Combinatorial Identities: Table III: Binomial Identities Derived from Trigonometric and Exponential Series

From the seven unpublished manuscripts of H. W. Gould
Edited and Compiled by Jocelyn Quaintance

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1 Basic Trigonometric Series

Remark 1.1 *Throughout this chapter, we assume n and a are nonnegative integers. We assume x and y are real or complex numbers.*

1.1 Telescoping Trigonometric Series

$$\sum_{k=1}^{n} \sin\frac{2k+1}{2}x = \frac{\cos(n+1)x - \cos x}{-2\sin\frac{x}{2}}, \qquad n \geq 1 \tag{1.1}$$

$$\sum_{k=1}^{n} \sin\frac{2k+1}{2}x = \frac{\sin\frac{(n+2)x}{2}\cdot\sin\frac{nx}{2}}{\sin\frac{x}{2}}, \qquad n \geq 1 \tag{1.2}$$

$$\sum_{k=1}^{n} \cos\frac{2k+1}{2}x = \frac{\sin(n+1)x - \sin x}{2\sin\frac{x}{2}}, \qquad n \geq 1 \tag{1.3}$$

$$\sum_{k=1}^{n} \cos\frac{2k+1}{2}x = \frac{\cos\frac{(n+2)x}{2}\cdot\sin\frac{nx}{2}}{\sin\frac{x}{2}}, \qquad n \geq 1 \tag{1.4}$$

$$\sum_{k=1}^{n} \frac{\sin\frac{k}{(k+1)!}x}{\cos\frac{x}{k!}\cdot\cos\frac{x}{(k+1)!}} = \tan x - \tan\frac{x}{(n+1)!}, \qquad n \geq 1 \tag{1.5}$$

$$\sum_{k=1}^{\infty} \frac{\sin \frac{k}{(k+1)!}x}{\cos \frac{x}{k!} \cdot \cos \frac{x}{(k+1)!}} = \tan x \tag{1.6}$$

$$\sum_{k=1}^{n} \frac{\tan 2^k x}{\cos 2^{k+1} x} = \tan 2^{n+1} x - \tan 2x, \qquad n \geq 1 \tag{1.7}$$

$$\sum_{k=1}^{n} \sec(k+1)x \cdot \sec kx = \frac{\tan(n+1)x - \tan x}{\sin x}, \qquad n \geq 1 \tag{1.8}$$

$$\sum_{k=1}^{n} \sin(x+(k-1)y) = \frac{\sin(x+\frac{n-1}{2}y) \cdot \sin \frac{ny}{2}}{\sin \frac{y}{2}}, \qquad n \geq 1 \tag{1.9}$$

$$\sum_{k=1}^{n} \sin kx = \frac{\sin \frac{(n+1)x}{2} \cdot \sin \frac{nx}{2}}{\sin \frac{x}{2}}, \qquad n \geq 1 \tag{1.10}$$

$$\sum_{k=1}^{n} \sin(2k-1)x = \frac{\sin^2 nx}{\sin x}, \qquad n \geq 1 \tag{1.11}$$

$$\sum_{k=1}^{n} k \cos \frac{(2k+1)x}{2} = \frac{(n+1)\sin(n+1)x \cdot \sin \frac{x}{2} - \sin \frac{(n+2)x}{2} \cdot \sin \frac{(n+1)x}{2}}{2 \sin^2 \frac{x}{2}} \tag{1.12}$$

$$\sum_{k=1}^{n} \cos(x+(k-1)y) = \frac{\cos(x+\frac{n-1}{2}y) \cdot \sin \frac{ny}{2}}{\sin \frac{y}{2}}, \qquad n \geq 1 \tag{1.13}$$

$$\sum_{k=1}^{n} \cos kx = \frac{\cos \frac{(n+1)x}{2} \cdot \sin \frac{nx}{2}}{\sin \frac{x}{2}}, \qquad n \geq 1 \tag{1.14}$$

$$\sum_{k=1}^{n} \cos(2k-1)x = \frac{\sin 2nx}{2 \sin x}, \qquad n \geq 1 \tag{1.15}$$

$$\sum_{k=0}^{n} \cos^3(x+ky) = \frac{\cos(3x+\frac{3ny}{2})\sin(\frac{3y(n+1)}{2})}{4 \sin \frac{3y}{2}} + \frac{3\cos(x+\frac{ny}{2})\sin \frac{(n+1)y}{2}}{4 \sin \frac{y}{2}} \tag{1.16}$$

1.2 Sums and Products Based on Double Angle Formulas

$$\prod_{k=0}^{n} \cos 2^k x = \frac{\sin 2^{n+1} x}{2^{n+1} \sin x} \tag{1.17}$$

$$\prod_{k=1}^{n} \cos \frac{x}{2^k} = \frac{\sin x}{2^n \sin \frac{x}{2^n}}, \qquad n \geq 1 \tag{1.18}$$

$$\prod_{k=1}^{\infty} \cos \frac{x}{2^k} = \frac{\sin x}{x} \tag{1.19}$$

$$\sum_{k=1}^{n} \sin^2 kx = \frac{n}{2} - \frac{\cos(n+1)x \cdot \sin nx}{2 \sin x}, \qquad n \geq 1 \tag{1.20}$$

$$\sum_{k=1}^{n} \cos^2 kx = \frac{n}{2} + \frac{\cos(n+1)x \cdot \sin nx}{2 \sin x}, \qquad n \geq 1 \tag{1.21}$$

1.3 Sums Based on Half Angle Formulas

$$\sum_{k=1}^{n} \csc 2^{k-1} x = \cot \frac{x}{2} - \cot 2^{n-1} x, \qquad n \geq 1 \tag{1.22}$$

$$\sum_{k=0}^{n} \csc \frac{x}{2^k} = \cot \frac{x}{2^{n+1}} - \cot x \tag{1.23}$$

$$\sum_{k=1}^{n} \frac{1}{2^{k-1}} \tan \frac{x}{2^{k-1}} = \frac{1}{2^{n-1}} \cot \frac{x}{2^{n-1}} - 2 \cot 2x, \qquad n \geq 1 \tag{1.24}$$

$$\sum_{k=1}^{\infty} \frac{1}{2^{k-1}} \tan \frac{x}{2^{k-1}} = \frac{1}{x} - 2 \cot 2x \tag{1.25}$$

1.4 Miscellaneous Trigonometric Series

$$\sum_{k=a}^{n-1} 3^k \sin^3 \frac{x}{3^{k+1}} = \frac{1}{4}\left(3^n \sin\frac{x}{3^n} - 3^a \sin\frac{x}{3^a}\right), \qquad n \geq 1 \tag{1.26}$$

$$\sum_{k=a}^{n-1} (-3)^k \cos^3 \frac{x}{3^{k+1}} = \frac{1}{4}\left((-3)^a \cos\frac{x}{3^a} - (-3)^n \cos\frac{x}{3^n}\right), \qquad n \geq 1 \tag{1.27}$$

