Combinatorial Identities: Table II: Advanced Techniques for Summing Finite Series

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1 Melzak's Formula

1.1 Statement of Melzak's Formula

Remark 1.1 *Throughout this chapter, we assume n and α are nonnegative integers, and x, y, and z are real or complex numbers. Furthermore, we use the convention that $f(x)$ represents a polynomial of degree n.*

Z. A. Melzak's Formula from Problem 4458 of The American Math. Monthly, 1951, P. 636

Let

$$f(x) = \sum_{i=0}^{n} a_i x^i. \tag{1.1}$$

Then,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{f(x-k)}{y+k} = \frac{f(x+y)}{y\binom{y+n}{n}}, \qquad y \neq 0, -1, ..., -n. \tag{1.2}$$

1.2 Basic Examples of Melzak's Formula

1.2.1 First Example: $f(x) \equiv 1$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{z+yk} = \frac{1}{z\binom{n+\frac{z}{y}}{n}}, \qquad z \neq 0, -y, -2y, ..., -yn \tag{1.3}$$

$$\sum_{k=0}^{\alpha n+1}(-1)^k\binom{\alpha n+1}{k}\frac{1}{\alpha^2 n-\alpha n+\alpha-1-\alpha k}=\frac{(-1)^{\alpha n}}{(\alpha n+1)\binom{\frac{\alpha^2 n-\alpha n+\alpha-1}{\alpha}}{\alpha n+1}},\qquad \alpha\geq 2 \tag{1.4}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{n^2-2n-(n-1)k}=\frac{(-1)^{n-1}}{n\binom{\frac{n^2-2n}{n-1}}{n}},\qquad n\geq 3 \tag{1.5}$$

$$\sum_{k=0}^{n}(-1)^k\binom{2n+1}{k}\frac{2n+1-2k}{(2n+1-nk)(2n^2-n-1-nk)}$$
$$=\frac{1}{n(2n+1)\binom{\frac{2n^2-n-1}{n}}{2n+1}},\qquad n\geq 2 \tag{1.6}$$

Inversion of Equation (1.3)

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{\binom{k+\frac{z}{y}}{k}}=\frac{z}{z+yn},\qquad z\neq -yn \tag{1.7}$$

$$\sum_{k=0}^{\alpha n+1}(-1)^k\binom{\alpha n+1}{k}\frac{1}{\binom{k+\frac{\alpha n+1}{\alpha}}{k}}=\frac{1}{1+\alpha},\qquad \alpha\geq 1 \tag{1.8}$$

$$\sum_{k=0}^{\alpha n+1}(-1)^k\binom{\alpha n+1}{k}\frac{1}{\binom{k-\frac{\alpha n+1}{\alpha}}{k}}=\frac{1}{1-\alpha},\qquad \alpha\geq 2 \tag{1.9}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{2^k k!}{\prod_{\alpha=0}^{k}(z+2\alpha)}=\frac{1}{z+2n},\qquad z\neq 0,-2,-4,...,-2n \tag{1.10}$$

$$\sum_{k=0}^{n}(-1)^k\binom{2n}{n-k}=\frac{1}{2}\binom{2n}{n}=\binom{2n-1}{n},\qquad n\geq 1 \tag{1.11}$$

$$\sum_{k=0}^{2n+1}(-1)^k\binom{2n+1}{k}\frac{1}{\binom{k+\frac{2n+1}{2}}{k}}=\frac{1}{3} \tag{1.12}$$

$$\sum_{k=0}^{2n+1}(-1)^k\frac{\binom{3n+2}{n+k+1}}{\binom{2n+2k+1}{n+k}}2^{2k}=\frac{1}{3}\frac{\binom{3n+2}{n+1}}{\binom{2n+1}{n}} \tag{1.13}$$

1.2.2 Second Example: $f(x) \equiv x^n$

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{(x-k)^n}{y+k} = \frac{(x+y)^n}{y\binom{y+n}{n}}, \qquad y \neq 0, -1, -2, ..., -n \tag{1.14}$$

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{(n-k)^n}{n+k} = \frac{2^n n^{n-1}}{\binom{2n}{n}} \tag{1.15}$$

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{(n-k)^n}{1+k} = (n+1)^{n-1} \tag{1.16}$$

$$(x+1)^n = \sum_{\alpha=0}^{n} \frac{(-1)^\alpha}{\alpha+1} \sum_{k=\alpha}^{n} \binom{n}{k}\binom{k}{\alpha}(x-k)^{n-k}(k-\alpha)^k, \qquad \textit{with } 0^0 \equiv 1 \tag{1.17}$$

1.2.3 Third Example: $f(x) \equiv \binom{x+n}{n}$

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k}\binom{x-k+n}{n} \frac{1}{y+k} = \frac{\binom{x+y+n}{n}}{y\binom{y+n}{n}}, \qquad y \neq 0, -1, -2, ..., -n \tag{1.18}$$

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k}\binom{2n-k}{n} \frac{1}{y+k} = \frac{\binom{2n+y}{n}}{y\binom{y+n}{n}}, \qquad y \neq 0, -1, -2, ..., -n \tag{1.19}$$

$$\sum_{k=0}^{n}(-1)^k \binom{2n}{k}\binom{2n-k}{n}^2 \frac{1}{y+k} = \frac{\binom{2n}{n}\binom{2n+y}{n}}{y\binom{y+n}{n}}, \qquad y \neq 0, -1, -2, ..., -n \tag{1.20}$$

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k}\binom{3n-k}{n} \frac{1}{2n-k} = \frac{(-1)^n}{n\binom{2n}{n}} \tag{1.21}$$

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k}\binom{n+k}{n} \frac{1}{y+k} = \sum_{k=0}^{n}(-1)^k \binom{n+k}{2k}\binom{2k}{k} \frac{1}{y+k} = (-1)^n \frac{\binom{y-1}{n}}{y\binom{y+n}{n}} \tag{1.22}$$

1.2.4 Fourth Example: $f(x) = \binom{x^2}{n}$

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{(y-k)^2}{n}\frac{1}{x+k} = \frac{\binom{(y+x)^2}{n}}{x\binom{x+2n}{2n}}, \qquad x \neq 0, -1, -2, ..., -2n \tag{1.23}$$

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{k^2}{n}\frac{1}{x+k} = \frac{\binom{x^2}{n}}{x\binom{x+2n}{2n}}, \qquad x \neq 0, -1, -2, ..., -2n \tag{1.24}$$

1.2.5 Fifth Example: $f(x) = \binom{mx}{n}$

Remark 1.2 *For this example, we assume m is a real or complex number.*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{m(x-k)}{n}\frac{1}{y+k} = \frac{\binom{m(x+y)}{n}}{y\binom{y+n}{n}}, \qquad y \neq 0, -1, -2, ..., -n \tag{1.25}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{-mk}{n}\frac{1}{y+k} = \frac{\binom{my}{n}}{y\binom{y+n}{n}}, \qquad y \neq 0, -1, -2, ..., -n \tag{1.26}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{mk}{n}\frac{1}{y+k} = (-1)^n\frac{\binom{-my+n-1}{n}}{y\binom{y+n}{n}}, \qquad y \neq 0, -1, -2, ..., -n \tag{1.27}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{z-yk}{n}\frac{z}{z-yk} = \begin{cases} 1, & n = 0 \\ 0, & n \geq 1 \end{cases} \tag{1.28}$$

1.2.6 Sixth Example: $f(x) = \binom{z+\alpha x}{r}x^p$

Remark 1.3 *In this example, we assume α is a real or complex number. We also assume r and p are positive integers.*

$$\sum_{k=0}^{r+p}(-1)^k\binom{r+p}{k}\binom{z+\alpha x-\alpha k}{r}\frac{(x-k)^p}{y+k} = \frac{(x+y)^p}{y\binom{y+r+p}{r+p}}\binom{z+\alpha x+\alpha y}{r} \tag{1.29}$$

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{z-\alpha k}{n}\frac{(x-k)^n}{y+k} = \frac{(x+y)^n}{y\binom{y+2n}{2n}}\binom{z+\alpha y}{n} \tag{1.30}$$

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{z-\alpha k}{n}\frac{k^n}{k+1} = (-1)^n\frac{1}{2n+1}\binom{z+\alpha}{n} \tag{1.31}$$

1.2.7 Seventh Example: $f(x) = \binom{x}{r}^p$

Remark 1.4 *For this example, we assume r and p are positive integers.*

$$\sum_{k=0}^{rp}(-1)^k\binom{rp}{k}\binom{x-k}{r}^p\frac{1}{y+k} = \frac{\binom{x+y}{r}^p}{y\binom{y+rp}{n}}, \qquad y \neq 0, -1, -2, ..., -rp \tag{1.32}$$

$$\sum_{k=0}^{rp}(-1)^k\binom{rp}{k}\binom{rp-k}{r}^p\frac{1}{k+r} = \frac{1}{r}\binom{rp+r}{r}^{p-1} \tag{1.33}$$

1.3 Advanced Applications of Melzak's Formula

1.3.1 First Application: $f(x) \equiv 1$

$$\frac{1}{\binom{n+x}{n}} = \sum_{k=0}^{n}(-1)^k\frac{x}{x+k} = \sum_{j=0}^{\infty}(-1)^j\frac{1}{x^{j+n}}\sum_{k=0}^{n}(-1)^{n+k}\binom{n}{k}k^{j+n} \tag{1.34}$$

1.3.2 Second Application: $f(x) = x^n$

$$\sum_{k=0}^{n}\binom{n}{k}(x-k)^k(1-x+k)^{n-k}\frac{y}{y+k} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(x+y)^k}{\binom{y+k}{k}} \tag{1.35}$$

A Variation of Equation (1.35)

$$\sum_{k=0}^{n}\binom{n}{k}(x-bk)^k(b-x+bk)^{n-k}\frac{y}{y+k} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}b^{n-k}\frac{(x+by)^k}{\binom{y+k}{k}}, \tag{1.36}$$

where b is a nonzero real or complex number.

Three Generalizations of Equation (1.35)

Remark 1.5 *For the following three identities, we assume a and b are nonzero real or complex numbers. We also assume p is a nonnegative integer.*

$$\sum_{k=0}^{p}\binom{p}{k}\frac{(by-bk)^k(a-by+bk)^{p-k}}{x+k} = \sum_{n=0}^{p}(-1)^n\binom{p}{n}a^{p-n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(by-bk)^n}{x+k} \tag{1.37}$$

$$\sum_{k=0}^{p}\binom{p}{k}\frac{(y-bk)^k(a-y+bk)^{p-k}}{x+k}=\sum_{n=0}^{p}(-1)^n\binom{p}{n}a^{p-n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(y-bk)^n}{x+k} \tag{1.38}$$

$$\sum_{k=0}^{p}\binom{p}{k}\frac{(y-bk)^k(1-y+bk)^{p-k}}{x+k}=\sum_{n=0}^{p}(-1)^n\binom{p}{n}\frac{(y+bx)^n}{x\binom{x+n}{n}} \tag{1.39}$$

1.3.3 Evaluation of $\sum_{k=1}^{n}\frac{1}{k}$

$$\sum_{k=1}^{n}\frac{1}{k}=\frac{f'(x)}{f(x)}-\sum_{k=1}^{n}(-1)^k\binom{n}{k}\frac{f(x-k)}{kf(x)}, \qquad n\geq 1 \tag{1.40}$$

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{f(x-k)}{k}=f(x)\sum_{k=1}^{n}\frac{1}{k}-f'(x), \qquad n\geq 1 \tag{1.41}$$

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\binom{x-k}{n}\frac{1}{k}=(x-n)\binom{x}{n}\sum_{k=1}^{n}\frac{1}{k(x+k-n)}, \qquad n\geq 1 \tag{1.42}$$

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\binom{n+k}{k}\frac{1}{k}=2\sum_{k=1}^{n}\frac{1}{k}, \qquad n\geq 1 \tag{1.43}$$

1.3.4 Fourth Application: Let $f(x)=\frac{\binom{2x+2n}{2n}}{\binom{x+n}{n}}$

Remark 1.6 *The following identity is the solution to Example III, (A), in "Théorie des Nombres" par Edouard Lucas, Tome Premier, Gauthier-Villars, Paris, Chapter XI, P. 187.*

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}\frac{1}{x+k}=\frac{\binom{2n}{n}\binom{2x+2n}{2n}}{x\binom{x+n}{n}^2}=\frac{2^{2n}\binom{n+x-\frac{1}{2}}{n}}{x\binom{x+n}{n}} \tag{1.44}$$

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}\frac{1}{x+k}=\frac{2^n(2x+1)(2x+3)...(2x+n-1)}{x(x+1)...(x+n)}, \qquad n\geq 1 \tag{1.45}$$

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}\frac{1}{k+1}=\frac{2n+1}{n+1}\binom{2n}{n} \tag{1.46}$$

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}\frac{1}{2k+1}=\frac{2^{4n}}{(2n+1)\binom{2n}{n}}=\frac{2^{2n}}{\binom{n+\frac{1}{2}}{n}} \tag{1.47}$$

1.3.5 Evaluation of $\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{\binom{b+k}{c}}$

Remark 1.7 *The following identity is due to R. Frisch. It can by found in Eugen Netto's "Lehrbuch der Combinatorik", Leipzig, 1901, Sec. Ed. 1927, reprint by Chelsea Publ. Co., New York, N.Y. 1958. This identity can also be found in R. Frisch's "Sur les semi-invariants, etc. ", Videnskaps-Akademiets Skrifter, II, No. 3, Oslo, 1926.*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{\binom{b+k}{c}} = \frac{c}{n+c}\frac{1}{\binom{n+b}{b-c}}, \tag{1.48}$$

provided b *and* c *are positive integers,* $b \geq c$*, and* $\binom{b+k}{c} \neq 0$ *for* $k = 0, 1, 2, ..., n$*.*

