{1, r, \ldots, r^{n-1}\} W(I + o(1)) \\ \times &\ \operatorname{diag}\left\{ r^{\frac{n-1}{2}} \exp \int^x \left(p^{1/n} + \frac{n^2-1}{24n^2}\frac{p'^2}{p^{2+1/n}}\right), \ldots \right\}, \end{aligned}$$

may be written as

$$\begin{aligned} Y_{\ell j} &= r^\ell e^{i\ell\theta_j}(1 + o(1)) r^{\frac{n-1}{2}} \exp\left[\int^x \left(re^{i\theta_j} + \frac{n^2-1}{24n^2}\frac{p'^2}{p^2} r^{-1} e^{-i\theta_j}\right)\right] = \\ &= r^{-\frac{n-1}{2}+\ell} \exp\left[i\ell\theta_j + \int^x \left(re^{i\theta_j} + \frac{n^2-1}{24n^2}\frac{p'^2}{p^2} r^{-1} e^{-i\theta_j}\right) + o(1)\right], \end{aligned}$$

$\ell, j = 0, \ldots, n-1$. Since $p(x)$ is real valued, the real and imaginary parts of a solution are solutions too. If we denote

$$I(x) = \int^x \left(r + \frac{n^2-1}{24n^2}\frac{p'^2}{p^2} r^{-1}\right), \qquad J(x) = \int^x \left(r - \frac{n^2-1}{24n^2}\frac{p'^2}{p^2} r^{-1}\right), \qquad (3.1)$$

we may get real solutions and their ℓth derivatives, $y^{(\ell)}$, $\ell = 0, \ldots, n-1$, of the form

$$
\begin{aligned}
&r^{-\frac{n-1}{2}+\ell} \exp\left[\cos\theta_j I(x) + o(1)\right] \cos\left[\sin\theta_j J(x) + \ell\theta_j + o(1)\right], \\
&r^{-\frac{n-1}{2}+\ell} \exp\left[\cos\theta_j I(x) + o(1)\right] \sin\left[\sin\theta_j J(x) + \ell\theta_j + o(1)\right].
\end{aligned}
\tag{3.2}
$$

Next we shall show how are the above solutions related to the theory developed in [2,3]. Let $S(c_0, c_1, \ldots, c_n)$ denote the number of sign changes in the sequence of the non-zero numbers $c_0, \ldots, c_n$, and for each solution y of (1.1), we define the functional

$$
S(y, x^+) \equiv \lim_{\epsilon \to 0+} S(y(x+\epsilon), -y'(x+\epsilon), y''(x+\epsilon), \ldots, (-1)^n y^{(n)}(x+\epsilon)).
$$

For each fixed solution y of (1.1), $S(y, x^+)$ may be considered as a function of x, and this function is used to study the qualitative behaviour of the solutions. It was proved in [2,3] that for each solution y, $S(y, x^+)$ is a nondecreasing function for $x \in [a, \infty)$ and there exist a ray (x_y, ∞) on which its value is ultimately constant. This terminal value is an integer $k, 0 \leq k \leq n$, which satisfies the parity condition.

$$
(-1)^{n-k} = \operatorname{sgn}[p].
$$

Such an integer is called *admissible.* For example, one has for the solutions of the equation $y^{(4)} = y$,

$$
\begin{aligned}
S(e^{-x}, x^+) &= S(e^{-x}, e^{-x}, e^{-x}, e^{-x}, e^{-x}) = 0, \\
S(\sin x, x^+) &= S(\cos x, x^+) = 2, \\
S(e^{x}, x^+) &= 4,
\end{aligned}
$$

for every x, while for the solutions of $y^{(4)} = -4y$,

$$
\begin{aligned}
S(e^{-x}\sin x, x^+) &= S(e^{-x}\cos x, x^+) = 1, \\
S(e^{x}\sin x, x^+) &= S(e^{x}\cos x, x^+) = 3.
\end{aligned}
$$

The quantity $S(y, x^+)$ was utilized in [2,3] to present a basis $\{u_0, \ldots, u_{n-1}\}$ of the solution space of (1.1) which reflects the oscillatory nature of the solutions and their behaviour near infinity. The following was shown:

(i) For every admissible integer k, the elements of this particular basis satisfy

$$
S(c_1 u_{k-1} + c_2 u_k, x^+) = k \text{ on } [a, \infty) \tag{3.3}
$$

for every c_1, c_2.

(ii) For every pair of admissible integers $r < k$, a corresponding minor of the Wronskian does not vanish:

$$W(u_{r-1}, u_r, \ldots, u_{k-1}, u_k) \neq 0. \tag{3.4}$$

(iii) If k is an admissible integer and u_{k-1}, u_k happen to be nonoscillatory then each of them satisfies

$$\begin{aligned} u^{(\ell)}u^{(\ell+1)} &> 0, \qquad \ell = 0, \ldots, k-1, \\ u^{(\ell)}u^{(\ell+1)} &< 0, \qquad \ell = k, \ldots, n-1. \end{aligned}$$

Hence, for every two admissible integers $r < k$, if u_{r-1}, u_r, and u_{k-1}, u_k are nonoscillatory then $u_{r-1}, u_r = o(u_{k-1}), o(u_k)$ near infinity. There arises the question how should one compare the rate of growth of oscillatory solutions? One possible attempt to answer this question in [3] was to show that if $r < k$ are two admissible integers, then

$$W(u_{r-1}, u_r) = \mathcal{O}(W(u_{k-1}, u_k)) \text{ as } x \to \infty, \tag{3.5}$$

without any assumption about the oscillatory nature of the solutions (and for a more general equation than (1.1)).