2 The Exponential Function with Trigonometric Series

Remark 2.1 *Throughout this chapter, we assume n and r are nonnegative integers. We let x, y, and z denote real or complex numbers. Furthermore, we reserve $i \equiv \sqrt{-1}$. Finally if x is a real number, we let $[x]$ denote the floor of x.*

2.1 Limit Definition for e

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}\sum_{r=0}^{n}\binom{n}{r}\left(\frac{1}{n}\right)^r \tag{2.1}$$

2.2 The Exponential Series and Various Applications

2.2.1 The Exponential Series

$$\boldsymbol{e^z = \sum_{r=0}^{\infty} \frac{z^r}{r!}} \tag{2.2}$$

2.2.2 Series from $e^{ix} = \cos x + i \sin x$

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = \cos x \tag{2.3}$$

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = \sin x \tag{2.4}$$

$$\sum_{n=0}^{\infty} \frac{x^n \cos ny}{n!} = e^{x \cos y} \cos(x \sin y) \tag{2.5}$$

$$\sum_{n=0}^{\infty} \frac{x^n \sin ny}{n!} = e^{x \cos y} \sin(x \sin y) \tag{2.6}$$

Remark 2.2 *The following identity is from Problem 415 of The Mathematics Magazine, May 1960. Solutions to this problem are found in The Mathematics Magazine, Vol. 34, No. 3, 1961, P. 178.*

$$\sum_{k=0}^{n} \binom{n}{k} \cos kx \cdot \sin(n-k)x = 2^{n-1} \sin nx \tag{2.7}$$

$$\cos^n x \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n}{2k} \tan^{2k} x = \cos nx \tag{2.8}$$

$$\cos^n x \sum_{k=0}^{[\frac{n-1}{2}]} (-1)^k \binom{n}{2k+1} \tan^{2k+1} x = \sin nx, \qquad n \geq 1 \tag{2.9}$$

2.3 Expansions of $(e^{ix} \pm 1)^n$

2.3.1 Expansions of $(e^{ix} + 1)^n$

$$\sum_{k=0}^{n} \binom{n}{k} \cos kx = 2^n \cos \frac{nx}{2} \left(\cos \frac{x}{2}\right)^n \tag{2.10}$$

$$\sum_{k=0}^{n} \binom{n}{k} \sin kx = 2^n \sin \frac{nx}{2} \left(\cos \frac{x}{2}\right)^n \tag{2.11}$$

2.3.2 Expansions of $(e^{ix} - 1)^n$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \cos kx = (-2)^n \cos \frac{n(x+\pi)}{2} \left(\sin \frac{x}{2}\right)^n \tag{2.12}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\sin kx=(-2)^n\sin\frac{n(x+\pi)}{2}\left(\sin\frac{x}{2}\right)^n \tag{2.13}$$

Inversion of Identity (2.12)

$$\sum_{k=1}^{n}(-1)^{k-1}2^k\left(\sin\frac{x}{2}\right)^k\cos\frac{k(x+\pi)}{2}+n=\sum_{k=1}^{n}(-1)^{k-1}\binom{n+1}{k+1}\cos kx \tag{2.14}$$

2.3.3 Applications of Equations (2.10), (2.11), (2.12), and (2.13)

$$\sum_{k=0}^{[\frac{n}{2}]}\binom{n}{2k}\cos kx=2^{n-1}\left(\cos^n\left(\frac{x}{4}\right)\cos\left(\frac{nx}{4}\right)+(-1)^n\sin^n\left(\frac{x}{4}\right)\cos\left(\frac{n\pi}{2}+\frac{nx}{4}\right)\right) \tag{2.15}$$

$$\begin{aligned}\sum_{k=0}^{[\frac{n-1}{2}]}\binom{n}{2k+1}\cos(2k+1)x&=2^{n-1}\cos^n\left(\frac{x}{2}\right)\cos\left(\frac{nx}{2}\right)\\&-2^{n-1}(-1)^n\sin^n\left(\frac{x}{2}\right)\cos\left(\frac{n(\pi+x)}{2}\right),\qquad n\geq 1\end{aligned} \tag{2.16}$$

$$\sum_{k=0}^{[\frac{n}{2}]}\binom{n}{2k}\sin kx=2^{n-1}\left(\cos^n\left(\frac{x}{4}\right)\sin\left(\frac{nx}{4}\right)+(-1)^n\sin^n\left(\frac{x}{4}\right)\sin\left(\frac{n\pi}{2}+\frac{nx}{4}\right)\right) \tag{2.17}$$

$$\begin{aligned}\sum_{k=0}^{[\frac{n-1}{2}]}\binom{n}{2k+1}\sin(2k+1)x&=2^{n-1}\cos^n\left(\frac{x}{2}\right)\sin\left(\frac{nx}{2}\right)\\&-2^{n-1}(-1)^n\sin^n\left(\frac{x}{2}\right)\sin\left(\frac{n(\pi+x)}{2}\right),\qquad n\geq 1\end{aligned} \tag{2.18}$$

2.4 The Geometric Series $\sum_{k=1}^{n}(ye^{ix})^k$

$$\sum_{k=1}^{n}y^k\cos kx=\frac{y^{n+2}\cos nx-y^{n+1}\cos(n+1)x+y\cos x-y^2}{y^2-2y\cos x+1},\qquad n\geq 1 \tag{2.19}$$

$$\sum_{k=1}^{n}y^k\sin kx=\frac{y^{n+2}\sin nx-y^{n+1}\sin(n+1)x+y\sin x}{y^2-2y\cos x+1},\qquad n\geq 1 \tag{2.20}$$

$$\sum_{k=1}^{\infty} y^k \cos kx = \frac{y\cos x - y^2}{y^2 - 2y\cos x + 1}, \qquad |y| < 1 \tag{2.21}$$

$$\sum_{k=1}^{\infty} y^k \sin kx = \frac{y\sin x}{y^2 - 2y\cos x + 1}, \qquad |y| < 1 \tag{2.22}$$

$$\sum_{k=1}^{\infty} \frac{y^k \cos kx}{k} = \frac{1}{2}\ln\frac{1}{1 - 2y\cos x + y^2}, \qquad |y| < 1, \tag{2.23}$$

if y is a complex number, use the prinple value of $\ln y$

$$\sum_{k=1}^{\infty} \frac{y^k \sin kx}{k} = \arctan\frac{y\sin x}{1 - y\cos x}, \qquad |y| < 1 \tag{2.24}$$

3 Advanced Trigonometric Series Expansions

Remark 3.1 *Throughout this chapter, we assume n and j are nonnegative integers, while x and y are real or complex numbers. We also let $[x]$ denote the floor of x (for real x).*

3.1 Two Identities Associated with Coefficients in Trigonometric Expansions

3.1.1 First Identity

$$\boldsymbol{\sum_{k=j}^{[\frac{n}{2}]} \binom{n+1}{2k+1}\binom{k}{j} = 2^{n-2j}\binom{n-j}{j}, \qquad j \le \left[\frac{n}{2}\right]} \tag{3.1}$$