Two restatement of Equation (1.48)

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{c}{k+c}\frac{1}{\binom{k+b}{b-c}} = \frac{1}{\binom{b+n}{c}} \tag{1.49}$$

provided b *and* c *are positive integers,* $b \geq c$*, and* $\binom{b+k}{b-c} \neq 0$ *for* $k = 0, 1, 2, ..., n$*.*

$$\sum_{k=0}^{\alpha n}(-1)^k\binom{\alpha n}{k}\frac{1}{\binom{\beta n+k}{k}} = \frac{\beta}{\alpha+\beta}, \quad \alpha \text{ and } \beta \text{ positive integers, } \alpha \neq \beta. \tag{1.50}$$

Specific Example of Equation (1.48)

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{\binom{3n+k}{2n}} = \frac{2}{3\binom{4n}{n}} \tag{1.51}$$

1.4 Generalized Melzak's Formula

Remark 1.8 *In the following identity, we evaluate* $x! = \Gamma(x+1)$*.*

$$\boldsymbol{\sum_{k=0}^{\infty}(-1)^k\binom{x}{k}\frac{1}{z+k} = \frac{\Gamma(z)\Gamma(x+1)}{\Gamma(x+z+1)} = \frac{1}{z\binom{x+z}{x}},} \tag{1.52}$$

where $Re(x) > -1$ *and* $z \neq 0, -1, -2, ...$

1.4.1 Applications of Equation (1.52)

$$\sum_{k=0}^{\infty}\binom{2k}{k}\frac{1}{2^{2k}}\frac{1}{z+k}=\frac{1}{z\binom{x-\frac{1}{2}}{z}} \tag{1.53}$$

$$\sum_{k=0}^{\infty}\binom{2k}{k}\frac{1}{2^{2k}(z+2k+1)}=\frac{\pi}{2^{z+1}}\binom{z}{\frac{z}{2}} \tag{1.54}$$

$$\sum_{k=0}^{\infty}\binom{2k}{k}\frac{1}{2^{2k}(2k+1)}=\frac{\pi}{2} \tag{1.55}$$

1.5 Derivative Version of Melzak's Formula

$$\boldsymbol{\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{f(x-k)}{(y+k)^2}=\frac{f(x+y)\sum_{k=0}^{n}\frac{1}{k+y}-\frac{d}{dy}f(x+y)}{y\binom{y+n}{n}},}$$
$$\boldsymbol{y\neq 0,-1,...,-n} \tag{1.56}$$

1.5.1 Applications of Equation (1.56)

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(x-k)^n}{(y+k)^2}=\frac{(x+y)^n\sum_{k=0}^{n}\frac{1}{k+y}-n(x+y)^{n-1}}{y\binom{y+n}{n}} \tag{1.57}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{k^n}{(1+k)^2}=(-1)^n\frac{\sum_{k=1}^{n+1}\frac{1}{k}-n}{n+1} \tag{1.58}$$

$$\sum_{k=0}^{n}\frac{1}{k+y}=\frac{y\binom{y+n}{n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(x-k)^n}{(y+k)^2}+n(x+y)^{n-1}}{(x+y)^n},\qquad y\neq 0,-1,...,-n \tag{1.59}$$

$$\sum_{k=1}^{n+1}\frac{1}{k}=(-1)^n(n+1)\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{k^n}{(k+1)^2}+n,\qquad 0^0\equiv 1 \tag{1.60}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{(y+k)^2}=\frac{1}{y\binom{y+n}{n}}\sum_{k=0}^{n}\frac{1}{k+y} \tag{1.61}$$

1.6 Melzak's Formula for Polynomials of Degree $n+1$

Remark 1.9 *In this section, we assume $f(x)$ is a polynomial of degree $n+1$, that is, $f(x) = \sum_{i=0}^{n+1} a_1 x^i$.*

$$\boldsymbol{\sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{f(x-k)}{y+k} = \frac{f(x+y)}{y\binom{y+n}{n}} - n!a_{n+1},} \tag{1.62}$$

$$\boldsymbol{y \neq 0, -1, ..., -n, \qquad n \geq 1}$$

1.6.1 Examples of Equation (1.62)

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{(x-k)^{n+1}}{y+k} = \frac{(x+y)^{n+1}}{y\binom{y+n}{n}} - n!, \qquad n \geq 1 \tag{1.63}$$

$$\sum_{k=0}^{rp-1}(-1)^k \binom{rp-1}{k} \binom{x-k}{r}^p \frac{1}{y+k} = \frac{\binom{x+y}{r}^p}{y\binom{y+rp-1}{y}} - \frac{(rp-1)!}{r!^p}, \tag{1.64}$$

r and p are positive integers, $rp \geq 2$, $\qquad y \neq 0, -1, -2, ..., 1-rp$

1.7 Partial Fraction Generalizations of Melzak's Formula

1.7.1 First Partial Fraction Decompostion

Let $f(x) = \sum_{i=0}^{n+1} a_i x^i$. Then,

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{f(y-k)}{(k+x)(k+z)} = \frac{1}{(n+1)(x-z)} \left(\frac{f(y+z)}{\binom{z+n}{n+1}} - \frac{f(y+x)}{\binom{x+n}{n+1}} \right), \; x \neq z. \tag{1.65}$$

1.7.2 Second Partial Fraction Decomposition

Assume u, v, and w are real or complex numbers. Let $f(x) = \sum_{k=0}^{n+2} a_i x^i$. Then,

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{f(y-k)}{(k+u)(k+v)(k+w)} = \frac{u-1}{(n+1)(n+2)(u-v)(u-w)} \left(\frac{f(y+u)}{\binom{u+n}{n+2}} - \frac{f(y+w)}{\binom{w+n}{n+2}} \right)$$
$$- \frac{v-1}{(n+1)(n+2)(u-v)(v-w)} \left(\frac{f(y+v)}{\binom{v+n}{n+2}} - \frac{f(y+w)}{\binom{w+n}{n+2}} \right), \; u \neq v \neq w. \tag{1.66}$$

Example of Equation (1.66)

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(n+1)(n+2)}{(k+3)(k+4)(k+7)} = \frac{1}{2(n+3)} - \frac{1}{\binom{n+4}{2}} + \frac{1}{2\binom{n+7}{5}} \tag{1.67}$$

1.7.3 Third Partial Fraction Decomposition

Let $f(x)=\sum_{i=0}^{n+\alpha-1} a_i x^i$*. Then,*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(y-k)}{\binom{k+\alpha}{k}} = -\sum_{k=1}^{\alpha}(-1)^k\binom{\alpha}{k}\frac{(y+k)}{\binom{k+n}{k}},\ \alpha\geq 1. \tag{1.68}$$

Two Applications of Equation (1.68)

First Application
Let $f(x)=\sum_{i=0}^{2n-1} a_i x^i$*. Then,*

$$f(y) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{f(y-k)+f(y+k)}{\binom{k+n}{k}}. \tag{1.69}$$

Second Application
Let $f(x)=\sum_{i=1}^{p} a_i x^{2i}$ *and assume* $1\leq p\leq n-1$*. Then,*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{f(k)}{\binom{k+n}{n}} = 0. \tag{1.70}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{k^{2p}}{\binom{k+n}{n}} = 0,\ \textit{provided } p \textit{ is a positive integer, } 1<2p<2n-1 \tag{1.71}$$

2 Lagrange Interpolation Formula

Remark 2.1 *Throughout this chapter, we assume, unless otherwise specified, that* n *and* j *are nonnegative integers, and* x*,* y*, and* z *are real or complex numbers.*

2.1 Two Basic Forms of the Lagrange Interpolation Formula

2.1.1 Product Form

Let $f(x)$ *be a polynomial of degree* n*, that is,* $f(x)=\sum_{i=0}^{n} a_i x^i$*. Then, for* $n\geq 1$*,*

$$\boldsymbol{f(x) = \sum_{k=0}^{n} f(x_k)\prod_{\substack{i=0\\ i\neq k}}^{n}\frac{x-x_i}{x_k-x_i}},\ \textit{where the } \boldsymbol{x_i} \textit{ are distinct.} \tag{2.1}$$

2.1.2 Derivative Form

Let $f(x)$ be a polynomial of degree n, that is, $f(x) = \sum_{i=0}^{n} a_i x^i$. Let $\varphi(x) = \prod_{i=0}^{n}(x - x_i)$. Then, for $n \geq 1$,

$$\boldsymbol{f(x) = \varphi(x) \sum_{k=0}^{n} \frac{f(x_k)}{(x - x_k)\varphi'(x_k)}}, \text{ where the } \boldsymbol{x_i} \text{ are distinct.} \tag{2.2}$$

Generalization of Equation (2.2)
Let $f(x)$ be a polynomial of degree n, that is, $f(x) = \sum_{i=0}^{n} a_i x^i$. Let $\varphi(x) = \prod_{i=0}^{n}(x - x_i)$. Then, for $n \geq 1$,

$$\boldsymbol{f(x + y) = \varphi(x) \sum_{k=0}^{n} \frac{f(y + x_k)}{(x - x_k)\varphi'(x_k)}}, \text{ where the } \boldsymbol{x_i} \text{ are distinct.} \tag{2.3}$$

Integral Form of Equation (2.2)
Assume x is a real number. Let $f(x)$ be a polynomial of degree n, that is, $f(x) = \sum_{i=0}^{n} a_i x^i$. Let $\varphi(x) = \prod_{i=0}^{n}(x - x_i)$. Then, for $n \geq 1$,

$$\int \frac{f(x)}{\varphi(x)}\, dx = \sum_{k=0}^{n} \frac{f(x_k)}{\varphi'(x_k)} \ln(x - x_k) + C, \text{ where the } x_i \text{ are distinct.} \tag{2.4}$$

Derivative Form of Equation (2.2)
Let $f(x)$ be a polynomial of degree n, that is, $f(x) = \sum_{i=0}^{n} a_i x^i$. Let $\varphi(x) = \prod_{i=0}^{n}(x - x_i)$. Then, for $n \geq 1$,

$$\left(\frac{f(x)}{\varphi(x)}\right)' = (-1)^n n! \sum_{k=0}^{n} \frac{f(x_k)}{(x - x_k)^{n+1}\varphi'(x_k)}, \text{ where the } x_i \text{ are distinct.} \tag{2.5}$$

2.2 Partial Fraction Expansion Via Equation (2.2) and Melzak's Formula

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{k^m}{\prod_{i=1}^{\alpha}(k - x_i)} = -\sum_{j=1}^{\alpha} \frac{x_j^{m-1}}{\binom{n-x_j}{n} \prod_{\substack{i=1 \\ i \neq j}}^{\alpha}(x_j - x_i)}, \text{ } m \text{ a positive integer} \tag{2.6}$$

2.3 Applications of Equation (2.3)

2.3.1 First Application: Let $x_i = i$

$$f(x + y) = \sum_{k=0}^{n} f(k + y) \prod_{\substack{j=0 \\ j \neq k}}^{n} \frac{x - j}{k - j} = \sum_{k=0}^{n} \binom{x}{k}\binom{n - x}{n - k} f(k + y) \tag{2.7}$$

$$\sum_{k=0}^{n} \binom{x}{k}\binom{n-x}{n-k}\binom{k+y}{n} = \binom{x+y}{n} \tag{2.8}$$

2.3.2 Second Application: Let $x_i = i^2$

Let $f(x)$ be a polynomial of degree n, that is, $f(x) = \sum_{i=0}^{n} a_i x^i$. Then, for $n \geq 1$

$$\sum_{k=0}^{n} (-1)^k \binom{2n}{n+k} \frac{f(y+k^2)}{x^2-k^2} = \frac{(-1)^n f(x^2+y)}{2x(x-n)\binom{x+n}{2n}} + \frac{1}{2}\binom{2n}{n}\frac{f(y)}{x^2}. \tag{2.9}$$

Let $f(x)$ be a polynomial of degree n, that is, $f(x) = \sum_{i=0}^{n} a_i x^i$. Then, for $n \geq 1$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{f(y+k^2)}{(x^2-k^2)\binom{n+k}{k}} = \frac{(-1)^n f(x^2+y)}{2x(x-n)\binom{x+n}{2n}\binom{2n}{n}} + \frac{f(y)}{2x^2}. \tag{2.10}$$

Specific Cases of Equation (2.9)

$$\sum_{k=0}^{n} (-1)^k \binom{2n}{n+k} = \binom{2n-1}{n}, \qquad n \geq 1 \tag{2.11}$$

$$\sum_{k=0}^{n} (-1)^k \binom{2n}{n+k} \frac{1}{x^2-k^2} = \frac{1}{2x^2}\binom{2n}{n} + \frac{1}{2x(x-n)\binom{x+n}{2n}} \tag{2.12}$$

$$\sum_{k=0}^{n} (-1)^k \binom{2n}{n+k} \frac{1}{n^3-k^2} = \frac{1}{2n^3}\binom{2n}{n} + \frac{(-1)^n}{2n^2\sqrt{n}(\sqrt{n}-1)\binom{n+n\sqrt{n}}{2n}}, \qquad n \geq 2 \tag{2.13}$$