The solutions (3.2) that result from our asymptotic formulas enjoy similar properties, provided they are arranged in a suitable order. Let us rename the n arguments of the nth root of sgn $[p]$ so that

$$-1 \leq \cos\theta_0 \leq \cos\theta_1 \leq \ldots \leq \cos\theta_{n-1} \leq 1.$$

It is easily seen that for an admissible k, that is when $(-1)^{n-k} = \operatorname{sgn}[p]$, one may take

$$\theta_{k-1} = \pi - \pi k/n, \qquad \theta_k = -\pi + \pi k/n,$$

and thus

$$\ldots = \cos\theta_{k-2} < \cos\theta_{k-1} = \cos\theta_k < \cos\theta_{k+1} = \ldots \tag{3.6}$$

If $k = 0$ is admissible, i.e. $(-1)^n p(x) > 0$, then $-1 = \cos\theta_0 < \cos\theta_1$, and if $k = n$ is admissible, that is $p(x) > 0$, then $\cos\theta_{n-2} < \cos\theta_{n-1} = 1$. (By convention, in (3.6) disregard θ_{-1}, θ_n, which are not defined at all). Hence, for an admissible integer k, $1 \leq k \leq n-1$, we may choose a pair of solutions v_{k-1}, v_k such that

$$v_{k-1}^{(\ell)} = r^{\frac{n-1}{2}+\ell} \exp\left[-\cos\frac{\pi k}{n} I(x) + o(1)\right] \cos\left[\sin\frac{\pi k}{n} J(x) + \ell(\pi - \frac{\pi k}{n}) + o(1)\right] \tag*{$(3.7)_{k-1}$}$$

$$v_k^{(\ell)} = r^{\frac{n-1}{2}+\ell} \exp\left[-\cos\frac{\pi k}{n} I(x) + o(1)\right] \sin\left[\sin\frac{\pi k}{n} J(x) + \ell(\pi - \frac{\pi k}{n}) + o(1)\right] \tag{3.7$_k$}$$

If $k = 0$ is admissible, and $-1 = \cos\theta_0 < \cos\theta_1$, we take

$$v_0^{(\ell)} = (-1)^\ell r^{-\frac{n-1}{2}+\ell} \exp\left[-I(x) + o(1)\right]. \tag{3.7$_0$}$$

$|v_0^{(\ell)}|$ are decreasing functions since $v_0^{(\ell)} v_0^{(\ell+1)} < 0$. Finally, if $k = n$ is admissible, and $\cos\theta_{n-2} < \cos\theta_{n-1} = 1$, we have

$$v_{n-1}^{(\ell)} = r^{-\frac{n-1}{2}+\ell} \exp\left[I(x) + o(1)\right]. \tag{3.7$_{n-1}$}$$

v_{n-1} and its derivatives are increasing functions. $(3.7)_j$ hold even for $\ell = n$, since $r^n(-1)^{n-k} = |p|\ \mathrm{sgn}[p] = p$.

The basis $\{v_0, \ldots, v_{n-1}\}$ that is defined by $(3.7)_0 - (3.7)_{n-1}$ is not identical with the basis used in [2,3], but they are closely related. Their similar properties are as following:

Theorem 2. *Let $r < k$ be two admissible integers. The solutions $v_0, \ldots, v_{n-1}$ satisfy*

$$S(c_1 v_{k-1} + c_2 v_k, x^+) = k \qquad \text{for every } c_1, c_2, \tag{3.8}$$

$$W(v_{r-1}, v_r, \ldots, v_{k-1}, v_k) \neq 0 \tag{3.9}$$

for sufficiently large values of x. Moreover,

$$\lim \frac{W(v_{r-1}, v_r)}{W(v_{k-1}, v_k)} = 0. \tag{3.10}$$

Proof. If we substitute (3.7) for $(-1)^\ell v_{k-1}^{(\ell)}$, then for large values of x

$$S(v_{k-1}, x^+) = S(\cos(\zeta + o(1)), \cos(\zeta - \frac{\pi k}{n} + o(1)), \ldots, \cos(\zeta - \pi k + o(1))$$

and

$$S(v_k, x^+) = S(\sin(\zeta + o(1)), \sin(zeta - \frac{\pi k}{n} + o(1)), \ldots, \sin(\zeta - \pi k + o(1)).$$

where $\zeta = \sin\frac{\pi k}{n} J(x)$. In the above sequences sign changes occur when their terms satisfy either $c_j c_{j-1} < 0$ or $c_j = 0$ and $c_{j-1} c_{j+1} < 0$. In both cases the $o(1)$ terms have no influence on the number of the sign changes and both of them equal k. The same holds as well for every linear combination of v_{k-1}, v_k. If 0 is an admissible integer for (1.1), then obviously $S(v_0,x^+)=0$, and if n is admissible, then $S(v_{n-1},x^+)=n$. Next,

$$W(v_{r-1},v_r,\ldots,v_{k-1},v_k)=r^{-(\ldots)}\exp\left(-2\sum_{\substack{r\le i\le k\\ i\equiv r(\mathrm{mod}\ 2)}}\cos(\pi i/n)I(x)+o(1)\right)$$
$$\times\begin{vmatrix}\cos\left(\sin\frac{\pi r}{n}J(x)+o(1)\right), & \sin\left(\sin\frac{\pi r}{n}J(x)+o(1)\right), & \ldots\\ \cos\left(\sin\frac{\pi r}{n}J(x)+\pi-\frac{\pi r}{n}+o(1)\right), & \sin\left(\sin\frac{\pi r}{n}J(x)+\pi-\frac{\pi r}{n}+o(1)\right), & \ldots\\ \vdots & \vdots & \end{vmatrix}.$$

The last determinant is independent of $J(x)$ and it equals, up to $o(1)$, to a nonzero constant. Thus, (3.9) holds. (3.10) is implied by

$$W(v_{k-1},v_k)=r^{-n}\exp\left(-2\cos\frac{\pi k}{n}I(x)+o(1)\right)\sin\left(\pi k/n+o(1)\right),$$

because $\cos\theta_k=-\cos\frac{\pi k}{n}>-\cos\frac{\pi r}{n}=\cos\theta_r$.