Applications of Equation (3.1)

$$\sum_{k=0}^{[\frac{n}{2}]} \binom{n+1}{2k+1} = 2^n \tag{3.2}$$

$$\sum_{k=1}^{[\frac{n}{2}]} \binom{n+1}{2k+1} k = (n-1)2^{n-2}, \qquad n \ge 1 \tag{3.3}$$

$$\sum_{k=0}^{n} \binom{4n+1}{2n-2k}\binom{k+n}{n} = 2^{2n}\binom{3n}{n} \tag{3.4}$$

3.1.2 Second Identity

$$\sum_{k=j}^{[\frac{n}{2}]}\binom{n}{2k}\binom{k}{j}=2^{n-2j}\binom{n-j}{j}-2^{n-1-2j}\binom{n-1-j}{j}, \qquad j\leq\left[\frac{n}{2}\right] \tag{3.5}$$

Restatement of Equation (3.5)

$$\sum_{k=j}^{[\frac{n}{2}]}\binom{n}{2k}\binom{k}{j}=\frac{n2^{n-2j-1}}{n-j}\binom{n-j}{j}, \qquad j\leq\left[\frac{n}{2}\right] \tag{3.6}$$

Applications of Equation (3.5)

$$\sum_{k=0}^{[\frac{n}{2}]}\binom{n}{2k}=2^{n-1} \tag{3.7}$$

$$\sum_{k=1}^{[\frac{n}{2}]}\binom{n}{2k}k=n2^{n-2}, \qquad n\geq 2 \tag{3.8}$$

Applications of Equation (3.6)

$$\sum_{k=0}^{n}\binom{4n}{2n-2k}\binom{k+n}{n}=\frac{2^{2n+1}}{3}\binom{3n}{n} \tag{3.9}$$

$$\sum_{k=0}^{[\frac{n}{2}]}\binom{n-k}{k}\frac{x^k}{n-k}=\frac{(1+\sqrt{4x+1})^n+(1-\sqrt{4x+1})^n}{n2^n}, \qquad n\geq 1 \tag{3.10}$$

Applications of Equation (3.10)

$$\sum_{k=0}^{[\frac{n}{2}]}\binom{n-k}{k}\frac{1}{n-k}=\frac{(1+\sqrt{5})^n+(1-\sqrt{5})^n}{n2^n}, \qquad n\geq 1 \tag{3.11}$$

Remark 3.2 *The following identity is equivalent of Example 44, p. 445 of Hardy's Pure Mathematics.*

$$\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\binom{n-k}{k}\frac{1}{n-k}=\begin{cases}(-1)^{n-1}\frac{1}{n}, & \text{if } n \text{ is not a multiple of 3}\\ (-1)^n\frac{2}{n}, & \text{if } n \text{ is a multiple of 3}\end{cases} \tag{3.12}$$

$$\sum_{k=0}^{[\frac{n}{2}]} \binom{n-k}{k} \frac{6^k}{n-k} = \frac{3^n + (-1)^n 2^n}{n}, \qquad n \geq 1 \tag{3.13}$$

$$\sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n-k}{k} \frac{4^{n-k}}{n-k} = \frac{2^{n+1}}{n}, \qquad n \geq 1 \tag{3.14}$$

$$\sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n-k}{k} \frac{4^{n-k}}{k+1} = \frac{4^{n+1} - 2^{n+1}}{n+2} \tag{3.15}$$

3.2 Expansion of $\frac{\sin(n+1)x}{\sin x}$

$$\frac{\sin(n+1)x}{\sin x} = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n-k}{k} (2\cos x)^{n-2k} \tag{3.16}$$

3.3 Expansion of $\cos nx$

$$\cos nx = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \cos^{n-2k} x \left(2^{n-2k} \binom{n-k}{k} - 2^{n-1-2k} \binom{n-k-1}{k} \right) \tag{3.17}$$

$$\cos nx = \frac{n}{2} \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n-k}{k} \frac{(2\cos x)^{n-2k}}{n-k}, \qquad n \geq 1 \tag{3.18}$$

3.4 Expansions of $\cos 2nx$

3.4.1 Using $\sin^{2k} x$

$$\cos 2nx = \sum_{k=0}^{n} (-1)^k \sin^{2k} x \sum_{j=0}^{k} \binom{2n}{2j} \binom{n-k}{k-j} \tag{3.19}$$

$$\sum_{j=0}^{k} \binom{2n}{2j} \binom{n-j}{k-j} = \frac{2^{2k}}{(2k)!} \prod_{j=0}^{k-1} (n^2 - j^2) = \frac{n}{n+k} \binom{n+k}{2k} 2^{2k} \tag{3.20}$$

Restatement of Equation (3.19)

$$\cos 2nx = \sum_{k=0}^{n} (-1)^k \frac{n}{n+k} \binom{n+k}{2k} 2^{2k} \sin^{2k} x, \qquad n \geq 1 \tag{3.21}$$

3.4.2 Using $\cos^{2k} x$

$$\cos 2nx = \sum_{k=0}^{n} (-1)^{n-k} \frac{n}{n+k} \binom{n+k}{2k} 2^{2k} \cos^{2k} x, \qquad n \geq 1 \tag{3.22}$$

3.4.3 Binomial Identities Resulting From the Coefficient of $\cos^{2k} x$ in Equation (3.22)

$$\sum_{k=j}^{n} (-1)^k \binom{k}{j} \frac{n}{n+k} \binom{n+k}{2k} 2^{2k} = (-1)^n \frac{n}{n+j} \binom{n+j}{2j} 2^{2j}, \qquad n \geq 1 \tag{3.23}$$

$$\sum_{k=0}^{n} (-1)^k \frac{n}{n+k} \binom{n+k}{2k} 2^{2k} = (-1)^n, \qquad n \geq 1 \tag{3.24}$$

$$\sum_{k=1}^{n} (-1)^k \frac{2^{2k}}{(2k)!} \prod_{j=0}^{k-1} (n^2 - j^2) = (-1)^n - 1, \qquad n \geq 1 \tag{3.25}$$

Generalization of Equation (3.20)

$$\boldsymbol{\sum_{k=0}^{n} \binom{2x}{2k} \binom{x-k}{n-k} = \frac{2^{2n}}{(2n)!} \prod_{k=0}^{n-1} (n^2 - k^2) = \frac{x}{x+n} \binom{x+n}{2n} 2^{2n}} \tag{3.26}$$