2.3.3 Third Application: Roots of Unity

Remark 2.2 *Throughout this subsection, we define w_n^{α}, for $\alpha = 1, 2, ..., n$, to be an n^{th} root of unity, i.e. $w_n = e^{\frac{2\pi i}{n}}$, where $i = \sqrt{-1}$. We will also assume $f(x)$ is a polynomial of degree $n-1$, namely, $f(x) = \sum_{j=0}^{n-1} a_j x^j$.*

$$\frac{f(x)}{x^n - 1} = \frac{1}{n}\sum_{k=1}^{n} \frac{f(w_n^k) w_n^k}{x - w_n^k}, \qquad n \geq 1 \tag{2.14}$$

$$\frac{x^r}{x^n-1} = \frac{1}{n}\sum_{k=1}^{n} \frac{(w_n^k)^{r+1}}{x-w_n^k}, \ n \geq 1, \ r \textit{ an integer, } 0 \leq r \leq n-1 \tag{2.15}$$

$$\frac{1}{x^n-1} = \frac{1}{n}\sum_{k=1}^{n} \frac{w_n^k}{x-w_n^k}, \qquad n \geq 1 \tag{2.16}$$

2.3.4 Fourth Application: Shifted Roots of Unity

Remark 2.3 *Throughout this subsection, we define w_n^α, for $\alpha = 1, 2, ..., n$, to be an n^{th} root of $z^n = -1$, i.e. $w_n = e^{\frac{2k+1}{n}\pi i}$, where $i = \sqrt{-1}$. We will also assume $f(x)$ is a polynomial of degree $n-1$, namely, $f(x) = \sum_{j=0}^{n-1} a_j x^j$.*

$$\frac{f(x)}{x^n + 1} = \frac{-1}{n} \sum_{k=1}^{n} \frac{f(w_n^k) w_n^k}{x - w_n^k}, \qquad n \geq 1 \tag{2.17}$$

$$\frac{x^r}{x^n + 1} = \frac{-1}{n} \sum_{k=1}^{n} \frac{(w_n^k)^{r+1}}{x - w_n^k}, \; n \geq 1, \; r \text{ an integer, } 0 \leq r \leq n-1 \tag{2.18}$$

$$\frac{1}{x^n + 1} = \frac{-1}{n} \sum_{k=1}^{n} \frac{w_n^k}{x - w_n^k}, \qquad n \geq 1 \tag{2.19}$$

3 Hagen/Rothe Coefficients

Remark 3.1 *Throughout this chapter, we assume n, α, and β are nonnegative integers. We also assume x, y, and z are real or complex numbers. For real x, we let $[x]$ denote the floor of x.*

Remark 3.2 *There are two main referencec for the material in this Chapter. The first is Johann Georg Hagen's Synopsis der hoeheren Mathematik, Berlin, 1891 (3 Volumes). The second is Heinrich August Rothe's, Formulae de serierum reversione, etc. Leipzig, 1793.*

3.1 Hagen/Rothe Coefficients of the First Type

We let Hagen/Rothe Coefficient of the First Type be denoted by $A_n(x, d)$, where,

$$\boldsymbol{A_n(x, z) = \frac{x}{x + nz} \binom{x + nz}{n}.} \tag{3.1}$$

Alternative Definition of $A_n(x, z)$

$$\boldsymbol{A_n(x, z) = \sum_{k=0}^{n-1} (-1)^{k+n+1} \binom{n}{k} \binom{x + kz}{n} \frac{x}{x + kz}, \qquad n \geq 1} \tag{3.2}$$

3.1.1 Generating Function for $A_k(\alpha, \beta)$

$$\sum_{k=0}^{\infty} A_k(\alpha,\beta) z^k = y^{\alpha}, \ \textit{where } z = \frac{y-1}{y^{\beta}}, \textit{ and } |z| < \left| \frac{(\beta-1)^{\beta-1}}{\beta^{\beta}} \right| \tag{3.3}$$

3.1.2 Properties of $A_n(x, z)$

n^{th} Difference Property

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} A_n(k,z) = \sum_{k=0}^{n}(-1)^k \binom{n}{k}\binom{k+nz}{n}\frac{k}{k+nz} = (-1)^n, \qquad n \geq 1 \tag{3.4}$$

Convolution Formula

$$\sum_{k=0}^{n} A_k(x,z)A_{n-k}(y,z) = A_n(x+y,z), \qquad \textit{i.e.} \tag{3.5}$$

$$\sum_{k=0}^{n}\binom{x+kz}{k}\binom{y+nz-kz}{n-k}\frac{x}{x+kz}\cdot\frac{y}{y+nz-kz} = \binom{x+y+nz}{n}\frac{x+y}{x+y+nz} \tag{3.6}$$

Generalization of Equation (3.5): Let p and q be real or complex numbers. Then,

$$\sum_{k=0}^{n} A_k(x,z)A_{n-k}(y,z)(p+qk) = \frac{p(x+y)+nxq}{x+y+zn}\binom{x+y+zn}{n}, \tag{3.7}$$

Hagen Identity

$$\sum_{k=0}^{n}\binom{x+zk}{k}\binom{y-zk}{n-k}\frac{p+qk}{(x+zk)(y-zk)} = \frac{p(z+y-zn)+nxq}{(x+y)x(y-zn)}\binom{x+y}{n} \tag{3.8}$$

Vosmansky Extension

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}A_{2n}(k,x)A_{2n}(2n-k,x) = (-1)^n x\binom{2n}{n}A_{2n}(n,x), \qquad i.e. \tag{3.9}$$

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{k+2nx}{2n}\binom{2n-k+2nx}{2n}\frac{k(2n-k)}{(k+2nx)(2n-k+2nx)} =$$
$$(-1)^n\binom{2n}{n}\binom{n+2nx}{2n}\frac{x}{1+2x} = \frac{(-1)^n}{2\cdot n!^2}\prod_{j=0}^{n-1}((2nx)^2-j^2), \ n \geq 1 \tag{3.10}$$

Restatement of Equation (3.9)

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}A_n(k,x)A_n(n-k,x) =$$

$$(-1)^{\left[\frac{n}{2}\right]}\binom{n}{\left[\frac{n}{2}\right]}\binom{\left[\frac{n}{2}\right]+nx}{n}\frac{\left[\frac{n}{2}\right]x}{\left[\frac{n}{2}\right]+nx}\frac{(-1)^n+1}{2} \tag{3.11}$$

3.2 Hagen/Rothe Coefficients of the Second Type

Remark 3.3 *To form the Hagen/Rothe Coefficient of the Second Type, simply delete the fraction factor of $A_n(x,z)$.*

3.2.1 Generating Function for Hagen/Rothe Coefficients of the Second Type

$$\sum_{k=0}^{\infty}\binom{\alpha+\beta k}{k}z^k = \frac{y^{\alpha+1}}{(1-\beta)y+\beta},\ \textit{where } z=\frac{y-1}{y^\beta}\textit{, and } |z|<\left|\frac{(\beta-1)^{\beta-1}}{\beta^\beta}\right| \tag{3.12}$$

3.2.2 Properties of Hagen/Rothe Coefficients of the Second Type

n^{th} Difference Formula

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x+yk}{n} = (-1)^n y^n \tag{3.13}$$

First Convolution Formula

$$\sum_{k=0}^{n}\binom{x+k\beta}{k}\binom{y+(n-k)\beta}{n-k} = \sum_{k=0}^{n}\binom{x+z+k\beta}{k}\binom{y-z+(n-k)\beta}{n-k} \tag{3.14}$$

Second Convolution Formula

$$\sum_{k=0}^{n}\binom{x+kz}{k}\binom{y+(n-k)z}{n-k}\frac{x}{x+kz} = \binom{x+y+nz}{n} \tag{3.15}$$

Variation of Equation (3.15)

$$\sum_{k=0}^{n}\binom{x+kz}{z}\binom{y-kz}{n-k}\frac{1}{x+kz} = \frac{1}{x}\binom{x+y}{n} \tag{3.16}$$

Remark 3.4 *The following identity, proposed by K. L. Chung, is from Problem 4211 of The American Math. Monthly, 1946, P. 397. Assume d is a real or complex number. Then,*

$$\sum_{k=1}^{n}\binom{dk}{k}\binom{nd-kd}{n-k}\frac{1}{dk-1}=\frac{1}{d-1}\binom{nd}{n},\ n\geq 1,\ d\neq 1. \tag{3.17}$$

Variation of Equation (3.17)

$$\sum_{k=0}^{n}\binom{n}{k}\frac{\binom{(d-1)n}{(d-1)k}}{\binom{dn}{dk}}\frac{1}{dk-1}=\frac{d-2}{1-d},\qquad d\neq 1 \tag{3.18}$$

3.3 Abel Coefficients

Remark 3.5 *In this section, we define the Abel Coefficients, $B(x,z)$, as*

$$\boldsymbol{B_n(x,z)=\frac{x}{x+nz}\frac{(x+nz)^n}{n!}.} \tag{3.19}$$

The Abel Coefficients are obtained from $A_n(x,z)$ by a simple limiting process. A reference for the material in this subsection is N. H. Abel's, "Demonstration d'une expression de laquelle la formule binôme est un cas particulier", Jour. reine u. angew. Math., Vol. 1 (1826), P. 159.

3.3.1 Generating Functions for $B_k(\alpha,\beta)$

$$\sum_{k=0}^{\infty}B_k(\alpha,\beta)z^k=x^\alpha=e^{\alpha w}:\ x=e^w,\ z=we^{-\beta w},\ |\beta we^{-\beta w}|<\frac{1}{e} \tag{3.20}$$

$$\sum_{k=0}^{\infty}B_k(\alpha,\beta)z^k\frac{\alpha+\beta k}{\alpha}=\frac{x^\alpha}{1-\beta\ln x}=\frac{e^{\alpha w}}{1-\beta w}:\ x=e^w,\ z=we^{-\beta w},\ |\beta we^{-\beta w}|<\frac{1}{e} \tag{3.21}$$

3.3.2 Properties of $B_n(x,z)$

Remark 3.6 *For the following three identities, we assume p, q, and t are real or complex numbers.*

$$\sum_{k=0}^{n}B_k(x,z)B_{n-k}(y,z)=B_n(x+y,z) \tag{3.22}$$

$$\sum_{k=0}^{n} B_k(x,z)B_{n-k}(y,z)(p+qk) = \frac{p(x+y)+qnx}{x+y} B_n(x+y,z) \tag{3.23}$$

$$\sum_{k=0}^{n} B_k(x,\beta)\frac{x+\beta k}{x} B_{n-k}(y,\beta)\frac{y+\beta(n-k)}{y} = \\ \sum_{k=0}^{n} B_k(x+t,\beta)\frac{x+t+\beta k}{x+t} B_{n-k}(y-t,\beta)\frac{y-t+\beta(n-k)}{y-t} \tag{3.24}$$

3.4 Generalization of $A_n(x,z)$

Remark 3.7 *In this section, we assume w is a real or complex number.*

$$\frac{w}{w+zn}\binom{x+zn}{n} = (-1)^n \frac{\binom{x-w}{n}}{\binom{\frac{w}{z}+n}{n}} \\ + \sum_{k=0}^{n-1}(-1)^{n+k-1}\binom{n}{k}\binom{x+zk}{n}\frac{w}{w+zk}, \qquad n \geq 1,\ z \neq 0 \tag{3.25}$$

Generating Function

$$\sum_{k=0}^{\infty}\binom{\alpha+\beta k}{k}\frac{w}{w+\beta k}z^k = x^\alpha \sum_{n=0}^{\infty}(-1)^n \frac{\binom{\alpha-w}{n}}{\binom{\frac{w}{\beta}+n}{n}}\left(\frac{x-1}{x}\right)^n, \qquad z = \frac{x-1}{x^\beta} \tag{3.26}$$

Remark 3.8 *In the following equation, assume*

$$F(n) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{\alpha+\beta k}{n} f(k),\ \ f(0) = 1. \tag{3.27}$$

Then,

$$\sum_{k=0}^{\infty}\binom{\alpha+\beta k}{k} z^k f(k) = x^\alpha \sum_{n=0}^{\infty}(-1)^n\left(\frac{x-1}{x}\right)^n F(n), \qquad z = \frac{x-1}{x^\beta} \tag{3.28}$$

3.5 Generalized Rothe Convolutions

Remark 3.9 *In this section, given a function $f(x)$, we assume there exists coefficients $C_k(\alpha,\beta)$ such that*

$$x^\alpha = \sum_{k=0}^{\infty} C_k(\alpha,\beta) z^k,\ \textit{where } z = \frac{f(x)}{x^\beta}. \tag{3.29}$$

Using convolution theory, we show that

$$\sum_{k=0}^{n} C_k(\alpha,\beta)C_{n-k}(\gamma,\beta) = C_n(\alpha+\gamma,\beta). \tag{3.30}$$

A reference for this material is J. G. Van der Corput's Over eenige identiteiten, Nieuw Archief voor Wiskunde, Series 2, Vol. 17, 1931, pp. 17-27.