(3.10) means that the solutions v_{k-1},v_k are , in a sense, "larger" near infinity than $v_{r-1},v_r,r<$ k. This desirable property could not been deduced in [2,3] without the additional assumptions of the present work.

The solutions $(3.7)_0-(3.7)_{n-1}$ give us an idea about the oscillatory behaviour of the solutions of (1.1). As $\lim_{x\to\infty}r'(x)/r^2(x)=0$,

$$I(x)=\int_{x_0}^{x}\left(r+\frac{n^2-1}{24n^2}\frac{p'^2}{p^2}r^{-1}\right)=\int_{x_0}^{x}r\left(1+\frac{n^2-1}{24}\left(\frac{r'}{r^2}\right)^2\right)$$

and

$$J(x)==\int_{x_0}^{x}r\left(1-\frac{n^2-1}{24}\left(\frac{r'}{r^2}\right)^2\right)$$

tend to infinity if and only if $\int^x r\to\infty$. Consequently,

Theorem 3. *Let the assumptions of Theorem 1 be satisfied. Each solution y of Eq. (1.1) is either*
(i) oscillatory, or satisfies
(ii) $|y^{(\ell)}|\to 0$, $\ell=0,\ldots,n-1$, (if 0 is admissible, $(-1)^n p>0$), or
(iii) $|y^{(\ell)}|\to\infty$ $\ell=0,\ldots,n-1$, (if n is admissible, $p>0$),
if and only if $\int^x r\to\infty$.
Moreover, for each solution $y(x)$ there is an admissible integer k such that

$$-\cos\frac{\pi k}{n}=\limsup\frac{\log r(x)^{(n-1)/2}y(x)}{I(x)},$$

and if $y(x)$ is oscillatory, there is a sufficiently large x_0 such that the mth zero of $y(x)$ in (x_0, ∞) is

$$x_m = J^{-1}\left(m\pi / \sin \frac{\pi k}{n}\right) + o(1),$$

where J^{-1} denotes the inverse of J.

In the Soviet literature the following definitions are well known (See [10]):

Eq. (1.1) is said to have *Property A* if for even n all its solutions are oscillatory, and for odd n they are either oscillatory or satisfy

$$|y^{(i)}| \to 0 \quad \text{as } x \to \infty \quad i = 0, \dots n-1. \tag{3.11}$$

Eq. (1.1) is said to have *Property B* if for even n all its solutions are either oscillatory or satisfy (3.11) or satisfy

$$|y^{(i)}| \to \infty \quad \text{as } x \to \infty \quad i = 0, \dots n-1, \tag{3.12}$$

and for odd n all the solutions are either oscillatory or satisfy (3.12). In terms of these definitions, Theorem 3 states that under the listed assumptions Eq.(1.1) has *Property A* if $p(x) > 0$, and *Property B* if $p(x) < 0$.

§**4.** This section deals with the case

$$\lim r^2(x)/r'(x) \quad \text{is finite and nonzero.}$$

This assumption is satisfied by the Euler equation $y^{(n)} + cx^{-n}y = 0$. It is studied in [1, p.93], but we formulate our result more explicitly.

$r'(x)/r^2(x) \to \text{const} \neq 0$ implies that $r(x) \to 0$ as $x \to \infty$. Introducing

$$s(x) \equiv -r^2(x)/r'(x), \qquad s \equiv \lim_{x\to\infty} s(x),$$

Eq. (1.4) may be rewritten as

$$\begin{aligned} A_1(x) &= (-r'/r)\left[\begin{pmatrix} 0 & 0 & \dots & & \\ & 1 & & & \\ \vdots & & 2 & & \\ & & & \ddots & \\ & & & & n-1 \end{pmatrix} + s(x)\begin{pmatrix} 0 & 1 & 0 & \dots & \\ & 0 & 1 & & \\ \vdots & & \ddots & & \\ & & & & 1 \\ 1 & & & & 0 \end{pmatrix}\right] \\ &= +[s(x) - s]\begin{pmatrix} 0 & 1 & 0 & \dots & \\ & 0 & 1 & & \\ \vdots & & \ddots & & \\ & & & & 1 \\ 1 & & & & 0 \end{pmatrix}\Bigg] \end{aligned} \tag{4.1}$$

Since $s(x) - s \to 0$, the last term of this expression is a small perturbation of the first one. In order to use Levinson's theorem we have to diagonalize

$$\begin{pmatrix} 0 & s & 0 & \dots & & \\ & 1 & s & & & \\ \vdots & & 2 & & & \\ & & & \ddots & s & \\ s & & & & n-1 \end{pmatrix} \tag{4.2}$$

Its characteristic equation is

$$\varphi(\lambda) \equiv \lambda(\lambda - 1)\dots(\lambda - n + 1) - s^n = 0, \tag{4.3}$$

and let its eigenvalues be $\lambda_0, \dots, \lambda_{n-1}$. If the λ_i's are *distinct* and $s \neq 0$, (4.2) may be diagonalized by the matrix

$$U = (\, \lambda_k(\lambda_k - 1)\dots(\lambda_k - \ell + 1)s^{-\ell}\,)_{\ell,k=0}^{n-1} \tag{4.4}$$

whose columns are the corresponding eigenvectors

$$(1, \lambda s^{-1}, \lambda(\lambda-1)s^{-2}, \dots, \lambda(\lambda-1)\dots(\lambda-n+2)s^{-n+1})^T.$$

Note that that $\det(U) = s^{-n(n-1)/2}V(\lambda_0, \dots, \lambda_{n-1})$, where $V(\lambda_0, \dots, \lambda_{n-1})$ is the determinant of Vandermonde, $\prod(\lambda_i - \lambda_j)$.