3.4.4 Applications of Equation (3.26)

$$\sum_{k=0}^{n} (-1)^k \binom{2n}{n-k} \binom{2n+2k+1}{2k} = (-1)^n (n+1) 2^{2n} \tag{3.27}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{2^{2k}}{\binom{2k}{k}} = \frac{1}{1-2n} \tag{3.28}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{x+k}{k} \frac{2^{2k}}{\binom{2k}{k}(x+k)} = (-1)^n \frac{\binom{2x}{2n}}{x\binom{x}{n}}, \qquad x \neq 0 \tag{3.29}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}\frac{2^{2k}}{\binom{2k}{k}(n+k)} = \frac{(-1)^n}{n}, \qquad n \geq 1 \tag{3.30}$$

n^{th} *Difference of the Harmonic Series*

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\sum_{j=1}^{2k}\frac{1}{j} = \frac{1}{2n} + \frac{2^{2n-1}}{n\binom{2n}{n}}, \qquad n \geq 1 \tag{3.31}$$

Inversion of Equation (3.31)

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{2^{2k}}{k\binom{2k}{k}} = 2\sum_{j=1}^{2n}\frac{1}{j} - \sum_{j=1}^{n}\frac{1}{j}, \qquad n \geq 1 \tag{3.32}$$

3.5 Expansions of $\frac{\sin(2n+1)x}{\sin x}$

3.5.1 Using $\sin^{2k} x$

$$\sin(2n+1)x = \sum_{k=0}^{n}(-1)^k \sin^{2k+1} x \sum_{j=0}^{k}\binom{2n+1}{2j+1}\binom{n-j}{k-j} \tag{3.33}$$

$$\begin{aligned}\sum_{j=0}^{k}\binom{2n+1}{2j+1}\binom{n-j}{k-j} &= \frac{2n+1}{(2k+1)!}\prod_{j=0}^{k-1}\left((2n+1)^2-(2j+1)^2\right) \\ &= 2^{2k}\frac{2n+1}{n-k}\binom{n+k}{2k+1}\end{aligned} \tag{3.34}$$

$$\begin{aligned}\sum_{j=0}^{k}\binom{2n+1}{2k-2j+1}\binom{n-k+j}{j} &= \frac{2n+1}{(2k+1)!}\prod_{j=0}^{k-1}\left((2n+1)^2-(2j+1)^2\right) \\ &= 2^{2k}\frac{2n+1}{n-k}\binom{n+k}{2k+1}\end{aligned} \tag{3.35}$$

Restatement of Equation (3.33)

$$\sin(2n+1)x = \sum_{k=0}^{n}(-1)^k 2^{2k}\frac{2n+1}{n-k}\binom{n+k}{2k+1}\sin^{2k+1} x \tag{3.36}$$

3.5.2 Using $\cos^{2k} x$

$$\frac{\sin(2n+1)x}{\sin x} = \sum_{k=0}^{n} (-1)^{n-k} \cos^{2k} x \sum_{j=0}^{k} \binom{2n+1}{2k-2j} \binom{n-k+j}{j} \tag{3.37}$$

$$\frac{\sin(2n+1)x}{\sin x} = \sum_{k=0}^{n} (-1)^{n-k} \cos^{2k} x \sum_{j=0}^{k} \binom{2n+1}{2j} \binom{n-j}{k-j} \tag{3.38}$$

3.5.3 Binomial Identities Resulting From Coefficient of $\sin^{2k} x$ in Equation (3.36)

$$\sum_{k=j}^{n} (-1)^k 2^{2k} \frac{2n+1}{n-k} \binom{n+k}{2k+1} \binom{n}{j} = (-1)^n \sum_{r=0}^{j} \binom{2n+1}{2r} \binom{n-r}{j-r} \tag{3.39}$$

$$\sum_{k=0}^{n} (-1)^k 2^{2k} \frac{2n+1}{n-k} \binom{n+k}{2k+1} = (-1)^n \tag{3.40}$$

Generalization of Equation (3.34)

$$\begin{aligned} \sum_{k=0}^{n} \binom{2x+1}{2k+1} \binom{x-k}{n-k} &= \frac{2x+1}{(2n+1)!} \prod_{k=0}^{n-1} \left((2x+1)^2 - (2k+1)^2\right) \\ &= 2^{2n} \frac{2x+1}{2n+1} \binom{x+n}{2n} \end{aligned} \tag{3.41}$$

3.5.4 Product Expansion for $\frac{\sin(2n+1)x}{\sin x}$

$$\frac{\sin(2n+1)x}{\sin x} = (2n+1) \prod_{k=1}^{n} \left(1 - \frac{\sin^2 x}{\sin^2 \frac{\pi k}{2n+1}}\right), \qquad n \geq 1 \tag{3.42}$$

3.6 Series for $\cos^n x$

$$2^{n-1} \cos^n x = \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{k} \cos(n-2k)x - \frac{1}{2} \binom{n}{[\frac{n}{2}]} \frac{(-1)^n + 1}{2} \cos x \tag{3.43}$$

3.6.1 Applications of Equation (3.43)

$$\sum_{k=0}^{[\frac{n}{2}]} \binom{n}{k} = 2^{n-1} + \frac{1}{2} \binom{n}{[\frac{n}{2}]} \frac{(-1)^n + 1}{2} \tag{3.44}$$

$$\cos^{2n+1} x = \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{2n+1}{k} \cos(2n+1-2k)x \tag{3.45}$$

$$\int \cos^{2n+1} x \, dx = \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{2n+1}{k} \frac{\sin(2n+1-2k)x}{2n+1-2k} + C \tag{3.46}$$

$$\cos^{2n} x = \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} \binom{2n}{k} \cos(2n-2k)x + \frac{1}{2^{2n}} \binom{2n}{n}, \qquad n \geq 1 \tag{3.47}$$

$$\int \cos^{2n} x \, dx = \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} \binom{2n}{k} \frac{\sin(2n-2k)x}{2n-2k} + \frac{1}{2^{2n}} \binom{2n}{n} x + C, \qquad n \geq 1 \tag{3.48}$$

$$\frac{\sin(2n+1)x}{\sin x} = 2 \sum_{k=0}^{n} \cos 2kx - 1 \tag{3.49}$$

3.7 Series for $\sin^n x$

$$2^{n-1} \sin^n x = \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{k} \cos\left((n-2k)x - \frac{n-2k}{2}\pi \right) - \frac{1}{2} \binom{n}{[\frac{n}{2}]} \frac{(-1)^n + 1}{2} \tag{3.50}$$

3.7.1 Applications of Equation (3.50)

$$\sin^{2n} x = \frac{(-1)^n}{2^{2n-1}} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \cos(2n-2k)x + \frac{1}{2^{2n}} \binom{2n}{n}, \qquad n \geq 1 \tag{3.51}$$

$$\int \sin^{2n} x \, dx = \frac{(-1)^n}{2^{2n-1}} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \frac{\sin(2n-2k)x}{2n-2k} + \frac{1}{2^{2n}} \binom{2n}{n} + C, \qquad n \geq 1 \tag{3.52}$$

$$\sin^{2n+1} x = \frac{(-1)^n}{2^{2n}} \sum_{k=0}^{n} (-1)^k \binom{2n+1}{k} \sin(2n+1-2k)x \tag{3.53}$$

$$\int \sin^{2n+1} x \, dx = \frac{(-1)^n}{2^{2n}} \sum_{k=0}^{n} (-1)^{k+1} \binom{2n+1}{k} \frac{\cos(2n+1-2k)x}{2n+1-2k} + C \tag{3.54}$$