3.5.1 Applications of Equation (3.30)

$$\sum_{k=1}^{n-1} \binom{\beta k}{k}\binom{\beta n-\beta k}{n-k}\frac{1}{k(n-k)} = \frac{2\beta}{n}\binom{\beta n}{n}\sum_{k=1}^{n-1}\frac{1}{\beta n-n+k}, \qquad n \geq 2 \tag{3.31}$$

$$\sum_{k=1}^{n-1} \binom{n}{k} k^{k-1}(n-k)^{n-k-1} = 2(n-1)n^{n-2}, \qquad n \geq 2 \tag{3.32}$$

3.6 Generalized Difference Series

3.6.1 Differential Series

Remark 3.10 *Throughout this subsection, we use the convention that $D_x^n f(x)$ is the n^{th} derivative with respect to x.*

Taylor Series

$$\boldsymbol{f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}\left[D_t^n f(t)\right]_{t=0}} \tag{3.33}$$

Abel Series

$$\boldsymbol{f(x) = \sum_{n=0}^{\infty} \frac{x(x-n)^{n-1}}{n!}\left[D_t^n f(t)\right]_{t=n}} \tag{3.34}$$

Generalized Differential Series

$$f(x) = \sum_{n=0}^{\infty} \frac{x}{x+yn}\frac{(x+yn)^n}{n!}\left[D_t^n f(t)\right]_{t=-yn} \tag{3.35}$$

3.6.2 Difference Series

Remark 3.11 *Throughout this section, we let $\Delta_{t,1}^{n} f(t)$ denote the n^{th} difference of $f(t)$, namely,*

$$\Delta_{t,1}^{n} f(t) = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} f(x+k). \tag{3.36}$$

Remark 3.12 *The material in this section can be found in N. E. Nörlund's Vorlesungen über Differenzenrechnung, Berlin, 1924. (Reprinted by Chelsea Publ. Co., New York, N. Y., 1954.) Also see N. Nielsen's Handbuch der Theorie der Gammafunktion, Leiptzig, 1906. We also refer the reader to R. C. Buck's "Interpolation Series", Trans. Amer. Math. Soc., Vol. 64, 1948, pp 283-298.*

Newton-Gregory Series

$$\boldsymbol{f(x) = \sum_{n=0}^{\infty} \binom{x}{n} \left[\Delta_{t,1}^{n} f(t)\right]_{t=0}} \tag{3.37}$$

Stirling Series

$$f(x) = f(0) + \sum_{n=1}^{\infty} \frac{x}{n} \binom{x-1+\frac{n}{2}}{n-1} \left[\Delta_{t,1}^{n} f(t)\right]_{t=\frac{-n}{2}} \tag{3.38}$$

Generalized Difference Series: Two Versions

$$\boldsymbol{f(x) = \sum_{n=0}^{\infty} \frac{x}{x+yn} \binom{x+yn}{n} \left[\Delta_{t,1}^{n} f(t)\right]_{t=-yn}} \tag{3.39}$$

$$f(x+y) = \sum_{n=0}^{\infty} \binom{\frac{x+\beta n}{z}}{n} \frac{x}{x+\beta n} z^{n} \left[\Delta_{t,z}^{n} f(t+y)\right]_{t=-\beta n} \tag{3.40}$$

4 Jensen's Formula

Remark 4.1 *In this chapter, we assume u, v, α, β, and γ, are real or complex numbers, while n and p are nonnegative integers. We also note that a different variation of Equation (4.1) is found in J. L. W. V. Jensen's "Sur une identité d'Abel et sur d'autres formules analogues", Acta Mathematica, Vol. 26, 1902, pp. 307-318.*

Jensen's Formula

$$\boldsymbol{\sum_{k=0}^{n} \binom{\alpha+\beta k}{k} \binom{\gamma-\beta k}{n-k} = \sum_{k=0}^{n} \binom{\alpha+\gamma-k}{n-k} \beta^{k}} \tag{4.1}$$

4.1 Specific Evaluations of Equation (4.1)

$$\sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k} = 2^{2n} \tag{4.2}$$

$$\sum_{k=0}^{n} \binom{2k}{k}\binom{n-2k}{n-k} = 2^{n+1} - 1 \tag{4.3}$$

$$\sum_{k=0}^{n} \binom{\alpha+\beta k}{k}\binom{p-\alpha-\beta k}{n-k} = \beta^{p+1}(\beta-1)^{n-p-1}, \qquad 0 \le p \le n-1 \tag{4.4}$$

$$\sum_{k=0}^{n} \binom{\alpha+\beta k}{k}\binom{n-\alpha-\beta k}{n-k} = \frac{\beta^{n+1}-1}{\beta-1}, \qquad \beta \neq 1 \tag{4.5}$$

$$v^n \sum_{k=0}^{n} \binom{\frac{u}{v}k}{k}\binom{n-\frac{u}{v}k}{n-k} = \frac{u^{n+1}-v^{n+1}}{u-v}, \qquad v \neq 0, \;\; u \neq v \tag{4.6}$$

$$F_{n+1} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}} = \left(\frac{1-\sqrt{5}}{2}\right)^{n} \sum_{k=0}^{n} \binom{\frac{1+\sqrt{5}}{1-\sqrt{5}}k}{k}\binom{n-\frac{1+\sqrt{5}}{1-\sqrt{5}}}{n-k}, \tag{4.7}$$

where $F_0 = 0, F_1 = 1, F_{n+1} = F_n + F_{n-1}$ *are the Fibonocci Numbers*

4.2 Limiting Case of Equation (4.1)

$$\boldsymbol{\sum_{k=0}^{n} \frac{(\alpha+\beta k)^k}{k!}\frac{(\gamma-\beta k)^{n-k}}{(n-k)!} = \beta^n \sum_{k=0}^{n} \frac{(\alpha+\gamma)^k}{\beta^k k!}} \tag{4.8}$$

5 Lagrange Inversion Theorem

Remark 5.1 *Throughout this chapter, we assume t, h, x, y, and z are real or complex numbers, while n is a nonnegative integer. We also let $D_t^n f(t)$ denote the n^{th} derivative of $f(t)$ with respect to t.*

5.1 Basic Statement of the Lagrange Inversion Theorem

Let $x = t + h\varphi(x)$. Then,

$$\boldsymbol{f(x) = \sum_{n=0}^{\infty} \frac{h^n}{n!} D_t^{n-1} \left(f'(t)(\varphi(t))^n\right)}. \tag{5.1}$$

Remark 5.2 *For an analytic proof of the Lagrange Inversion Theorem, also known as Bürmann's Theorem, we refer the reader to three sources. The first is Modern Analysis, Fourth Edition, by Whittaker and Watson, 1927, P. 133. We also suggest An Introduction to the Theory of Functions of a Complex Variable by E. T. Copson, Oxford University Press, First Ed. 1935, reprint 1944, pp. 123-125. The third source is T. W. Chaundy's "The Validity of Legrange's Expansion", Quarterly Journal of Mathematics, Oxford Series, Vol. 2, 1931, pp. 284-297.*

5.2 Derivative Formulas Associated with the Lagrange Inversion Theorem

Remark 5.3 *The formulas in this section are given as exercises in Recueil d'Exercises sur le Calcul Infinitésimal, Second Edition, by Frederic Frenet, Paris, 1866, Page 15 and Pages 75-76.*

Let $z = x + yf(z)$. Let $D_x^n f(x)$ denote the n^{th} derivative of $f(x)$ with respect to x. Furthermore, assume $F(z)$ is any continuous, differentiable function. Let $z = x + yf(x)$. Then,

$$D_y z = f(z) D_x z \tag{5.2}$$

$$D_x z = \frac{1}{1 - yf'(z)} \tag{5.3}$$

$$D_y^n z = D_x^{n-1} \left((f(x)^n D_x z\right) \tag{5.4}$$

Duhamel's Theorem

$$D_y^n F(z) = D_x^{n-1} \left(F'(z)(f(x))^n D_x z\right) \tag{5.5}$$

Lagrange Inversion Formula

$$\boldsymbol{F(z) = \sum_{n=0}^{\infty} \frac{y^n}{n!} D_x^{n-1} \left(F'(x)(f(x))^n\right)} \tag{5.6}$$

6 Basic Operators

Remark 6.1 *Throughout this chapter, we assume n and p are nonnegative integers. We also assume x is a real or complex number, while h is a nonzero real or complex number. We also let D_x denote the derivative with respect to x.*

6.1 The Operator $(x\Delta_h)^n$

Remark 6.2 *Recall that*

$$\Delta_h f(x) = \frac{f(x+h)-f(x)}{h}, \tag{6.1}$$

is the (first) difference operator.

$$\boldsymbol{(x\Delta_h)^n f(x) = \sum_{j=0}^{n} \binom{j+\frac{x}{h}-1}{j} h^j \Delta_h^j f(x) \sum_{k=0}^{j} (-1)^{j+k} \binom{j}{k} k^n} \tag{6.2}$$

6.1.1 Specific Evaluations of Equation (6.2)

$$(x\Delta_1)^n f(x) = \sum_{j=0}^{n} \binom{j+x-1}{j} \Delta_1^j f(x) \sum_{k=0}^{j} (-1)^{j+k} \binom{j}{k} k^n \tag{6.3}$$

$$(x\Delta_{\frac{1}{2}})^n f(x) = \sum_{j=0}^{n} \binom{j+2x-1}{j} \frac{1}{2^j} \Delta_{\frac{1}{2}}^j f(x) \sum_{k=0}^{j} (-1)^{j+k} \binom{j}{k} k^n \tag{6.4}$$

$$(x\Delta_2)^n f(x) = \sum_{j=0}^{n} \binom{j+\frac{x}{2}-1}{j} 2^j \Delta_2^j f(x) \sum_{k=0}^{j} (-1)^{j+k} \binom{j}{k} k^n \tag{6.5}$$

$$(x\Delta_h)^n x^p = \sum_{j=0}^{n} \binom{\frac{x}{h}+j-1}{j} B_{j,j}^n \sum_{r=0}^{p} \binom{p}{r} x^{p-r} h^r B_{j,j}^r, \text{ where} \tag{6.6}$$

$$B_{j,j}^n = \sum_{k=0}^{j} (-1)^{j+k} \binom{j}{k} k^n$$

$$(x\Delta_1)^n \binom{x}{p} = \sum_{j=0}^{n} \binom{x+j-1}{j} \binom{x}{p-j} \sum_{k=0}^{j} (-1)^{j+k} \binom{j}{k} k^n \tag{6.7}$$

6.2 The Operator $(xE_h)^n$

Remark 6.3 *Recall that E_h is the shift operator, that is $E_hf(x) = f(x+h)$.*

$$(xE_h)^n f(x) = h^n n!\binom{\frac{x}{h}+n-1}{n} f(x+nh) = h^n n!\binom{\frac{x}{h}+n-1}{n} E_h^n f(x) \tag{6.8}$$

6.3 The Operator $(x^r D_x)^n$

6.3.1 The Operator $(xD_x)^n$

Grunert's Formula
Let S be function of x. Then,

$$\boldsymbol{(xD_x)^n S = \sum_{j=0}^{n}\sum_{k=0}^{j} \frac{(-1)^{j+k}}{j!}\binom{j}{k} k^n x^j D_x^j S.} \tag{6.9}$$

6.3.2 The Operator $(x^2 D_x)^n$

Let S be function of x. Then,

$$\boldsymbol{(x^2 D_x)^n S = \sum_{j=0}^{n} \frac{n!}{j!}\binom{n-1}{j-1} x^{n+j} D_x^j S.} \tag{6.10}$$

6.3.3 The Operator $\left(\frac{1}{x}D_x\right)^n$

Let S be function of x. Then,

$$\left(\frac{1}{x}D_x\right)^n S = \sum_{k=0}^{n}(-1)^{n-k}\frac{kn!}{(2n-k)k!}\binom{2n-k}{n}\frac{x^{k-2n}}{2^{n-k}} D_x^k S. \tag{6.11}$$

6.3.4 Recursive Formula for $(x^r D_x)^n$

Let S be a function of x. Let r be any real number. Define $T^0_{r,0} \equiv 1$. Assume $T^n_{r,0} \equiv 0$, for $n \geq 1$; Also define $T^n_{r,j} \equiv 0$ for $j > n$. Then,

$$(x^r D_x)^n S = \sum_{j=0}^{n} T^n_{r,j} x^{(r-1)n+j} D_x^j S, \text{ where } T^{n+1}_{r,j+1} = ((r-1)n+j+1)T^n_{r,j+1} + T^n_{r,j}. \tag{6.12}$$

Specific Evaluations of Equation (6.12)

$$\sum_{k=0}^{n} \frac{p!}{(p-k)!} T_{r+1,k}^{n} = r^n n! \binom{\frac{p}{r}+n-1}{n} \tag{6.13}$$

$$\sum_{k=0}^{n} \binom{n}{k}\binom{p}{k} k = n\binom{p+n-1}{n}, \text{ where } p \text{ is a real or complex number} \tag{6.14}$$

Generalization of Equation (6.13)

$$\sum_{k=0}^{n} \binom{x}{k} k! T_{r+1,k}^{n} = r^n n! \binom{\frac{x}{r}+n-1}{n} \tag{6.15}$$

Specific Evaluations of Equation (6.15)

$$\sum_{k=0}^{n} (-1)^k \binom{x}{k}\binom{2n-k}{n} \frac{2^k k}{2n-k} = 2^{2n} \binom{\frac{-x}{2}+n-1}{n} \tag{6.16}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n+k}{k}\binom{x}{n-k} \frac{n-k}{2^k(n+k)} = 2^n \binom{\frac{x}{2}}{n} \tag{6.17}$$

$$\sum_{k=0}^{n} \binom{2k}{k}\binom{2n-k}{n} \frac{k}{2^k(2n-k)} = (-1)^n 2^{2n} \binom{\frac{-1}{4}}{n} \tag{6.18}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n+k}{k}\binom{x+1}{n-k} 2^{-n-k} = \binom{\frac{x}{2}}{n} \tag{6.19}$$