The transformation $Z_1 = UZ_2$ reduces then our system into $Z_2' = A_2 Z_2$ with

$$A_2 = U^{-1}A_1 U = (-r'/r)\left[\, \mathrm{diag}\{\lambda_0, \dots, \lambda_{n-1}\} + [s(x) - s]U^{-1}PU\right],$$

where P is the permutation matrix

$$\begin{pmatrix} 0 & 1 & 0 & \dots & \\ & 0 & 1 & & \\ \vdots & & \ddots & & \\ & & & & 1 \\ 1 & & & & 0 \end{pmatrix}.$$

Next we calculate $M = U^{-1}PU$. Multiplying U from the left by P shifts its rows cyclically upward and the last row becomes $(1, \dots, 1)$. But by (4.3) we may replace the 1 at the (n,k)th entry by $\lambda_k(\lambda_k - 1)\dots(\lambda_k - n + 1)s^{-n}$, $k = 0, 1, \dots, n-1$, so finally M equals

$$\begin{pmatrix} 1 & 1 & \dots \\ \lambda_0/s & \lambda_1/s & \\ \lambda_0(\lambda_0-1)/s^2 & \lambda_1(\lambda_1-1)/s^2 & \\ \vdots & \vdots & \\ \lambda_0(\lambda_0-1)\dots(\lambda_0-n+2)/s^{n-1} & & \end{pmatrix}^{-1} \begin{pmatrix} \lambda_0/s & \lambda_1/s & \dots \\ \lambda_0(\lambda_0-1)/s^2 & \lambda_1(\lambda_1-1)/s^2 & \\ \vdots & \vdots & \\ \lambda_0(\lambda_0-1)\dots(\lambda_0-n+1)/s^n & & \end{pmatrix}. \tag{4.5}$$

The (r,q)th term of $M, M_{r,q},\ r,q=0,\dots,n-1$, equals

$$|U|^{-1}\times \begin{vmatrix} 1 & 1 & \dots & 1 & \lambda_q/s & 1 & \dots \\ \lambda_0/s & \lambda_1/s & & \lambda_{r-1}/s & \lambda_q(\lambda_q-1)/s^2 & \lambda_{r+1}/s & \\ \vdots & \vdots & & \vdots & & \vdots & \\ \lambda_0(\lambda_0-1)\dots(\lambda_0-t+2)/s^{t-1} & & \dots & & \lambda_q(\lambda_q-1)\dots(\lambda_q-t+1)/s^t & & \\ \vdots & & & & \vdots & & \\ \lambda_0(\lambda_0-1)\dots(\lambda_0-n+2)/s^{n-1} & & \dots & & \lambda_q(\lambda_q-1)\dots(\lambda_q-n+1)/s^n & & \end{vmatrix},$$

where the rth column of the determinant consists of the qth column of the right-most matrix in the product (4.5). This follows easily by expanding the last determinant according to its rth column. When we factor out λ_q/s from the rth column of the last determinant and $s^{-\ell}$ from its ℓth row, there remains $\lambda_q s^{-1-n(n-1)/2}V(\lambda_0,\dots,\lambda_{r-1},\lambda_q-1,\lambda_{r+1},\dots,\lambda_{n-1})$. Thus,

$$\begin{aligned} M_{r,q} &= s^{-1}\lambda_q V(\lambda_0,\dots,\lambda_{r-1},\lambda_q-1,\lambda_{r+1},\dots,\lambda_{n-1})\big/V(\lambda_0,\dots,\lambda_{n-1}) \\ &= s^{-1}\lambda_q\prod_{i\neq r}(\lambda_q-1-\lambda_i)\big/\prod_{i\neq r}(\lambda_r-\lambda_i). \end{aligned}$$

Since $\varphi(x)=(x-\lambda_0)\dots(x-\lambda_{n-1})$, the last expression may be written as

$$s^{-1}\lambda_q\varphi(\lambda_q-1)/\varphi'(\lambda_r)(\lambda_q-1-\lambda_r).$$

On the other hand, since λ_q is a zero of $\varphi(\lambda)$, it follows by (4.3) that $\lambda_q\varphi(\lambda_q-1)=-ns^n$, so that

$$M_{r,q}=-ns^{n-1}/\varphi'(\lambda_r)(\lambda_q-1-\lambda_r).$$

In particular, $M_{r,r} = ns^{n-1}/\varphi'(\lambda_r)$. Consequently, the eigenvalues of $A_2(x)$ are

$$\begin{aligned}\mu_r(x) &= -r'/r\left[\lambda_r - [s(x)-s]M_{rr} + \mathcal{O}([s(x)-s]^2)\right] \\ &= -r'/r\left[\lambda_r - [s(x)-s]ns^{n-1}/\varphi'(\lambda_r)\right] + \mathcal{O}((r'/r)[s(x)-s]^2),\end{aligned}$$