4 Advanced Trigonometric Product Expansions

Remark 4.1 *For this chapter, we assume n is a nonnegative integer, while x, y, and z are real or complex numbers. We also assume, that whenever x is a real number, $[x]$ denotes the floor of x.*

4.1 Product Expansion of $\cos nx - \cos ny$

$$\boldsymbol{\cos nx - \cos ny = 2^{n-1} \prod_{k=0}^{n-1} \left(\cos x - \cos \left(y + \frac{2k\pi}{n} \right) \right), \qquad n \geq 1} \tag{4.1}$$

4.1.1 Applications of Equation (4.1)

$$\cos nx + 1 = 2^{n-1} \prod_{k=0}^{n-1} \left(\cos x - \cos \frac{2k+1}{n} \pi \right), \qquad n \geq 1 \tag{4.2}$$

$$\prod_{k=0}^{n-1} \cos \left(y + \frac{2k\pi}{n} \right) = \frac{1}{2^{n-1}} \left((-1)^{[\frac{n}{2}]} \frac{1+(-1)^n}{2} - (-1)^n \cos ny \right), \qquad n \geq 1 \tag{4.3}$$

$$\prod_{k=0}^{2n-1} \cos \left(y + \frac{k\pi}{n} \right) = \frac{(-1)^n - \cos 2ny}{2^{2n-1}}, \qquad n \geq 1 \tag{4.4}$$

$$\prod_{k=0}^{2n} \cos \left(y + \frac{2k\pi}{2n+1} \right) = \frac{\cos(2n+1)y}{2^{2n}} \tag{4.5}$$

$$\prod_{k=0}^{2n} \cos \frac{2k\pi}{2n+1} = \frac{1}{2^{2n}} \tag{4.6}$$

$$\prod_{k=0}^{2n} \cos \frac{2k+1}{2n+1}\pi = -\frac{1}{2^{2n}} \tag{4.7}$$

$$\prod_{k=0}^{4n+1} \cos \frac{k\pi}{2n+1} = -\frac{1}{2^{4n}} \tag{4.8}$$

4.1.2 Product Expansion of $\sin nx$

$$\boldsymbol{\sin nx = 2^{n-1} \prod_{k=0}^{n-1} \sin\left(x + \frac{k\pi}{n}\right), \qquad n \geq 1} \tag{4.9}$$

Applications of Equation (4.9)

$$\sin^2 \frac{ny}{2} = 2^{2n-2} \prod_{k=0}^{n-1} \sin^2\left(\frac{y}{2} + \frac{k\pi}{n}\right), \qquad n \geq 1 \tag{4.10}$$

$$\prod_{k=0}^{n-1} \sin\left(\frac{k\pi + x}{n}\right) = \frac{\sin x}{2^{n-1}}, \qquad n \geq 1 \tag{4.11}$$

$$\prod_{k=1}^{n-1} \cos \frac{k\pi}{n} = \frac{(-1)^{\left[\frac{n}{2}\right]}}{2^{n-1}} \left(\frac{1-(-1)^n}{2}\right), \qquad n \geq 2 \tag{4.12}$$

$$\prod_{k=1}^{2n} \cos \frac{k\pi}{2n+1} = \frac{(-1)^n}{2^{2n}}, \qquad n \geq 1 \tag{4.13}$$

$$\prod_{k=1}^{n-1} \sin \frac{k\pi}{n} = \frac{n}{2^{n-1}}, \qquad n \geq 2 \tag{4.14}$$

$$\prod_{k=1}^{n-1} \Gamma\left(\frac{k}{n}\right) = \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}}, \qquad n \geq 1 \tag{4.15}$$

$$\prod_{k=1}^{n-1} \cot \frac{k\pi}{n} = \frac{(-1)^{\left[\frac{n}{2}\right]}}{n} \left(\frac{1-(-1)^n}{2}\right), \qquad n \geq 2 \tag{4.16}$$

$$n \cot nx = \sum_{k=0}^{n-1} \cot \left(x + \frac{k\pi}{n} \right), \qquad n \geq 1 \tag{4.17}$$

$$n^2 \csc^2 nx = \sum_{k=0}^{n-1} \csc^2 \left(x + \frac{k\pi}{n} \right), \qquad n \geq 1 \tag{4.18}$$

4.2 Various Product Expansions Involving Equations (4.11) and (4.14)

4.2.1 Expansions Involving Equation (4.14)

$$\prod_{k=1}^{n-1} \left(\sin \frac{k\pi}{n} \right)^k = \frac{\sqrt{n^n}}{2^{\frac{n(n-1)}{2}}}, \qquad n \geq 2 \tag{4.19}$$

Remark 4.2 *The following identity, proposed by J. E. Wilkins, Jr., is found in Problem E1044 of The American Math. Monthly, Vol. 59, No. 10, December 1952.*

$$\prod_{k=1}^{n-1} \left(2 \sin \frac{k\pi}{n} \right)^k = \sqrt{n^n}, \qquad n \geq 2 \tag{4.20}$$

4.2.2 Expansion Involving Equation (4.11)

$$\prod_{k=1}^{n-1} \left(\sin \frac{k\pi - x}{n} \sin \frac{k\pi + x}{n} \right)^k = \frac{1}{2^{n(n-1)}} \left(\frac{\sin x}{\sin \frac{x}{n}} \right)^n, \qquad n \geq 2 \tag{4.21}$$

Applications of Equation (4.21)

$$\prod_{k=1}^{n-1} \left(\cos \frac{k\pi}{n} \right)^{2k} = \frac{(-1)^{\frac{n(n-1)}{2}}}{2^{n(n-1)}} (-1)^{n[\frac{n}{2}]} \left(\frac{1-(-1)^n}{2} \right)^n, \qquad n \geq 2 \tag{4.22}$$

$$n^2 \cot x - n \cot \frac{x}{n} = \sum_{k=1}^{n-1} k \left(\cot \frac{k\pi + x}{n} - \cot \frac{k\pi - x}{n} \right), \qquad n \geq 2 \tag{4.23}$$

$$\sum_{k=1}^{n-1} k \csc \frac{k\pi + x}{n} \cdot \csc \frac{k\pi - x}{n} = \frac{n \cot \frac{x}{n} - n^2 \cot x}{\sin \frac{2x}{n}}, \qquad n \geq 2 \tag{4.24}$$

4.2.3 Product Expansion for $\tan x$

$$\tan x = \prod_{k=0}^{2n-1} \left(\sin \frac{k\pi + 2x}{2n} \right)^{(-1)^k}, \qquad n \geq 1 \tag{4.25}$$