6.4 The Operator $(D_x x^r)^n$

6.4.1 Operator $(D_x x)^n$

Let S be function of x. Then,

$$(D_x x)^n S = \sum_{j=0}^{n} \sum_{k=0}^{j+1} \frac{(-1)^{k+j+1}}{(j+1)!} \binom{j+1}{k} k^{n+1} x^j D_x^j S. \tag{6.20}$$

6.4.2 Recursive Formula for $(D_x x^r)^n$

Let S be a function of x. Let r be any real number. Define $T^0_{r,0} \equiv 1$. Assume $T^n_{r,0} \equiv 0$, for $n \geq 1$. Also define $T^n_{r,j} \equiv 0$ for $j > n$. Then,

$$(D_x x^r)^n S = \sum_{j=0}^{n} T^{n+1}_{r,j+1} x^{(r-1)n+j} D^j_x S, \text{ where } T^{n+1}_{r,j+1} = ((r-1)n + j + 1)T^n_{r,j+1} + T^n_{r,j}. \quad (6.21)$$

6.4.3 Combined with $x^r D_x$

$$(D_x x - x D_x)^n f(x) = f(x) \quad (6.22)$$

$$(D_x x^r - x^r D_x)^n f(x) = r^n x^{(r-1)n} f(x) \quad (6.23)$$

6.5 The Operator $(x D_x x)^n$

Let S be a function of x. Then,

$$(x D_x x)^n S = \sum_{k=0}^{n} \frac{n!}{k!} \binom{n}{k} x^{k+n} D^k_x S. \quad (6.24)$$

6.6 The Operator $(x D^p_x)^n$

Let S be a function of x. Let r be any real number. Define $T^0_{r,0} \equiv 1$; $T^n_{r,0} \equiv 0$, for $n \geq 1$; $T^n_{r,j} \equiv 0$ for $j > n$. Then,

$$(x D^p_x)^n S = \sum_{j=0}^{n} T^n_{p,j} x^j D^{pn-n+j}_x S, \text{ where } T^{n+1}_{p,j+1} = ((p-1)n + j + 1)T^n_{p,j+1} + T^n_{p,j}. \quad (6.25)$$

7 Intermediate Operator Techniques

Remark 7.1 *Throughout this chapter, we assume n and α are nonnegative integers, while x and z are real or complex numbers. We also use the convention that $D^n_x f(x)$ denotes the n^{th} derivative of f with respect to x.*

7.1 Change of Variable Formula

Assume y is a function of z and $x = e^z$. Then,

$$\boldsymbol{D^n_x y = e^{-nz} D_z (D_z - 1) ... (D_z - n + 1) y = \frac{n!}{x^n} \binom{D_z}{n} y.} \quad (7.1)$$

7.2 The Operator $(xD_x)^n$

Grunert's Formula
Let y be function of z. Assume $x = e^z$. Then,

$$\boldsymbol{(xD_x)^n y = D_z^n y = \sum_{j=0}^{n} \sum_{k=0}^{j} (-1)^{j+k} \binom{j}{k} k^n \binom{D_z}{j} y} \tag{7.2}$$

7.3 The Operator $\left(\frac{1}{x}D_x\right)^n$

Let y be a function of z. Assume $x = e^z$. Then,

$$\left(\frac{1}{x}D_x\right)^n y = \frac{n!}{x^{2n}} \sum_{j=0}^{n} C_j^n 2^{n-j} D_z^j y = \frac{n!}{x^{2n}} \sum_{j=0}^{n} C_j^n 2^{n-j} (xD_x)^j y, \tag{7.3}$$

where C_j^n is the coefficient of x^j in $\binom{x}{n}$, i.e. $\binom{x}{n} = \sum_{j=0}^{n} C_j^n x^j$.

7.3.1 Application of Equation (7.3)

Assume C_j^n is the coefficient of x^j the series expansion of $\binom{x}{n}$. Define $a_j(n)$ as

$$a_j(n) = \frac{(-1)^j}{j!} \sum_{k=0}^{j} (-1)^k \binom{j}{k} k^n.$$

Let S be a function of x. Then,

$$\left(\frac{1}{x}D_x\right)^n S = n! \sum_{k=0}^{n} \sum_{j=k}^{n} x^{k-2n} 2^{n-j} C_j^n a_k(j) D_x^k S. \tag{7.4}$$

7.4 The Operator $(x^n D_x^n)^\alpha$

Remark 7.2 *A reference for the formulas in this section is "On a Class of Finite Sums" by Leonard Carlitz, The American Math. Monthly, Nov. 1930, Vol. 37, No. 9, pp. 472-479.*

Assume S is a function of x. Then,

$$\begin{aligned}(x^n D_x^n)^\alpha S = n!^\alpha &\sum_{k=1}^{n(\alpha-1)+1} \frac{x^{k+n-1} D_x^{k+n-1} S}{(k+n-1)!} \\ &* \sum_{j=0}^{k+n-1} (-1)^j \binom{k+n-1}{j} \binom{k+n-1-j}{n}^\alpha, \ \alpha \geq 1.\end{aligned} \tag{7.5}$$

7.4.1 Specific Evaluations of Equation (7.5)

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^{\alpha}=\frac{1}{x^{2n}}\sum_{k=0}^{2n}\frac{(-1)^k}{k!^{\alpha}}(x^kD_x^k)^{\alpha}x^{2n} \tag{7.6}$$

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^{\alpha+1}=\frac{1}{x^{2n}}\sum_{k=0}^{2n}\frac{(-1)^k}{k!^{\alpha}}\binom{2n}{k}(x^kD_x^k)^{\alpha}x^{2n} \tag{7.7}$$

7.5 The Operator $(D_x^n x^n)^{\alpha}$

Let S be a function of x. Assume $n\alpha \leq m$. Then,

$$(D_x^n x^n)^{\alpha}S=n!^{\alpha}\sum_{k=0}^{m}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\binom{k-j+n}{n}^{\alpha}\frac{x^k}{k!}D_x^kS. \tag{7.8}$$

7.6 The Operator $D_x^{D_x}$

$$\left(D_x^{D_x}\right)f(x)=\sum_{n=0}^{\infty}\frac{z^n}{n!}D_z^n\sum_{k=0}^{n}(-1)^{n-k}D_x^kf(x), \qquad \textit{where } z=e^x \tag{7.9}$$

$$\left(D_x^{D_x}\right)f(x)=\sum_{n=0}^{\infty}\sum_{j=0}^{n}C_j^n\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}D_x^{k+j}f(x), \tag{7.10}$$

where C_j^n is the coefficient of x^j the series expansion of $\binom{x}{n}$.

7.7 Derivatives of a Class of Transcendental Funtions

Remark 7.3 *For this section, we define $E_0(x)=x$, $E_1(x)=e^x$, $E_2(x)=e^{e^x}$, and $E_n(x)=e^{E_{n-1}(x)}$. We define $a_j(n)$ as namely,*

$$a_j(n)=\frac{(-1)^j}{j!}\sum_{k=0}^{j}(-1)^k\binom{j}{k}k^n.$$

Finally, for this section only, we assume r is a nonnegative integer.

$$D_x^nE_r(x)=E_r(x)\sum_{i=1}^{n-1}\sum_{k_i=0}^{k_{i+1}}a_{k_i}(k_{i+1})e^{k_iE_{r-1-i}(x)}, \qquad n\geq 2,\ \ k_n=n \tag{7.11}$$

7.7.1 Specific Examples of Equation (7.11)

$$D_x^n e^{e^x} = e^{e^x} \sum_{j=0}^{n} a_j(n) e^{jx} \tag{7.12}$$

$$D_x^n e^{e^{e^x}} = e^{e^{e^x}} \sum_{k=0}^{j} a_k(j) e^{kx} \sum_{\alpha=0}^{k} a_\alpha(k) e^{\alpha e^x} \tag{7.13}$$

$$D_x^n e^{e^{e^{e^x}}} = e^{e^{e^{e^x}}} \sum_{j=0}^{n} a_j(n) e^{jx} \sum_{\beta=0}^{j} a_\beta(j) e^{\beta e^x} \sum_{\alpha=0}^{\beta} a_\alpha(\beta) e^{\alpha e^{e^x}} \tag{7.14}$$

7.8 Derivatives of x^x

Remark 7.4 *In this section, we define the sequence* $\{F_x(k)\}_{k=0}^{\infty}$ *by* $F_0(x) = 1$, $F_1(x) = 1$, *and* $F_k(x) = -D_x F_{k-1}(x) + \frac{k-1}{x} F_{k-2}(x)$, *whenever* $k \geq 2$. *We assume* x *is nonzero.*

$$D_x^n \left(x^x\right) = x^x \sum_{k=0}^{n} (-1)^k \binom{n}{k} (1 + \ln x)^{n-k} F_k(x) \tag{7.15}$$

8 Advanced Operator Techniques

Remark 8.1 *Throughout this chapter, we assume* n *and* r *are nonnegative integers, while* x *is a nonzero real or complex number. We assume, unless otherwise specified, that* y *is a function of* x. *We will use the convention that* $D_x^n y \equiv D^n y$ *is* n^{th} *derivative of* y *with respect to* x.

8.1 The Non-Linear Operator $(x + D)^n$

8.1.1 Definition of $(x + D)^n$

$$\boldsymbol{(x + D)^n y \equiv \underline{D}_1^n y = \prod_{k=1}^{n} (x + D) y, \qquad D_1^0 \equiv 1} \tag{8.1}$$

8.1.2 An Expansion of $\underline{D}_1^n$

$$\underline{D}_1^n y = \sum_{j=0}^{n} \binom{n}{j} \left[(x + D)^{j-1} x\right] D^{n-j} y, \tag{8.2}$$

8.1.3 Alternative Expansion for $\underline{D}_1^n$

$$\underline{D}_1^n y = \sum_{j=0}^{n} \binom{n}{j} A_j(x) D^{n-j} y, \tag{8.3}$$

where the $A_j(x)$ are polynomials of degree j in x such that

$$A_{n+1}(x) = xA_n(x) + nA_{n-1}(x), \qquad A_0(x) = 1. \tag{8.4}$$

8.1.4 Properties of $A_n(x)$

$$D_x^r A_n(x) = r! \binom{n}{r} A_{n-r}(x) \tag{8.5}$$

$$A_n(x) = n \int_0^x A_{n-1}(z)\, dz + A_n(0) \tag{8.6}$$

$$D_x^n e^{x^2} = (\sqrt{2})^n e^{x^2} A_n(\sqrt{2}x) \tag{8.7}$$

8.1.5 Explicit Formulas for $A_n(x)$

$$A_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k A_{n-k}(0) \tag{8.8}$$

Remark 8.2 *Ror real x, let $[x]$ denote the floor of x.*

$$A_n(x) = \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{k} \binom{n-k}{k} \frac{k!}{2^k} x^{n-2k} \tag{8.9}$$

$$A_n(x) = \frac{1}{(i\sqrt{2})^n} \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n}{k} \binom{n-k}{k} k! (i\sqrt{2}x)^{n-2k}, \qquad i = \sqrt{-1} \tag{8.10}$$

8.1.6 Properties of $(x + D)^n$

Order Property

$$(x + D)^{n+1} y = (x + D)(x + D)^n y = (x + D)^n (x + D) y \tag{8.11}$$

Product Property: Assume u and v are functions of x. Then,

$$(x + D)^n uv = \sum_{k=0}^{n} \binom{n}{k} D^k v (x + D)^{n-k} u. \tag{8.12}$$

8.2 The Non-Linear Operator $(x - D)^n$

8.2.1 Definition of $(x - D)^n$

$$\boldsymbol{(x - D)^n y \equiv \underline{D}_2^n y = \prod_{k=1}^{n} (x - D) y, \qquad D_2^0 \equiv 1} \tag{8.13}$$

8.2.2 An Expansion of $\underline{D}_2^n$

$$\underline{D}_2^n y = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \left[\frac{1}{a} (x - D)^{n-k} a \right] D^j y, \ \textit{a any nonzero constant} \tag{8.14}$$

8.2.3 Alternative Expansion for $\underline{D}_2^n$

$$\underline{D}_2^n y = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \bar{A}_{n-j}(x) D^j y, \tag{8.15}$$

where the $\bar{A}_j(x)$ are polynomials of degree j in x such that

$$\bar{A}_{n+1}(x) = x \bar{A}_n(x) - n \bar{A}_{n-1}(x), \qquad \bar{A}_0(x) = 1. \tag{8.16}$$

8.2.4 Properties of $\bar{A}_n(x)$

$$D_x^r \bar{A}_n(x) = r! \binom{n}{r} \bar{A}_{n-r}(x) \tag{8.17}$$

$$\bar{A}_n(x) = n \int_0^x \bar{A}_{n-1}(z) \, dz + \bar{A}_n(0) \tag{8.18}$$

8.2.5 Explicit Formulas for $\bar{A}_n(x)$

$$\bar{A}_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k \bar{A}_{n-k}(0) \tag{8.19}$$

$$\bar{A}_n(x) = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n}{k} \binom{n-k}{k} \frac{k!}{2^k} x^{n-2k} \tag{8.20}$$

$$\bar{A}_n(x) = \frac{1}{(\sqrt{2})^n} \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n}{k} \binom{n-k}{k} k! (\sqrt{2}x)^{n-2k} \tag{8.21}$$

$$\bar{A}_n(x) = (-1)^n e^{\frac{x^2}{2}} D_x^n e^{\frac{-x^2}{2}} \tag{8.22}$$

8.2.6 Properties of $(x - D)^n$

Order Property

$$(x-D)^{n+1} y = (x-D)(x-D)^n y = (x-D)^n (x-D) y \tag{8.23}$$

Product Property: Assume u and v are functions of x. Then,

$$(x-D)^n uv = \sum_{k=0}^{n} \binom{n}{k} D^k v (x-D)^{n-k} u. \tag{8.24}$$

8.3 Advanced Properties of $A_n(x)$ and $\bar{A}_n(x)$

8.3.1 Addition Properties

$$A_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} A_k(y) \tag{8.25}$$

$$\bar{A}_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \bar{A}_k(y) \tag{8.26}$$