$r = 0,\ldots,n-1$. By the identity $s^n(x) - s^n = ns^{n-1}[s(x)-s] + \mathcal{O}([s(x)-s]^2)$ this becomes

$$\mu_r(x) = -r'/r\,[\lambda_r - [s^n(x)-s^n]/\varphi'(\lambda_r)] + \mathcal{O}((r'/r)[s(x)-s]^2).$$

Finally, defining κ and $\Delta(x)$ by

$$\kappa + \Delta(x) = s^n(x) = p^{n+1}/(-p'/n)^n, \qquad \kappa = s^n = \lim_{x\to\infty} s^n(x), \tag{4.6}$$

we have

$$\mu_r = -r'/r(\lambda_r + \Delta(x)/\varphi'(\lambda_r)) + \mathcal{O}((r'/r)[s(x)-s]^2).$$

If the remainder term is integrable, we obtain by Levinson's theorem that

$$Z_2(x) = (I + o(1))\ \mathrm{diag}\ \left\{\exp\int^x \mu_0(t)dt,\ldots\right\}.$$

Summarizing the previous details and calculating Z_1, Y leads to the following result:

Theorem 4. *Let* $(-\frac{r'}{r^2})^n = \frac{p^{n+1}}{(-p'/n)^n} = \kappa + \Delta(x)$,
where $\kappa \neq 0$ *is a constant and* $\Delta(x) \to 0$ *as* $x \to \infty$. *If the polynomial*

$$\varphi(\lambda) \equiv \lambda(\lambda-1)\ldots(\lambda-n+1) - \kappa = 0 \tag{4.7}$$

has n *distinct zeros* $\lambda_0,\ldots,\lambda_{n-1}$, *and* $(r'/r)\Delta^2(x) \in \mathbf{L}_1(a,\infty)$, *then*

$$\begin{aligned}Y &= \mathit{diag}\{1, r, \ldots, r^{n-1}\}U(I + o(1)) \\ &\quad\times \mathit{diag}\left\{r^{-\lambda_0}\exp\left(\frac{1}{\varphi'(\lambda_0)}\int^x \frac{r'}{r}\Delta\right),\ldots\right\},\end{aligned} \tag{4.8}$$

where U *is the constant matrix defined in (4.4), or explicitly*

$$Y_{\ell j} = \left(\frac{\lambda_j(\lambda_j-1)\ldots(\lambda_j-\ell+1)}{\kappa^{\ell/n}} + o(1)\right) r(x)^{-\lambda_j+\ell}\exp\left(\frac{1}{\varphi'(\lambda_j)}\int^x \frac{r'}{r}\Delta\right), \tag{4.9}$$

$\ell, j = 0,\ldots,n-1$.

Remark. Our asymptotic formula refines the result of [1, Section 2.10]. For $n = 2$ it reduces to that of [1, Section 2.6].

Examples. For $p(x) = ax^{-n}$, we have $\kappa = a,\ \Delta(x) \equiv 0$.

Let now $p(x) = a/x^n + b/(x^n \log x)$. For this case $\kappa = a,\ \Delta(x) = b/\log x + \mathcal{O}(\log^{-2} x),\ r'/r = x^{-1}(1 + \mathcal{O}(\log^{-2} x))$, and $(r'/r)\Delta^2(x) \in \mathbf{L}_1(a, \infty)$. n solutions of the scalar equation are given by

$$y_k = x^{\lambda_k}(\log x)^{b/\varphi'(\lambda_k)}(1 + o(1)), \qquad k = 0, \dots, n-1.$$

§**5.** When $p(x)$ is real valued, Theorem 4 enables again an accurate analysis of the oscillation or nonoscillation of $Eq(1.1)$. When an eigenvalue λ is real, the corresponding solution (4.9) is obviously nonoscillatory. However, much more information is available. Let $\lambda = \alpha + i\beta$ be a root of (4.7) and $\eta + i\nu = 1/\varphi'(\lambda)$. Since $p(x)$ is real valued, the real part of a solution (4.9) is a solution as well. Since

$$\begin{aligned} &\Re\left\{\lambda(\lambda-1)\dots(\lambda-\ell+1)r(x)^{-\lambda+\ell}\right\} \\ &\quad = \left|\prod_{j=0}^{\ell-1}(\lambda - j)\right| r^{-\alpha+l} \cos\left(-\beta \log r + \arg \prod_{j}^{\ell-1}(\lambda - j)\right), \end{aligned}$$

real solutions of (1.1) and their ℓth derivatives, $y^{(\ell)},\ \ell = 0, \dots, n-1$, may be written as

$$\begin{aligned} &\left(\left|\prod_{j=0}^{\ell-1}(\lambda - j)\right| + o(1)\right) s^{-\ell} r^{-\alpha+l} \exp\left(\eta \int^x \frac{r'}{r}\Delta\right) \\ &\qquad\qquad \cos\left(\beta \log r - \vartheta_\ell + \nu \int^x \frac{r'}{r}\Delta + o(1)\right), \\ &\left(\left|\prod_{j=0}^{\ell-1}(\lambda - j)\right| + o(1)\right) s^{-\ell} r^{-\alpha+l} \exp\left(\eta \int^x \frac{r'}{r}\Delta\right) \\ &\qquad\qquad \sin\left(\beta \log r - \vartheta_\ell + \nu \int^x \frac{r'}{r}\Delta + o(1)\right), \end{aligned} \tag{5.1}$$

where $\vartheta_\ell = \arg \prod_{j=0}^{\ell-1}(\lambda - j),\ |\arg(\lambda - j)| < \pi$. Note that $\int^x \frac{r'}{r}\Delta = o(\log r)$ since $\Delta(x) \to 0$. To classify these solutions and study their oscillation, let us summarize some properties of (4.7):