Applications of Equation (4.25)

$$\sum_{k=0}^{2n-1} (-1)^k \cot \frac{k\pi + 2x}{2n} = \frac{2n}{\sin 2x}, \qquad n \geq 1 \tag{4.26}$$

$$\sum_{k=0}^{2n-1} (-1)^k \cot \frac{2k+1}{4n}\pi = 2n, \qquad n \geq 1 \tag{4.27}$$

$$\sum_{k=0}^{2n-1} (-1)^k \cot \frac{3k+1}{6n}\pi = \frac{4n\sqrt{3}}{3}, \qquad n \geq 1 \tag{4.28}$$

$$\sum_{k=0}^{2n-1} (-1)^k \csc^2 \frac{k\pi + 2x}{2n} = 4n^2 \csc 2x \cdot \cot 2x, \qquad n \geq 1 \tag{4.29}$$

$$\sum_{k=0}^{2n-1} \cot \frac{4k+1}{4n}\pi = 2n, \qquad n \geq 1 \tag{4.30}$$

$$\sum_{k=0}^{n-1} \cot \frac{4k+1}{4n}\pi = n, \qquad n \geq 1 \tag{4.31}$$

Remark 4.3 *The following identity is Problem 4220 of The American Math Monthly, Vol. 58, No.1, May 1952.*

$$\sum_{k=0}^{n-1} (-1)^k \tan \frac{2k+1}{4n}\pi = (-1)^{n+1} n, \qquad n \geq 1 \tag{4.32}$$

$$\sum_{k=0}^{n-1} \tan \frac{4k+1}{4n}\pi = (-1)^{n+1} n, \qquad n \geq 1 \tag{4.33}$$

4.3 Expansions of $\cot z$

$$\begin{aligned}\cot z &= \frac{1}{2n+1}\cot\frac{z}{2n+1}\\ &\quad+\sum_{k=1}^{n}\left(\frac{1}{2n+1}\cot\frac{z+k\pi}{2n+1}+\frac{1}{2n+1}\cot\frac{z-k\pi}{2n+1}\right)\end{aligned} \tag{4.34}$$

$$\cot z = \frac{1}{z}+\sum_{k=1}^{\infty}\left(\frac{1}{z+k\pi}+\frac{1}{z-k\pi}\right), \qquad z \textit{ not a multiple of } \pi \tag{4.35}$$

$$\pi\cot\pi z = \frac{1}{z}+\sum_{k=1}^{\infty}\left(\frac{1}{z+k}+\frac{1}{z-k}\right), \qquad z \textit{ not integral} \tag{4.36}$$

4.3.1 Applications of Equation (4.36)

$$\pi\csc\pi z = \frac{1}{z}+\sum_{k=1}^{\infty}(-1)^k\frac{2z}{z^2-k^2}, \qquad z \textit{ not integral} \tag{4.37}$$

$$\pi^2\csc^2\pi z = \sum_{k=-\infty}^{\infty}\frac{1}{(z-k)^2}, \qquad z \textit{ not integral} \tag{4.38}$$

$$\pi^3\cot\pi z\csc^2\pi z = \sum_{k=-\infty}^{\infty}\frac{1}{(z-k)^3}, \qquad z \textit{ not integral} \tag{4.39}$$

$$\pi^4\left(\csc^4\pi z-\frac{2}{3}\csc^2\pi z\right) = \sum_{k=-\infty}^{\infty}\frac{1}{(z-k)^4}, \qquad z \textit{ not integral} \tag{4.40}$$

$$\pi\tan\frac{\pi z}{2} = \sum_{k=0}^{\infty}\frac{4z}{(2k+1)^2-z^2} \tag{4.41}$$

$$\pi\sec\pi z = \sum_{k=0}^{\infty}(-1)^k\frac{2k+1}{\left(\frac{2k+1}{2}\right)^2-z^2} \tag{4.42}$$

4.4 Expansions of $z \cot z$ via the Bernoulli Numbers

Remark 4.4 *In this section, we let $\mathcal{B}_n$ denote the n^{th} Bernoulli number.*

$$z \cot z = \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k} \mathcal{B}_{2k}}{(2k)!} z^{2k}, \qquad |z| < \pi \tag{4.43}$$

$$z \cot z = 1 - 2 \sum_{j=1}^{\infty} \frac{z^{2j}}{\pi^{2j}} \sum_{k=1}^{\infty} \frac{1}{k^{2j}}, \qquad |z| < \pi \tag{4.44}$$

4.4.1 Applications of Equations (4.43) and (4.44)

$$\sum_{k=1}^{\infty} \frac{1}{k^{2n}} = (-1)^{n-1} \frac{2^{2n-1} \pi^{2n}}{(2n)!} \mathcal{B}_{2n}, \qquad n \geq 1 \tag{4.45}$$

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^{2n}} = (-1)^{n-1} \frac{(2^{2n}-1) \pi^{2n}}{2(2n)!} \mathcal{B}_{2n}, \qquad n \geq 1 \tag{4.46}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^{2n}} = (-1)^{n-1} \frac{(2^{2n}-1) \pi^{2n}}{(2n)!} \mathcal{B}_{2n}, \qquad n \geq 1 \tag{4.47}$$

$$\tan z = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{2^{2k}(2^{2k}-1) \mathcal{B}_{2k}}{(2k)!} z^{2k-1}, \qquad |z| < \frac{\pi}{2} \tag{4.48}$$

$$\frac{z}{\sin z} = \sum_{k=0}^{\infty} (-1)^n \frac{2(1-2^{2k-1}) \mathcal{B}_{2k}}{(2k)!} z^{2k}, \qquad |z| < \pi, \ z \neq 0 \tag{4.49}$$

5 Series Associated with the Beta Function

Remark 5.1 *We assume m, n, k, and r are nonnegative integers, while x, y, and t are real or complex numbers. If necessary, we use the Gamma function to evaluate $x!$ as $\Gamma(x) = (x-1)!$. Finally, recall that $[x]$ denotes the floor of x (for real x).*

5.1 Formulas from $\int_0^{\frac{\pi}{2}} \sin^x t \cos^y t \, dt$

$$\int_0^{\frac{\pi}{2}} \sin^x t \, \cos^y t \, dt = \frac{\pi}{2^{x+y+1}} \frac{x!y!}{\left(\frac{x}{2}\right)! \left(\frac{y}{2}\right)! \left(\frac{x+y}{2}\right)!} \tag{5.1}$$

$$\int_0^{\frac{\pi}{2}} \sin^{2k} x \, \cos^{2n} x \, dx = \frac{\pi \binom{2k}{k}\binom{2n}{n}}{2^{2n+2k+1}\binom{n+k}{k}} \tag{5.2}$$

$$\int_0^{\frac{\pi}{2}} \sin^n x \, dx = \int_0^{\frac{\pi}{2}} \cos^n x \, dx = \frac{\pi}{2^{n+1}} \binom{n}{\frac{n}{2}} \tag{5.3}$$