8.3.2 In the Expansion of x^n

$$x^n = \sum_{k=0}^{n} \binom{n}{k} A_{n-k}(x)\bar{A}_k(0) \tag{8.27}$$

$$x^n = \sum_{k=0}^{n} \binom{n}{k} \bar{A}_{n-k}(x)A_k(0) \tag{8.28}$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} \bar{A}_{n-k}(x)A_k(y) \tag{8.29}$$

$$(x-y)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \bar{A}_{n-k}(x)A_k(y) \tag{8.30}$$

8.3.3 Operator Evaluations

$$A_n(x-D)y = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{n-k}D^k y \tag{8.31}$$

$$\bar{A}_n(x+D)y = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}D^k y \tag{8.32}$$

8.3.4 Convolution Formulas

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} A_k(x)A_{2n-k}(x) = \frac{(2n)!}{n!} \tag{8.33}$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \bar{A}_k(x)\bar{A}_{2n-k}(x) = (-1)^n \frac{(2n)!}{n!} \tag{8.34}$$

8.4 The Linear Operator $(D + a)^n$

Remark 8.3 *Throughout this section, we assume a and b are nonzero real or complex numbers. We will also assume u and v are functions of x.*

8.4.1 Definition of $(D + a)^n$

$$\boldsymbol{(D + a)^n u = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} D^k u} \tag{8.35}$$

8.4.2 Product Rule

$$(D + a)^n uv = \sum_{k=0}^{n} \binom{n}{k} D^k v (D + a)^{n-k} u \tag{8.36}$$

$$(D + a - b)^n uv = \sum_{k=0}^{n} \binom{n}{k} (D + a)^k u (D - b)^{n-k} v \tag{8.37}$$

$$D^n(uv) = \sum_{k=0}^{n} \binom{n}{k} (D + a)^k u (D - a)^{n-k} v \tag{8.38}$$

8.4.3 Polynomial Operator $f(D + a)$

Remark 8.4 *Throughout this section, we assume $f(x)$ is a polynomial of degree n, namely, $f(x) = \sum_{k=0}^{n} a_k x^k$. We will also use the notation $f^{(k)}(x)$ to denote $D_x^k f(x)$.*

$$f(D + a - b)uv = \sum_{k=0}^{n} \frac{(D + a)^k u}{k!} f^{(k)}(D - b)v \tag{8.39}$$

$$f(D)uv = \sum_{k=0}^{n} \frac{(D + a)^k u}{k!} f^{(k)}(D - a)v \tag{8.40}$$

Exponential Shift Theorem

$$f(D - a + b)e^{ax}v = e^{ax} f(D + b)v \tag{8.41}$$

Applications of Equation (8.41)

$$f(D)(e^{ax}v) = e^{ax}f(D+a)v \tag{8.42}$$

$$D^n(e^x v) = e^x(D+1)^n v = e^x \sum_{k=0}^{n} \binom{n}{k} D^k v \tag{8.43}$$

Remark 8.5 *An alternative derivation for the exponential shift theorem and generalized Leibniz formula are given by C. J. Coe, "The Generalized Leibniz Formula", The American Math. Monthly, Sept. 1950, Vol. 57, No. 7, pp. 459-466.*

9 Shifted Legendre Polynomials $\tilde{P}_n(x)$

Remark 9.1 *In this chapter, we assume n and r are nonnegative integers, while x and a are real or complex numbers. If x is a real number, we use $[x]$ to denote the floor of x.*

9.1 Definition of $\tilde{P}_n(x)$

$$\boldsymbol{\tilde{P}_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 x^{n-k} = \sum_{k=0}^{n} \binom{n}{k}\binom{2n-k}{n}(x-1)^k} \tag{9.1}$$

9.2 Expansion of $\tilde{P}_n(x)$

$$\tilde{P}_n(x) = \sum_{j=0}^{n}(-1)^j x^j \sum_{k=j}^{n}(-1)^k \binom{n}{k}\binom{2n-k}{n}\binom{k}{j} \tag{9.2}$$

9.2.1 Applications of Equation (9.2)

$$\sum_{k=j}^{n}(-1)^k \binom{n}{k}\binom{2n-k}{n}\binom{k}{j} = (-1)^j \binom{n}{j}^2 \tag{9.3}$$

$$\sum_{k=1}^{n}(-1)^{k-1} \binom{n}{k}\binom{2n-k}{n} k = n^2, \qquad n \geq 1 \tag{9.4}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2n-k}{n}\binom{n+k}{n} \tag{9.5}$$

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^3=\sum_{k=0}^{2n}(-1)^k\binom{2n+k}{2n-k}\binom{2k}{k}\binom{4n-k}{4n} \tag{9.6}$$

$$\sum_{k=0}^{2n+1}(-1)^k\binom{2n+1}{k}=0, \qquad \textit{where } p \textit{ is an integer} \tag{9.7}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3=(-1)^n\binom{2n}{n}^2$$
$$+\sum_{k=0}^{n}\binom{n+k-1}{n}\sum_{j=0}^{k}(-1)^j\binom{2n+1}{j}\binom{3n-j+k}{n}\binom{n-1-j+k}{n},\ n\geq 1 \tag{9.8}$$

$$\sum_{k=0}^{2n+1}\binom{2n+k}{2n+1}\sum_{j=0}^{k-1}(-1)^j\binom{4n+3}{j}\binom{6n+3-j+k}{2n+1}\binom{2n-j+k}{2n+1}=\binom{4n+2}{2n+1}^2 \tag{9.9}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2\frac{1}{\binom{rn+n}{k}}=\frac{\binom{rn}{n}^2}{\binom{rn+n}{2n}\binom{2n}{n}} \tag{9.10}$$

$$\sum_{j=0}^{n}\binom{n}{j}^2 f(j)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2n-k}{n}\sum_{j=0}^{k}(-1)^j\binom{k}{j}f(j) \tag{9.11}$$

Examples of Equation (9.11)

$$\sum_{j=0}^{n}\binom{n}{j}^2 j^r=\sum_{k=0}^{r}\binom{n}{k}\binom{2n-k}{n}\sum_{j=0}^{k}(-1)^{j+k}\binom{k}{j}j^r \tag{9.12}$$

$$\sum_{j=0}^{n}\binom{n}{j}^2\frac{1}{2j+1}=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2n-k}{n}\frac{1}{\binom{k+\frac{1}{2}}{k}} \tag{9.13}$$

$$\sum_{k=0}^{n}\binom{n}{k}\binom{-n-1}{n-k}\frac{1}{\binom{k+r}{k}}=r(-1)^n\sum_{j=0}^{n}\binom{n}{j}^2\frac{1}{j+r},\qquad r\geq 1 \tag{9.14}$$

$$\sum_{j=0}^{n}\binom{n}{j}^2\binom{j}{n-k}=\binom{n}{k}\binom{n+k}{k}=\binom{n}{k}\binom{2n-k}{n},\qquad 0\leq k\leq n \tag{9.15}$$

Remark 9.2 *The following identity, due to E. T. Bell, is Problem 3457 of the American Math. Monthly, 1930, pp. 507-508.*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}\binom{2k}{k}2^{2n-2k}=\frac{1+(-1)^n}{2}\binom{n}{\left[\frac{n}{2}\right]}^2 \tag{9.16}$$

9.3 Expansion of $\tilde{P}_n^2(x)$

$$\tilde{P}_n^2(x)=\sum_{k=0}^{2n}x^k\sum_{j=\left[\frac{k}{n+1}\right](k-n)}^{k-\left[\frac{k+1}{n+1}\right](k-n)}\binom{n}{j}^2\binom{n}{k-j}^2 \tag{9.17}$$

9.3.1 Applications of Equation (9.17)

$$\sum_{k=0}^{n}\binom{n}{k}^4=\sum_{j=0}^{n}\binom{n}{j}\binom{2n-j}{n}\sum_{k=0}^{j}(-1)^k\binom{k+n}{n}\binom{n}{j-k}\binom{n-k+j}{n} \tag{9.18}$$

$$\sum_{k=0}^{n}\left[\binom{n}{k}^4-(-1)^k\binom{n}{k}^3\right]=\sum_{j=0}^{n-1}\binom{n}{j}\binom{2n-j}{n}$$
$$*\sum_{k=0}^{j}(-1)^k\binom{k+n}{n}\binom{n}{j-k}\binom{n-k+j}{n},\ n\geq 1 \tag{9.19}$$

$$\sum_{k=0}^{n}\binom{n}{k}^4=\sum_{j=0}^{n}(-1)^j\binom{n}{j}\binom{2n-j}{n}\sum_{k=0}^{j}(-1)^k\binom{n}{k}^2\binom{j}{k} \tag{9.20}$$

9.4 Three Expansions of $(x-1)^{2n}\tilde{P}_{2n}(x)$

9.4.1 First Expansion of $(x-1)^{2n}\tilde{P}_{2n}(x)$

$$(x-1)^{2n}\tilde{P}_{2n}(x) = \sum_{k=0}^{4n} x^{4n-k} \sum_{j=[\frac{k}{2n+1}](k-2n)}^{k-[\frac{k+1}{2n+1}](k-2n)} (-1)^{k-j}\binom{2n}{j}^2\binom{2n}{k-j} \tag{9.21}$$

9.4.2 Second Expansion of $(x-1)^{2n}\tilde{P}_{2n}(x)$

$$(x-1)^{2n}\tilde{P}_{2n}(x) = \sum_{j=0}^{m}(-1)^{j+2n}x^j \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{4n-k}{2n}\binom{2n+k}{j},\ m \geq 4n \tag{9.22}$$

Applications of Equations (9.21) and (9.22)

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^3 = \sum_{k=0}^{2n}(-1)^k\binom{2n+k}{2n-k}\binom{2k}{k}\binom{4n-k}{2n} \tag{9.23}$$

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^3 = (-1)^n\binom{2n}{n}\binom{3n}{2n} \tag{9.24}$$

$$\sum_{k=0}^{\infty}(-1)^k\frac{x^k}{k!^3}\sum_{j=0}^{\infty}\frac{x^j}{j!^3} = \sum_{k=0}^{\infty}(-1)^k\frac{(3k)!}{(2k)!^3k!^3}x^{2k} \tag{9.25}$$

$$\sum_{k=0}^{n}(-1)^k\binom{2n}{k}^2\binom{2n}{n-k} = \sum_{k=0}^{n}(-1)^{k+n}\binom{2n}{k+n}\binom{3n-k}{2n}\binom{3n+k}{3n} \tag{9.26}$$

$$\sum_{k=0}^{n}(-1)^k\binom{2n}{k}^2\binom{2n}{n-k} = \sum_{k=0}^{n}(-1)^k\binom{2n}{k}\binom{2n+k}{2n}\binom{4n-k}{3n} \tag{9.27}$$

$$\sum_{k=0}^{n}(-1)^k\binom{2n}{k}^2\binom{2n}{n-k} = \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{4n-k}{2n}\binom{2n+k}{n} \tag{9.28}$$

9.4.3 Third Expansion of $(x-1)^{2n}\tilde{P}_{2n}(x)$

$$(x-1)^{2n}\tilde{P}_{2n}(x)=\sum_{i=0}^{2n}\binom{2n}{i}^2\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{2n+j}(-1)^{2n+j-k}\binom{2n+j}{k}x^k \tag{9.29}$$

Application of Equation (9.29)

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^3=\sum_{k=0}^{2n}\binom{2n}{k}^2\sum_{j=0}^{k}(-1)^j\binom{k}{j}\binom{2n+j}{2n} \tag{9.30}$$

9.5 Expansion of $(a+x)^{2n}\tilde{P}_{2n}(x)$

$$(a+x)^{2n}\tilde{P}_{2n}(x)=\sum_{k=0}^{4n}x^{4n-k}\sum_{j=[\frac{k}{2n+1}](k-2n)}^{k-[\frac{k+1}{2n+1}](k-2n)}\binom{2n}{j}^2\binom{2n}{k-j}a^{k-j},\ a\neq 0 \tag{9.31}$$

9.6 Expansion of $\tilde{P}^2_{2n}(x)$

$$\tilde{P}^2_{2n}(x)=\sum_{k=0}^{4n}x^k\sum_{j=[\frac{k}{2n+1}](k-2n)}^{k-[\frac{k+1}{2n+1}](k-2n)}\binom{2n}{j}^2\binom{2n}{k-j}^2 \tag{9.32}$$

9.6.1 Application of Equation (9.32)

$$\begin{aligned}\sum_{j=0}^{n}\binom{2n}{j}^3\binom{2n}{j+n}^2&=\sum_{j=0}^{n}\binom{2n}{j+n}\binom{3n-j}{2n}\\&*\sum_{k=0}^{j}(-1)^k\binom{3n+k}{3n}\binom{2n}{j-k}\binom{2n-k+j}{2n}\end{aligned} \tag{9.33}$$