Lemma 3. *The algebraic function $\lambda(\kappa)$ which is defined by*

$$F(\lambda, \kappa) \equiv \lambda(\lambda - 1)\dots(\lambda - n + 1) - \kappa = 0 \tag{5.2}$$

has only real critical points $\kappa_1, ..., \kappa_{n-1}$, which are the local maxima and minima of the polynomial $x(x-1)\dots(x-n+1)$. Exactly 2 branches of $\lambda_0(\kappa), \dots, \lambda_{n-1}(\kappa)$ meet at each critical point. For each critical value κ_0, the corresponding double root $\lambda(\kappa_0)$ of (4.7) lies in the real interval $(k-1, k)$, where k is an integer that satisfies the parity condition $(-1)^{n-k} = \operatorname{sgn}[\kappa_0]$. Let the two branches which meet at this point be named $\lambda_{k-1}(\kappa), \lambda_k(\kappa)$. As κ varies on the half real line that contains κ_0,

$$\lim \lambda_{k-1}(\kappa) = k-1, \qquad \lim \lambda_k(\kappa) = k,$$

while as $|\kappa| \to \infty$,

$$\begin{aligned} \lambda_k(\kappa) &= e^{i\theta_k}|\kappa|^{1/n} + \frac{n-1}{2} + \frac{n^2-1}{24}e^{-i\theta_k}|\kappa|^{-1/n} + \dots, \\ \lambda_{k-1}(\kappa) &= \overline{\lambda_k(\kappa)}, \end{aligned} \tag{5.3}$$

where $\theta_0, \dots, \theta_{n-1}$ are arranged as in (3.6).

Proof. The branch points (λ, κ) of (5.2) are solutions of $dF/d\lambda = \sum_{j=0}^{n-1} \prod_{\ell \neq j}(\lambda - \ell) = 0$, that is, after division by $\prod_{\ell=0}^{n-1}(\lambda - \ell)$,

$$\sum_{j=0}^{n-1} \frac{1}{\lambda - j} = 0,$$

and its $n-1$ zeros lie in the intervals $(0,1), \dots, (n-2, n-1)$. The following details are simple. (5.3) is obtained by asymptotic solution of (5.2), but since the formulas will not be used here, the details are omitted. As the branches do not intersect at non-real points, the arguments θ_k in (4.12) are as those in (3.6).

Figure 1. Six branches of $\lambda(\lambda-1)\dots(\lambda-5) - \kappa = 0$ for $\kappa \in [-600, 0]$.

From now on we enumerate the eigenvalues $\lambda_0, \ldots, \lambda_{n-1}$ according to the following convention: For every admissible integer k, $1 \le k \le n-1$, the two roots of (4.7) which are located on the two branches of $\lambda(\kappa)$ of (5.2) which intersect in the interval $(k-1, k)$ will be named λ_{k-1}, λ_k. If $k = 0$ is admissible, λ_0 will be the only root of (4.7) in $(-\infty, 0)$, and if $k = n$ is admissible, then λ_{n-1} will be the root in $(n-1, \infty)$.

Accordingly, a basis $v_0, \ldots, v_{n-1}$ of solutions will be chosen as following: For an admissible integer $k, 1 \le k \le n-1$, if λ_{k-1}, λ_k are complex conjugates, $\lambda_k = \alpha_k + i\beta_k$, $1/\varphi'(\lambda_k) = \eta_k + i\nu_k$, $\vartheta_{\ell,k} = \arg_j^\ell \prod_0^1 (\lambda_k - j)$, then

$$v_{k-1}^{(\ell)} = \left(\left| \prod_{j=0}^{\ell-1} (\lambda_k - j) \right| + o(1) \right) s^{-\ell} r^{-\alpha_k + \ell} \exp\left(\eta \int^x \frac{r'}{r} \Delta \right) \times \cos\left(\beta_k \log r - \vartheta_{\ell,k} + \nu \int^x \frac{r'}{r} \Delta + o(1) \right), \tag{5.4$_{k-1}$}$$

$$v_k^{(\ell)} = \left(\left| \prod_{j=0}^{\ell-1} (\lambda_k - j) \right| + o(1) \right) s^{-\ell} r^{-\alpha_k + \ell} \exp\left(\eta \int^x \frac{r'}{r} \Delta \right) \times \sin\left(\beta_k \log r - \vartheta_{\ell,k} + \nu \int^x \frac{r'}{r} \Delta + o(1) \right), \tag{5.4$_k$}$$

and if λ_{k-1}, λ_k are real valued, then

$$v_j^{(\ell)} = (\lambda_j(\lambda_j - 1) \ldots (\lambda_j - \ell + 1) + o(1)) \, s^{-\ell} r(x)^{-\lambda_j + \ell} \exp\left(\frac{1}{\varphi'(\lambda_j)} \int^x \frac{r'}{r} \Delta \right), \tag{5.4$_j$}$$

$j = k-1, k$. If $k = 0$ or $k = n$ are admissible, then $v_0^{(\ell)}$ or $v_{n-1}^{(\ell)}$ will be given by the last formula with $j = 0$ or $j = n-1$, respectively. $(5.4)_j$, $j = 0, \ldots n-1$, hold even for $\ell = n$, because $\lambda_j(\lambda_j - 1) \ldots (\lambda_j - n + 1) s^{-n} = 1$.