5.2 Applications of Equation (5.2)

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}\binom{2k}{k} \frac{1}{\binom{n+k}{k}} = 1 \tag{5.4}$$

$$\sum_{k=0}^{2n} \binom{2n}{k}\binom{-\frac{1}{2}}{k} \frac{2^{2k}}{\binom{n+k}{k}} = 1 \tag{5.5}$$

$$\sum_{k=0}^{2n} (-1)^k \binom{3n}{n+k}\binom{2k}{k} = \binom{3n}{n} \tag{5.6}$$

$$\sum_{k=0}^{2n} \binom{3n}{2n-k}\binom{-\frac{1}{2}}{k} 2^{2k} = \binom{3n}{n} \tag{5.7}$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}\binom{2n+2k}{n+k} 3^{2n-k} = \binom{2n}{n} \tag{5.8}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{k}\frac{z^{2k}}{\binom{m+k}{k}} = \frac{2^{2m+1}}{\pi\binom{2m}{m}}\int_0^{\frac{\pi}{2}}(1-4z^2\sin^2 x)^n\ \cos^{2m}x\,dx \tag{5.9}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{k}\frac{z^{2k}}{\binom{m+k}{k}} = \frac{2^{2m+1}}{\pi\binom{2m}{m}}\int_0^{\frac{\pi}{2}}(1-4z^2\cos^2 x)^n\ \sin^{2m}x\,dx \tag{5.10}$$

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{2k}{k}\frac{1}{\binom{n+k}{k}2^{2k}} = \frac{\binom{6n}{3n}}{2^{4n}\binom{2n}{n}} \tag{5.11}$$

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{2k}{k}\frac{1}{\binom{n+r+k}{k}} = \frac{2^{2n+2r+1}}{\pi\binom{2n+2r}{n+r}}\int_0^{\frac{\pi}{2}}\cos^{2r}x\,(\cos 3x)^{2n}\,dx \tag{5.12}$$

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{2k}{k}\frac{1}{\binom{n+r+k}{k}} = \frac{2^{2n+2r+1}}{\pi\binom{2n+2r}{n+r}}\int_0^{\frac{\pi}{2}}\sin^{2r}x\,(\sin 3x)^{2n}\,dx \tag{5.13}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{k}\frac{1}{2^{2k}} = \frac{1}{2^{2n}}\binom{2n}{n} \tag{5.14}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{k} = (-1)^n\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{k}3^{n-k} \tag{5.15}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{k}\frac{1}{2^{2k}\binom{m+k}{k}} = \frac{\binom{2m+2n}{m+n}}{2^{2n}\binom{2m}{m}} \tag{5.16}$$

5.3 Generalization of Equation (5.12)

$$\sum_{k=0}^{2n+1}(-1)^k\binom{2n+1}{k}\binom{2k}{k}\frac{1}{\binom{n+r+k}{k}} = \frac{2^{2n+2r+1}}{\pi\binom{2n+2r}{n+r}}\int_0^{\frac{\pi}{2}}\cos^{2r-1}x\,(\cos 3x)^{2n+1}\,dx \tag{5.17}$$

5.3.1 Various Applications of 5.17 and 5.12

$$\sum_{k=0}^{2n+1}(-1)^k\binom{2n+1}{k}\binom{2k}{k}\frac{1}{\binom{n+1+k}{k}}=0 \tag{5.18}$$

$$\sum_{k=0}^{2n+1}(-1)^k\binom{2n+1}{k}\binom{2k}{k}\frac{1}{\binom{n+2+k}{k}}=\frac{n+2}{2(2n+3)} \tag{5.19}$$

$$\sum_{k=0}^{2n+1}(-1)^k\binom{2n+1}{k}\binom{2k}{k}\frac{1}{\binom{n+k}{k}}=-1 \tag{5.20}$$

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{2k}{k}\frac{1}{\binom{n+1+k}{k}}=\frac{n+1}{2n+1} \tag{5.21}$$

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{2k}{k}\frac{1}{\binom{n+2+k}{k}}=\frac{3(n+1)(n+2)}{2(2n+1)(2n+3)} \tag{5.22}$$

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{2k}{k}\frac{1}{\binom{n-1+k}{k}}=3, \qquad n\geq 1 \tag{5.23}$$

$$\sum_{k=0}^{2n+1}(-1)^k\binom{2n+1}{k}\binom{2k}{k}\frac{1}{\binom{n-1+k}{k}}=-\frac{5n+2}{n}, \qquad n\geq 1 \tag{5.24}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{k}\frac{1}{\binom{[\frac{n-2}{2}]+k}{k}}=3-\left(8+\frac{4}{n-1}\right)\frac{1-(-1)^n}{2}, \qquad n\geq 2 \tag{5.25}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{k}\frac{1}{\binom{[\frac{n-1}{2}]+k}{k}}=2(-1)^n+1, \qquad n\geq 1 \tag{5.26}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{k}\frac{1}{\binom{[\frac{n}{2}]+k}{k}}=(-1)^n \tag{5.27}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{k}\frac{1}{\binom{[\frac{n+1}{2}]+k}{k}}=\frac{(-1)^n+1}{2} \tag{5.28}$$

5.3.2 Application of Equation (5.21)

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{2n+2k}{n+k}\frac{3^{2n-k}}{n+k+1}=\binom{2n}{n} \tag{5.29}$$

5.3.3 Application of Equation (5.20)

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{2k}{k}\frac{2k+1}{\binom{n+k}{k}(n+k+1)}=1 \tag{5.30}$$

6 Complex Roots of Unity in Series

Remark 6.1 *Throughout this chapter, we let* $i \equiv \sqrt{-1}$. *We assume, unless otherwise specified, that* n *and* k *are nonnegative integers, while* x, y *and* z *denote real or complex numbers. We also let* $[x]$ *denote the floor of* x *(*x *real).*

6.1 Definition of w_n and Basic Orthogonality Relations

6.1.1 Definition of n^{th} Roots of Unity

Let

$$\boldsymbol{w_n = \cos\frac{2\pi}{n} + i\sin 2\pi n.} \tag{6.1}$$

The n^{th} *roots of unity are*

$$\boldsymbol{w_n^k = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}, \qquad k = 0, 1, 2, ..., n-1.} \tag{6.2}$$

6.1.2 Orthogonality Relations

$$\boldsymbol{\frac{1}{n}\sum_{k=0}^{n-1} w_n^{kr} = \begin{cases} 1, & r = \alpha n \\ 0, & r \neq \alpha n, \end{cases} \quad n \geq 2, \qquad r \textit{ and } \alpha \textit{ integers}} \tag{6.3}$$

$$\sum_{k=0}^{n-1}(-1)^k w_n^{rk} = \begin{cases} 0, \textit{ if } n \textit{ is even, } n \geq 1 \\ \frac{2}{1+w_n^r} \textit{ if } n \textit{ is odd} \end{cases} \qquad r \textit{ an integer} \tag{6.4}$$

Applications of Equation (6.3)

$$\sum_{k=0}^{\infty}\frac{x^{kn}}{(kn)!} = \frac{1}{n}\sum_{k=0}^{n-1} e^{xw_n^k}, \qquad n \geq 1 \tag{6.5}$$