9.7 Expansions of $(x-1)^n\tilde{P}_n(x)$

$$(x-1)^n\tilde{P}_n(x)=\sum_{k=0}^{2n}(-1)^kx^k\sum_{j=0}^{2n-k}(-1)^j\binom{n}{j}^2\binom{n}{2n-k-j} \tag{9.34}$$

$$(x-1)^n\tilde{P}_n(x)=\sum_{k=0}^{2n}(-1)^kx^k\sum_{j=0}^{n}(-1)^{n+j}\binom{n}{j}\binom{2n-j}{n}\binom{n+j}{k} \tag{9.35}$$

9.7.1 Application of Equations (9.34) and (9.35)

$$\sum_{j=0}^{2n-k}(-1)^j\binom{n}{j}^2\binom{n}{2n-k-j}=\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\binom{2n-j}{n}\binom{n+j}{k} \tag{9.36}$$

$$\sum_{j=0}^{2n-k}(-1)^j\binom{n}{j}^2\binom{n}{2n-k-j}=\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}\binom{n+j}{n}\binom{2n-j}{k} \tag{9.37}$$

10 Legendre Polynomials $P_n(x)$

Remark 10.1 *In this chapter, we assume n and r are nonnegative integers, while x, y, and t are real or complex numbers. If x is a real number, we use $[x]$ to denote the floor of x. We also use the convention that $D_x^n f(x)$ is the n^{th} derivative of $f(x)$ with respect to x.*

10.1 Three Ways to Define $P_n(x)$

10.1.1 Generating Function for $P_n(x)$

$$\sum_{n=0}^{\infty}P_n(x)t^n=(1-2xt+t^2)^{-\frac{1}{2}} \tag{10.1}$$

10.1.2 Rodrigues Formula

$$P_n(x)=\frac{1}{2^n n!}D_x^n(x^2-1)^n \tag{10.2}$$

10.1.3 Explicit Summation for $P_n(x)$

$$\boldsymbol{P_n(x)=\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\binom{n}{k}\binom{2n-2k}{n}\frac{x^{n-2k}}{2^n}} \tag{10.3}$$

10.1.4 Two Alternative Summations for $P_n(x)$

$$P_n(x)=\left(\frac{x-1}{2}\right)^n\sum_{k=0}^{n}\binom{n}{k}^2\left(\frac{x+1}{x-1}\right)^k \tag{10.4}$$

$$P_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{2n-k}{n} \left(\frac{x-1}{2} \right)^{n-k} \tag{10.5}$$

10.2 Binomial Identities from the Basic Definitions of $P_n(x)$

$$\sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n}{k} \binom{2n-2k}{n} = 2^n \tag{10.6}$$

$$\sum_{k=0}^{[\frac{n-2}{2}]} (-1)^k \binom{n}{k} \binom{2n-2k-2}{n} = 2^n - n - 1, \qquad n \geq 2 \tag{10.7}$$

$$\sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n}{k} \binom{2n-2k}{n} \frac{1}{2^{2k}} = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k}^2 3^k = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \binom{2n-k}{n} 2^k \tag{10.8}$$

$$\sum_{k=0}^{n} \binom{n}{k} \binom{2n-k}{n} = \sum_{k=0}^{n} \binom{n}{k}^2 2^k = \left(\frac{3}{2} \right)^n \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n}{k} \binom{2n-2k}{n} \frac{1}{9^k} \tag{10.9}$$

$$\begin{aligned} \left(\frac{5}{2} \right)^n \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n}{k} \binom{2n-2k}{n} \frac{1}{5^{2k}} &= 2^n \sum_{k=0}^{n} \binom{n}{k}^2 \left(\frac{3}{2} \right)^k \\ &= 2^n \sum_{k=0}^{n} \binom{n}{k} \binom{2n-k}{n} \frac{1}{2^k} \end{aligned} \tag{10.10}$$

Remark 10.2 *The following two identities, due to L. Carlitz, are the solutions to Problem 352 of The Math. Magazine, Vol. 32, No. 1, Sept.-Oct. 1958, pp. 47-48. We use the convention that* $((x^n))f(x)$ *is the coefficient of* x^n *in the power series expansion of* $f(x)$*.*

$$((x^n))(1-x^2)^n P_n \left(\frac{1+x}{1-x} \right) = \sum_{k=0}^{n} \binom{n}{k}^3 \tag{10.11}$$

$$((x^n))(1-x)^{2n} \left(P_n \left(\frac{1+x}{1-x} \right) \right)^2 = \sum_{k=0}^{n} \binom{n}{k}^4 \tag{10.12}$$

10.3 Connections to Shifted Legendre Polynomials $\tilde{P}_n(x)$

$$P_n(x) = \left(\frac{x-1}{2}\right)^n \tilde{P}_n\left(\frac{x+1}{x-1}\right), \qquad x \neq 1 \tag{10.13}$$

$$\tilde{P}_n(t) = (t-1)^n P_n\left(\frac{t+1}{t-1}\right), \qquad t \neq 1 \tag{10.14}$$

10.3.1 Expansions of $\tilde{P}_n(x)$ Using Equation (10.14)

$$(x-1)^n \tilde{P}_n(x) = \frac{1}{2^n} \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n}{k} \binom{2n-2k}{n} (x+1)^{n-2k} (x-1)^{n+2k} \tag{10.15}$$

$$(x-1)^{2n} \tilde{P}_{2n}(x) = \frac{1}{2^{2n}} \sum_{k=0}^{n} (-1)^k \binom{2n}{k} \binom{4n-2k}{2n} (x+1)^{2n-2k} (x-1)^{2n+2k} \tag{10.16}$$

Applications of Equations (10.15) and (10.16)

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = \frac{1}{2^{2n}} \sum_{k=0}^{n} (-1)^k \binom{2n}{k} \binom{4n-2k}{2n} \sum_{j=2k}^{2n} (-1)^j \binom{2n+2k}{j} \binom{2n-2k}{2n-j} \tag{10.17}$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = (-1)^n \frac{(6n)!}{2^{2n}(2n)!^3} \sum_{k=0}^{n} \binom{2n}{k} \binom{2n}{n-k} \frac{1}{\binom{6n}{4n-2k}} \tag{10.18}$$

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = \frac{(-1)^n}{2^{2n}\binom{2n}{n}} \sum_{k=0}^{n} \binom{n+k}{n} \binom{2n+2k}{n+k} \binom{4n-2k}{2n-k} \binom{2n-k}{n} \tag{10.19}$$

10.4 Laplace Integral for $P_n(x)$ and Various Applications

10.4.1 Laplace Integral for $P_n(x)$

$$P_n(x) = \int_0^1 (x + \sqrt{x^2-1} \cos 2\pi t)^n \, dt \tag{10.20}$$

10.4.2 Good's Formula for $P_n(x)$

Remark 10.3 *The following formula is found in " A new finite series for Legendre polynomials", by I. J. Good, Proc. Cambridge Philosophical Society, Vol. 51, 1955, pp. 385-388.*

Good's Formula

$$P_n(x) = \frac{1}{r}\sum_{k=0}^{r-1}\left(x + \sqrt{x^2-1}\cos\frac{2\pi k}{r}\right), \text{ for integral } r > n,\ r \neq 0 \tag{10.21}$$

Identities from the Proof of Good's Formula

$$P_n(x) = \sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\frac{(x^2-1)^{\frac{k}{2}}(x-(x^2-1)^{\frac{1}{2}})^{n-k}}{2^k} \tag{10.22}$$

$$P_n(x) = \sum_{k=0}^{[\frac{n}{2}]}\binom{n}{2k}\binom{2k}{k}\frac{x^{n-2k}(x^2-1)^k}{2^{2k}} \tag{10.23}$$

$$\sum_{k=j}^{[\frac{n}{2}]}\binom{n-j}{k-j}\binom{n-k}{k}2^{n-2k} = \binom{2n-2j}{n} \tag{10.24}$$

10.4.3 A Convolution Identity

Remark 10.4 *The following identity is from "Inverse Elliptic Functions and Legendre Polynomials" by R. P. Kelisky in The American Math. Monthly, Vol. 66, 1959, pp. 480-485.*

$$2^{2n}x^nP_n\left(\frac{x+x^{-1}}{2}\right) = \sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}x^{2k} = \frac{2^{2n+1}}{\pi}\int_0^{\frac{\pi}{2}}(x^2\sin^2 t + \cos^2 t)^n\,dt \tag{10.25}$$

Applications of Equation (10.25)

$$(-1)^nx^nP_n\left(\frac{x+x^{-1}}{2}\right) = \sum_{k=0}^{n}\binom{-\frac{1}{2}}{k}\binom{-\frac{1}{2}}{n-k}x^{2k} \tag{10.26}$$

$$(-1)^nx^nP_n\left(\frac{x+x^{-1}}{2}\right) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{k-\frac{1}{2}}{n}x^{2k} \tag{10.27}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}\binom{2k}{k+j}\frac{1}{2^{2k}} = \frac{(-1)^n+(-1)^j}{2^{2n+1}}\binom{n+j}{\frac{n+j}{2}}\binom{n-j}{\frac{n-j}{2}}, \tag{10.28}$$

j *integral,* $0 \le j \le n$

10.5 Derivatives of $P_n(x)$

$$D_x^r P_n(x) = \frac{r!}{2^n} \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n}{k} \binom{2n-2k}{n} \binom{n-2k}{r} x^{n-2k-r} \tag{10.29}$$

$$D_x^r P_n(x)|_{x=0} = r! \frac{(-1)^{\frac{n-r}{2}}}{2^n} \binom{n}{\frac{n-r}{2}} \binom{n+r}{n} \frac{1+(-1)^{n-r}}{2} \tag{10.30}$$

10.5.1 Applications of Equation (10.30)

$$\sum_{k=j}^{n} (-1)^k \binom{n}{k} \binom{2k}{k-j} \frac{1}{2^k} = \frac{(-1)^j}{2^n} \binom{n}{\frac{n-j}{2}} \frac{1+(-1)^{n-j}}{2}, \qquad 0 \le j \le n \tag{10.31}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{k} \frac{1}{2^k} = \frac{1}{2^n} \binom{n}{\frac{n}{2}} \frac{1+(-1)^n}{2} = \frac{1}{2^{2n}} \binom{2n}{n} \tag{10.32}$$

10.5.2 Grosswald's Formula

Remark 10.5 *The identities of this section are found in Emil Grosswald's "On Sums Involving Binomial Coefficients", American Math. Monthly, Vol. 60, No. 3, March 1953, pp. 179-181.*

Grosswald's Formula

$$D_x^r P_n(x)|_{x=0} = \frac{r!}{2^r} \binom{n}{r} \sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \binom{n+k+r}{n} \frac{1}{2^k} \tag{10.33}$$

Applications of Equation (10.33)

$$\sum_{k=0}^{n-r} (-1)^k \binom{n}{k+r} \binom{n+k+r}{k} \frac{1}{2^k} = (-1)^{\frac{n-r}{2}} 2^{r-n} \binom{n}{\frac{n+r}{2}} \frac{1+(-1)^{n-r}}{2} \tag{10.34}$$

$$\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \binom{n+k+r}{n} \frac{1}{2^k} = (-1)^{\frac{n-r}{2}} \frac{2^{r-n}}{\binom{n}{r}} \binom{n}{\frac{n-r}{2}} \binom{n+r}{n} \frac{1+(-1)^{n-r}}{2} \tag{10.35}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k+2r}{n} \frac{1}{2^k} = (-1)^{\frac{n}{2}} \frac{2^{-n}}{\binom{n+r}{r}} \binom{n+r}{\frac{n}{2}} \binom{n+2r}{r} \frac{1+(-1)^n}{2} \tag{10.36}$$

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{4n+k}{3n}\frac{1}{2^k}=(-1)^n\binom{4n}{n}\frac{1}{2^{2n}} \tag{10.37}$$

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{6n+k}{4n}\frac{1}{2^k}=(-1)^n\binom{4n}{n}\binom{6n}{2n}\frac{1}{2^{2n}\binom{4n}{2n}} \tag{10.38}$$

$$\sum_{k=0}^{3n}(-1)^k\binom{3n}{k}\binom{5n+k}{4n}\frac{1}{2^k}=(-1)^{\frac{3n}{2}}\frac{1}{2^{3n}\binom{4n}{n}}\binom{4n}{\frac{3n}{2}}\binom{5n}{n}\frac{1+(-1)^n}{2} \tag{10.39}$$

Extensions of Equation (10.34)

Remark 10.6 *The following two identities are from T. S. Nanjundiah's "On a Formula of Grosswald", American Math. Monthly, Vol. 61, No. 10, December 1954, pp. 700-702.*

$$\sum_{k=0}^{n}(-2)^k\binom{\mu}{k}\binom{2\mu-2k}{n-k}=\frac{(-1)^n+1}{2}\binom{\mu}{\frac{n}{2}}, \qquad u \textit{ a real number} \tag{10.40}$$

$$\sum_{k=0}^{n}(-2)^k\binom{\mu}{k}\binom{2\mu-k}{n-k}=(-1)^{\frac{n}{2}}\frac{(-1)^n+1}{2}\binom{\mu}{\frac{n}{2}}, \qquad u \textit{ a real number} \tag{10.41}$$

10.5.3 Double Summation Formula for $P_n(x)$

$$P_n(x)=\sum_{r=0}^{n}\binom{n}{r}\frac{x^r}{2^r}\sum_{k=0}^{n-r}(-1)^k\binom{n-r}{k}\binom{n+k+r}{n}\frac{1}{2^k} \tag{10.42}$$