The distribution of the zeros of real solutions of (1.1) follows easily from (5.4).

Theorem 5. *Let the assumptions of Theorem 4 be satisfied. For each solution $y(x)$ there is an admissible integer k and a corresponding eigenvalue λ_k such that*

$$\Re\{\lambda_k\} = \limsup \frac{\log y(x)}{\log r(x)}.$$

If λ_k is real and is in the interval $(k-1, k)$, y is nonoscillatory and satisfies $y^{(\ell)} y^{(\ell+1)} > 0$ for $\ell = 0, \ldots, k-1$ and $y^{(\ell)} y^{(\ell+1)} < 0$ for $\ell = k, \ldots, n-1$, hence the equation is eventually $(k, n-k)$-disfocal. If $y(x)$ is oscillatory, there is a sufficiently large x_0 such that the m-th zero of $y(x)$ in (x_0, ∞) is given by

$$x_m = H_k^{-1}(m\pi) + o(1),$$

where H_k^{-1} denotes the inverse of $H_k(x) = \Im \int_x^x \frac{r'}{r}\left(\lambda_k + \Delta(x)/\varphi'(\lambda_k)\right)$.

Theorem 6. *Let $p(x)$ be real valued and satisfies the assumptions of Theorem 4. The basis of solutions $v_0, \dots, v_{n-1}$ which is defined above satisfies (3.8)-(3.10).*

Proof. If λ_j is real and is located in the interval $(k-1, k)$, then (3.8) trivially holds, since $v_j^{(\ell)} v_j^{(\ell+1)} > 0$ for $\ell = 0, \dots, k-1$ and $v_j^{(\ell)} v_j^{(\ell+1)} < 0$ for $\ell = k, \dots, n-1$. If λ_j is complex valued, $S(v_{k-1}, x^+)$ equals

$$S\left(\cos(\zeta + o(1)), -\cos(\zeta - \vartheta_{1,k} + o(1)), \dots, (-1)^n \cos(\zeta - \vartheta_{n,k} + o(1))\right), \qquad (5.5)$$

where $\zeta = \beta \log r + \nu \int^x \frac{r'}{r}\Delta$, and $\vartheta_{\ell,k} = \arg \prod_{j=0}^{\ell-1}(\lambda_k - j)$. In order to see that $S(v_{k-1}, x^+) = k$ for sufficiently large values of x, let us consider λ in the right hand side of (5.5) as a function of κ,

$$S\left(\cos(\zeta), -\cos(\zeta - \vartheta_1(\kappa)), \dots, (-1)^n \cos(\zeta - \vartheta_n(\kappa))\right) \qquad (5.6)$$

$\vartheta_\ell(\kappa) = \arg \prod_{j=0}^{\ell-1}(\lambda(\kappa) - j)$, and let κ vary continuously through real values until $\lambda(\kappa)$ approaches the its critical real value in $(k-1, k)$. $\vartheta_\ell(\kappa)$ is obviously a continuous function of κ, $\ell = 0, \dots$,n. The integer valued function (5.6) may vary with κ only if one of the following happens:

(a) An entry $\cos(\zeta - \vartheta_\ell(\kappa))$ changes its sign as κ varies, while $\cos(\zeta - \vartheta_{\ell-1}(\kappa))\cos(\zeta - \vartheta_{\ell+1}(\kappa)) > 0$. This is impossible, since both $|\vartheta_\ell(\kappa) - \vartheta_{\ell-1}(\kappa)| = |\arg(\lambda(\kappa) - \ell + 1)|$ and $|\vartheta_{\ell+1}(\kappa) - \vartheta_\ell(\kappa)|$ are less than π.

(b) Several consecutive entries vanish for the same κ, $\cos(\zeta - \vartheta_\ell(\kappa)) = \cos(\zeta - \vartheta_{\ell+1}(\kappa)) = \ldots = 0$. This, too, cannot happen since $\vartheta_{\ell+1}(\kappa) - \vartheta_\ell(\kappa)$ is not a multiple of π. Consequently (5.6) must have a fixed value as long as $\lambda(\kappa)$ is not real. To determine its fixed value, it is sufficient to let κ approach the critical value κ_0, such that $\lambda = \lambda(\kappa_0) \in (k-1, k)$, (or to let $|\kappa| \to \infty$). For this critical value $\arg(\lambda - j) = 0$, $j = 0, \dots, k-1$, while $\arg(\lambda - j) = \pi$, $j = k, \dots, n-1$. Hence $\vartheta_\ell = 0$, $\ell = 0, \dots, k$, and $\vartheta_\ell = (k - \ell)\pi$, $\ell = k+1, \dots, n$. (5.6) becomes now

$$S\left(\cos(\zeta), -\cos(\zeta), \dots, (-1)^{k-1}\cos(\zeta), (-1)^{k+1}\cos(\zeta), \dots, (-1)^{k+1}\cos(\zeta)\right) = k.$$

This shows that v_{k-1}, v_k satisfy (3.8). (3.9) is verified as it was in §**3**.

Finally, $W(v_{k-1}, v_k) = |\lambda_k| r^{-2\alpha_k + 1} \exp(\eta \int^x \frac{r'}{r}\Delta) \sin(\vartheta_{1,k})$ and $\int^x \frac{r'}{r}\Delta = o(\log r)$ imply $W(v_{r-1}, v_r)/W(v_{k-1}, v_k) = |\lambda_r/\lambda_k| r(x)^{-2Re(\lambda_r - \lambda_k) + o(1)}$, and (3.10) follows from $\Re\lambda_r < Re\lambda_k$ and $r(x) \to 0$.