Remark 6.2 *Let z be a complex number. We let $\bar{z}$ denote the conjugate of z.*

$$\frac{1}{n}\sum_{k=0}^{n-1} w_n^{k\alpha}\bar{w}_n^{k\beta} = \binom{0}{\alpha-\beta},\ n\geq 2,\ \ 0\leq \alpha,\beta\leq n-1 \tag{6.6}$$

α and β are nonnegative integers

Let $f_n(x)=\sum_{j=1}^{n-1} a_j x^j$. Then,

$$f_n(x) = \frac{1}{n}\sum_{j=1}^{n-1}\sum_{k=0}^{n-1}\bar{w}_n^{kj} f_n(xw_n^k), \qquad n\geq 2 \tag{6.7}$$

6.2 Complex Roots of Unity in Evaluation of Series

6.2.1 Evaluation of $\sum_{k=0}^{n} f(kr)$

Remark 6.3 *In this section, we assume r is a positive integer. We also assume f is a real or complex valued function whose domain contains the set of nonnegative integers.*

$$\sum_{k=0}^{n} f(kr) = \frac{1}{r}\sum_{k=0}^{rn}\sum_{j=1}^{r} w_r^{jk} f(k) \tag{6.8}$$

$$\sum_{k=0}^{n} f(kr) = \frac{1}{r}\sum_{k=0}^{rn}\sum_{j=1}^{r}\cos\frac{2\pi jk}{r} f(k), \qquad f \text{ real valued} \tag{6.9}$$

$$\sum_{k=0}^{[\frac{n}{r}]}\binom{n}{rk} f(kr) = \frac{1}{r}\sum_{k=0}^{n}\sum_{j=1}^{r}\binom{n}{k}\cos\frac{2\pi jk}{r} f(k), \qquad f \text{ real valued} \tag{6.10}$$

6.2.2 Applications of Equation (6.8)

$$\sum_{k=0}^{[\frac{n}{r}]}\binom{n}{rk} = \frac{1}{r}\sum_{k=1}^{r}(1+w_r^k)^n \tag{6.11}$$

$$\sum_{k=0}^{[\frac{n}{2}]}\binom{n}{2k} = \begin{cases} 2^{n-1}, & n\geq 1 \\ 1, & n=0 \end{cases} \tag{6.12}$$

$$\sum_{k=0}^{[\frac{n}{r}]} \binom{n}{rk} = \frac{2^n}{r} \sum_{j=1}^{r} \left(\cos \frac{\pi j}{r}\right)^n \cos \frac{n\pi j}{r} \tag{6.13}$$

$$\sum_{k=0}^{[\frac{n}{3}]} \binom{n}{3k} = \frac{1}{3} \left(2^n + 2\cos \frac{n\pi}{3}\right) \tag{6.14}$$

$$\sum_{k=0}^{n} \binom{3n}{3k} = \frac{1}{3}(2^{3n} + 2(-1)^n) \tag{6.15}$$

$$\sum_{k=0}^{[\frac{n}{4}]} \binom{n}{4k} = \frac{1}{4} \left(2^n + 2^{\frac{n}{2}+1} \cos \frac{n\pi}{4}\right) \tag{6.16}$$

$$\sum_{k=0}^{n} \binom{4n}{4k} = \frac{1}{4}(2^{4n} + (-1)^n 2^{2n+1}) \tag{6.17}$$

$$\sum_{k=1}^{n} (-1)^k \left(\cos \frac{\pi k}{n}\right)^n = \frac{n}{2^{n-1}}, \qquad n \geq 1 \tag{6.18}$$

$$\sum_{k=0}^{[\frac{n}{r}]} \binom{n}{rk} x^{rk} = \frac{1}{r} \sum_{k=1}^{r} (1 + x w_r^k)^n \tag{6.19}$$

$$\sum_{k=0}^{[\frac{n-a}{r}]} \binom{n}{a+kr} x^{a+kr} = \frac{1}{r} \sum_{k=1}^{r} (w_r^k)^{-a} (1 + x w_r^k)^n, \;\; 0 \leq a \leq n, \;\; a \leq r-1, \;\; a \textit{ an integer} \tag{6.20}$$

$$\sum_{k=0}^{[\frac{n-a}{r}]} \binom{n}{a+kr} = \frac{1}{r} \sum_{k=1}^{r} \left(2\cos \frac{\pi k}{r}\right)^n \cos \frac{(n-2a)k\pi}{r}, \tag{6.21}$$

$0 \leq a \leq n, \;\; a \leq r-1, \;\;$ *a an integer*

$$\sum_{k=0}^{[\frac{n-1}{3}]} \binom{n}{3k+1} = \frac{1}{3}\left(2^n + 2\cos\frac{(n-2)\pi}{3}\right), \qquad n \geq 1 \tag{6.22}$$

$$\sum_{k=0}^{[\frac{2n}{3}]} \binom{n}{3k-n} = \frac{2^n + 2(-1)^n}{3} \tag{6.23}$$

Remark 6.4 *The following identity is W. J. Taylor's Problem 4152 Page 163 of The American Math. Monthly, 1945.*

$$\frac{1}{2n}\sum_{k=1}^{2n}\left(2\cos\frac{\pi k}{2n}\right)^{2n}\cos\frac{\alpha k\pi}{n} = \binom{2n}{n-\alpha}, \text{ } n \text{ and } \alpha \text{ integers, } n \geq 1, -n < \alpha < n \tag{6.24}$$

$$\frac{2^{2n}}{2n}\sum_{k=1}^{2n}\left(\cos\frac{k\pi}{2n}\right)^{2n} = \binom{2n}{n}, \qquad n \geq 1 \tag{6.25}$$

$$\sum_{k=1}^{n}\left(\cos\frac{k\pi}{n}\right)^{2n} = \frac{n}{2^{2n}}\left(\binom{2n}{n} + 2\right), \qquad n \geq 1 \tag{6.26}$$

$$\sum_{k=1}^{n}\left(\cos\frac{(2k-1)\pi}{2n}\right)^{2n} = \frac{n}{2^{2n}}\left(\binom{2n}{n} - 2\right), \qquad n \geq 1 \tag{6.27}$$

6.2.3 Convolution Formula via Equation (6.8)

Remark 6.5 *In this section, we assume g is a real or complex valued function whose domain contains the set of nonnegative integers. We will also assume r is a nonnegative integer.*

$$\sum_{k=0}^{\infty} x^{rk} f(k) \sum_{j=0}^{\infty} x^j g(j) = \sum_{j=0}^{\infty}\sum_{k=0}^{[\frac{j}{r}]} f(k) g(j-rk) \tag{6.28}$$

$$e^x\left(e^{\frac{x^n}{n}} - 1\right) = \sum_{j=1}^{\infty} x^j \sum_{k=1}^{[\frac{j}{n}]} \frac{1}{n^k k!(j-kn)!}, \qquad n \geq 2 \tag{6.29}$$