§**6.** The final section deals with the case

$$\lim_{x \to \infty} r^2(x)/r'(x) = 0.$$

This case requires a slight modification of the discussion of the previous section, which excludes $s(x) \to 0$. According to (4.1), let

$$A_1(x) = (-r'/r) \begin{pmatrix} 0 & s(x) & 0 & \dots & \\ & 1 & s(x) & & \\ \vdots & & 2 & & \\ & & & \ddots & s(x) \\ s(x) & & & & n-1 \end{pmatrix}, \quad s(x) = -r(x)^2/r'(x).$$

The eigenvalues of $(-r/r')A_1(x)$ are the roots of the equation

$$\lambda(\lambda-1)\dots(\lambda-n+1) - \Delta(x) = 0, \quad \Delta(x) = s^n(x),$$

namely,

$$\lambda_j(x) = j + (-1)^{n-j+1}\Delta(x)/j!\,(n-j+1)! + O(\Delta^2), \quad j = 0, \dots, n-1. \tag{6.1}$$

$(-r/r')A_1(x)$ may be diagonalized by

$$T(x) = \left(\lambda_j(x)(\lambda_j - 1)\dots(\lambda_j - \ell + 1)s^{j-\ell}(x)/j!\right)_{\ell,j=0}^{n-1} \tag{6.2}$$

whose columns

$$s^j/j!\left(1, \lambda_j(x)s^{-1}, \lambda_j(\lambda_j-1)s^{-2}, \dots, \lambda_j(\lambda_j-1)\dots(\lambda_j-n+2)s^{-n+1}\right)^T$$

are eigenvectors of $A_1(x)$. We have $T_{\ell j} = (1/(j-\ell)! + O(\Delta))s(x)^{j-\ell}$ for $\ell \le j$ and $T_{\ell j} = \mathcal{O}(\Delta)s(x)^{j-\ell}$ for $j < \ell$. Together this shows that $T(x) = I + \mathcal{O}(s)$ and $T^{-1}T' = \mathcal{O}(s')$. Therefore the transformation $Z_1 = TZ_2$ reduces our system into $Z_2' = A_2 Z_2$ with

$$A_2(x) = (-r'/r)\left[\operatorname{diag}\{\lambda_0(x), \dots, \lambda_{n-1}(x)\} + \mathcal{O}(s')\right].$$

In order to use Levinson's theorem, we need that $(r'/r)s' = (r'/r)(-r^2/r')' = -2r' + rr''/r' \in \mathbf{L}_1(a, \infty)$. Then

$$Z_2 = (I + o(1)) \operatorname{diag}\left\{\exp \int (-r'/r)\lambda_j\right\},$$
$$Y(x) = \operatorname{diag}\{1, r, \dots, r^{n-1}\}T(x)Z_2.$$

As in the previous section, we conclude the following

Theorem 7. *Let $\Delta(x) \equiv p^{n+1}/(-p'/n)^n \to 0$ as $x \to \infty$. If $rr''/r' - 2r' \in \mathbf{L}_1(a,\infty)$, then system (1.2) has a solution*

$$\begin{aligned} Y = \ & diag\{1, r, \ldots, r^{n-1}\}T(x)(I + o(1)) \\ \times \ & diag \left\{ r^{-j} \exp\left(\frac{(-1)^{n-j+1}}{j!\,(n-j-1)!} \int^x \frac{r'}{r}\Delta \right) \right\}_{j=0}^{n-1}, \end{aligned}$$

where $T(x)$ is the matrix defined in (6.2).

By observing that $T_{0,j} = s^j/j!$ and substituting for r, s and Δ, we get a basis of solutions for the scalar Eq. (1.1):

$$\begin{aligned} y_j(x) = (p/p')^j \exp\left(\frac{(-1)^{j-1}}{j!\,(n-j-1)!} \int^x \frac{p^n}{(p'/n)^{n-1}} \right) (1 + o(1)), \\ j = 0, \ldots, n-1. \end{aligned} \tag{6.3}$$

These solutions are obviously nonoscillatory, hence Eq. (1.1) is eventually disconjugate.

Remark. If $\varphi(\lambda) \equiv \lambda(\lambda-1)\ldots(\lambda-n+1)$, then $\frac{1}{\varphi'(\lambda_j)} = \frac{(-1)^{n-j+1}}{j!\,(n-j-1)!}$, though the assumptions in Theorems 4 and 7 are different.

If $p(x)$ and $r(x) = p^{1/n}(x)$ are real valued, the assumption that $s(x) = -r^2/r' \to 0$ includes that r' is defined on some neighborhood of infinity and $r'(x) \neq 0$ there. The monotonic function $r(x)$ satisfies $|r'/r^2| > 1/\epsilon$ for $x > x_\epsilon$, hence $1/r \to \infty$ as $x \to \infty$ and $r(x)$ decreases monotonically to 0. Therefore $r' \in \mathbf{L}_1$ and it suffices to assume that $rr''/r' \in \mathbf{L}_1$.

Example. Theorem 6 may be applied, for example, for $p(x) = b/x^{n+\epsilon}$, $\epsilon > 0$, $b/(x^n \log^\beta x)$, $\beta > 0$, and $b/(x^n \log(\log x))$. For the second of these, the solutions are

$$\begin{aligned} y_j = & x^{j + c(j)/\log^\beta x}(1 + o(1)), \qquad j = 0, \ldots, n-1, \\ & c(j) = (-1)^{n-j+1} b/(\beta-1)j!\,(n-j-1)! \end{aligned}$$

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