10.5.4 Extensions of Equation (10.29)

$$D_x^r(x)P_n(x)|_{x=\pm 1}=(\pm 1)^{n+r}\binom{n}{r}\binom{n+r}{r}\frac{r!}{2^r} \tag{10.43}$$

Identities from the Proof of Equation (10.43)

$$\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\binom{n}{k}\binom{2n-2k}{n+r}=2^{n-r}\binom{n}{r} \tag{10.44}$$

$$\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\binom{n}{k}\binom{2n-2k}{n}\binom{n-2k}{r}=2^{n-r}\binom{n}{r}\binom{n+r}{r} \tag{10.45}$$

$$\sum_{k=0}^{n}(-1)^k\binom{2n}{k}\binom{4n-2k}{2n}\binom{2n-2k}{n}=2^{n}\binom{2n}{n}\binom{3n}{2n} \tag{10.46}$$

Application of Equation (10.44)

$$\sum_{r=0}^{n}\binom{n}{r}^2\frac{1}{2^r}=\frac{1}{2^n}\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\binom{n}{k}\binom{3n-2k}{2n} \tag{10.47}$$

Generalization of Equation (10.44)

$$\boldsymbol{\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\binom{x}{k}\binom{2x-2k}{n-2k}=2^{n}\binom{x}{n}} \tag{10.48}$$

$$\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\binom{n+x}{k}\binom{2n+2x-2k}{n-2k}=2^{n}\binom{x+n}{n} \tag{10.49}$$

10.6 Differential and Difference Recurrence Relations

10.6.1 Differential Recurrence Relations

$$nP_n(x)-xP_n^{'}+P_{n-1}^{'}(x)=0,\qquad n\geq 1 \tag{10.50}$$

$$nP_{n-1}(x)-P_n^{'}(x)+xP_{n-1}^{'}(x)=0,\qquad n\geq 1 \tag{10.51}$$

$$(2n+1)P_n(x)-P_{n+1}^{'}(x)+P_{n-1}^{'}(x)=0,\qquad n\geq 1 \tag{10.52}$$

$$nP_{n-1}(x)+(x^2-1)P_n^{'}(x)-nxP_n(x)=0,\qquad n\geq 1 \tag{10.53}$$

$$(1-x^2)P_n^{''}(x)-2xP_n^{'}(x)+n(n+1)P_n(x)=0 \tag{10.54}$$

$$P_n^{'}(x)=\frac{n}{1-x^2}\left(P_{n-1}(x)-xP_n(x)\right),\qquad n\geq 1 \tag{10.55}$$

10.6.2 Difference Recurrence Relation

$$(n+1)P_{n+1}(x)-(2n+1)xP_n(x)+nP_{n-1}(x)=0, \qquad n\geq 1 \tag{10.56}$$

10.6.3 Christoffel's Identity

Remark 10.7 *A reference for Christoffel's Identity is* <u>*Fourier Series and Orthogonal Polynomials*</u> *by Dunham Jackson, 1941. The reader is also referred to E. B. Christoffel's, "Über die Gaussische Quadratur und eine Verallgemeinerung derselben", Journal für die reine und angewandte Mathematik, Vol. 55, 1858, pp. 61-82.*

Christoffel's Identity

$$(x-y)\sum_{k=0}^{n}(2k+1)P_k(x)P_k(y)=(n+1)\left(P_{n+1}(x)P_n(y)-P_n(x)P_{n+1}(y)\right) \tag{10.57}$$

10.7 Evaluation of $\int_{-1}^{1} P_n(x)P_r(x)\,dx$

$$\int_{-1}^{1} \boldsymbol{P_n(x)P_r(x)\,dx}=\begin{cases}\mathbf{0}, & \boldsymbol{n\neq r}\\ \frac{\mathbf{2}}{\mathbf{2}\boldsymbol{n}+\mathbf{1}}, & \boldsymbol{n=r}\end{cases} \tag{10.58}$$

$$\int_{-1}^{1} P_n(x)P_r(x)\,dx=\frac{2(-1)^{n+r}}{n+r+1}\sum_{k=0}^{n+r}\frac{(-1)^k}{\binom{n+r}{k}}\sum_{j=0}^{k}\binom{n}{j}^2\binom{r}{k-j}^2 \tag{10.59}$$

10.7.1 Formulas Associated with Equation (10.59)

$$\sum_{k=0}^{n+r}(-1)^k\sum_{j=0}^{k}\binom{n}{j}^2\binom{r}{k-j}^2=(-1)^{[\frac{n}{2}]+[\frac{r}{2}]}\binom{n}{[\frac{n}{2}]}\binom{r}{[\frac{r}{2}]}\frac{1+(-1)^n}{2}\frac{1+(-1)^r}{2} \tag{10.60}$$

$$\sum_{k=0}^{2n+2r}(-1)^k\sum_{j=0}^{k}\binom{2n}{j}^2\binom{2r}{k-j}^2=(-1)^{n+r}\binom{2n}{n}\binom{2r}{r} \tag{10.61}$$

10.8 Evaluation of $\int_{-1}^{1} x^r P_n(x)\,dx$

$$\int_{-1}^{1} x^r P_n(x)\,dx = \begin{cases} \frac{1}{2^n}\frac{\binom{r}{n}}{\binom{\frac{n+r+1}{2}}{n}}\frac{1+(-1)^{r+n}}{r-n-1}, & r \geq n \geq 0 \\ 0, & n > r \geq 0 \end{cases} \tag{10.62}$$

$$\int_{-1}^{1} \boldsymbol{P_n(x)\,dx} = \begin{cases} \mathbf{0}, & \boldsymbol{n \geq 1} \\ \mathbf{2}, & \boldsymbol{n = 0} \end{cases} \tag{10.63}$$

$$\int_{-1}^{1} \boldsymbol{x^n P_n(x)\,dx = \frac{2^{n+1}}{\binom{2n}{n}(2n+1)}} \tag{10.64}$$

10.8.1 An Indefinite Integral

$$\int P_n(x)\,dx = \begin{cases} \frac{xP_n(x)-P_{n-1}(x)}{n+1}, & n \geq 1 \\ x, & n = 0 \end{cases} \tag{10.65}$$

10.9 Polynomial Expansions via Legendre Polynomials

If $f(x)$ is any polynomial of degree n in x,

$$f(x) = \sum_{k=0}^{n} P_k(x)\frac{2k+1}{2}\int_{-1}^{1} P_k(x)f(x)\,dx \tag{10.66}$$

10.9.1 Examples of Equation (10.66)

$$x^n = \sum_{k=0}^{n} \binom{n}{k}\frac{(2k+1)(1+(-1)^{n+k})}{2^{k+1}(n-k+1)\binom{\frac{n+k+1}{2}}{k}}P_k(x) \tag{10.67}$$

$$x^{2n} = \sum_{k=0}^{n} \frac{(4k+1)\binom{2n}{n+k}}{(2n+2k+1)\binom{2n+2k}{n+k}}2^{2k}P_{2k}(x) \tag{10.68}$$

$$\sum_{k=0}^{n} \frac{(4k+1)\binom{2n}{n+k}}{(2n+2k+1)\binom{2n+2k}{n+k}}2^{2k} = 1 \tag{10.69}$$

$$x^{2n+1} = \sum_{k=0}^{n} \frac{\binom{2n}{n+k}}{\binom{2n+2k}{n+k}} \frac{(2n+1)(4k+3)}{(2n+2k+1)(2n+2k+3)} 2^{2k} P_{2k+1}(x) \tag{10.70}$$

$$x^{n} = \sum_{k=0}^{[\frac{n}{2}]} \frac{(2n-4k+1)\binom{n-k}{k}}{(2n-2k+1)\binom{2n-2k}{n-2k}} 2^{n-2k} P_{n-2k}(x) \tag{10.71}$$

$$\sum_{k=0}^{[\frac{n}{2}]} \frac{(2n-4k+1)\binom{n-k}{k}}{(2n-2k+1)\binom{2n-2k}{n-2k}} 2^{n-2k} = \sum_{k=0}^{[\frac{n}{2}]} \frac{(2n-4k+1)\binom{n}{k}}{(2n-2k+1)\binom{2n-2k}{n-2k}} 2^{n-2k} = 1 \tag{10.72}$$

Remark 10.8 *An equivalent form of Equation (10.72), namely,*

$$\sum_{k=0}^{[\frac{n}{2}]} 2^{2n-4k+1}(2n-4k+1)\frac{(n-k+\frac{1}{2})!(n-k)!^2}{k!(2n-2k+1)!^2} = \frac{\sqrt{\pi}}{2^n n!},$$

is found in N. Arley, " Om summation av en række", Mat. Tidsskr. B. Københaun, 1937, pp. 42-44.

Remark 10.9 *The following identity, due to René Lagrange, [Acta Mathematica, Vol. 52, 1929], is a generalization of Equation (10.72).*

$$\sum_{k=0}^{n} \frac{(x+y+1-2k)\binom{x}{k}\binom{y-k}{n-k}}{(x+y+1-k)\binom{x+y-k}{n}} = 1 \tag{10.73}$$

Applications of Equation (10.73)

$$\sum_{k=0}^{n} \frac{(2n-4k+1)\binom{n}{k}}{(2n-2k+1)\binom{2n-2k}{n-k}} 2^{2n-2k} = 1 \tag{10.74}$$

$$\sum_{k=[\frac{n}{2}]}^{n} \frac{(2n-4k+1)\binom{n}{k}}{(2n-2k+1)\binom{2n-2k}{n-k}} 2^{2n-2k} = 1 - 2^n \tag{10.75}$$

10.9.2 Application of Equation (10.66) to Integration

If $f(x)$ is any polynomial of degree n in x such that $f(x) = \sum_{k=0}^{n} a_k P_k(x)$, then

$$\int_{-1}^{1} f^2(x)\,dx = \sum_{k=0}^{n} \frac{2}{2k+1} a_k^2. \tag{10.76}$$

Applications of Equation (10.76)

$$\sum_{k=0}^{n} \binom{n}{k}^2 \frac{(2k+1)(1+(-1)^{n+k})}{\binom{\frac{n+k+1}{2}}{k}^2 2^{2k+1}(n-k+1)^2} = \frac{1}{2n+1} \tag{10.77}$$

$$\sum_{k=0}^{n} \binom{2n}{2k}^2 \frac{(4k+1)}{\binom{n+k+\frac{1}{2}}{2k}^2 2^{4k}(2n-2k+1)^2} = \frac{1}{4n+1} \tag{10.78}$$

$$\sum_{k=0}^{n} \frac{\binom{2n}{n+k}^2}{\binom{2n+2k}{n+k}^2} \frac{(4k+1)2^{4k}}{(2n+2k+1)^2} = \frac{1}{4n+1} \tag{10.79}$$

11 Hermite Polynomials $H_n(x)$

Remark 11.1 *In this chapter, we assume n and r are nonnegative integers. We assume x, y, z, and t are real or complex numbers. We use the convention that $D_x^r f(x)$ is the r^{th} derivative of $f(x)$. Finally, if x is a real number, we let $[x]$ denote the floor of x.*

11.1 Three Definitions for $H_n(x)$

11.1.1 Exponential Generating Function

$$\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = e^{2xt-t^2} \tag{11.1}$$

11.1.2 Operator Definitions

$$H_n\left(\frac{x}{\sqrt{2}}\right) = (-1)^n(\sqrt{2})^n e^{\frac{x^2}{2}} D_x^n e^{\frac{-x^2}{2}} \tag{11.2}$$

$$\boldsymbol{H_n(x) = (-1)^n e^{x^2} D_x^n e^{-x^2}} \tag{11.3}$$

$$\boldsymbol{H_n(x) = (-1)^n (D_x - 2x)^n 1} \tag{11.4}$$

11.1.3 Explicit Formula for $H_n(x)$

$$\boldsymbol{H_n(x) = \sum_{k=0}^{[\frac{n}{2}]}(-1)^k \binom{n}{k}\binom{n-k}{k} k!(2x)^{n-2k}} \tag{11.5}$$

11.2 Derivatives of $H_n(x)$

$$D_x^r H_n(x) = r! \sum_{k=0}^{[\frac{n}{2}]}(-1)^k \binom{n}{k}\binom{n-k}{k} k! 2^{n-2k} \binom{n-2k}{r} x^{n-2k-r} \tag{11.6}$$

$$D_x^r H_n(x)|_{x=0} = (-1)^{\frac{n-r}{2}} 2^r \frac{n!}{\left(\frac{n-r}{2}\right)!} \frac{1+(-1)^{n-r}}{2} \tag{11.7}$$

11.2.1 Taylor's Formula for $H_n(x)$

$$H_n(x) = \sum_{k=0}^{n} x^k \frac{(-1)^{\frac{n-k}{2}} 2^k n!}{k! \left(\frac{n-k}{2}\right)!} \cdot \frac{1+(-1)^{n-k}}{2} \tag{11.8}$$

11.3 Operator Analysis with Hermite Polynomials

$$e^{\frac{x^2}{2}}(x - D_x)^n \left(e^{\frac{-x^2}{2}} f(x)\right) = (-1)^n (D_x - 2x)^n f(x) \tag{11.9}$$

$$\begin{aligned} \left(e^{x^2} D_x e^{-x^2}\right)^n f(x) = e^{x^2} D_x^n \left(e^{-x^2} f(x)\right) &= (D_x - 2x)^n f(x) \\ &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} H_k(x) D_x^{n-k} f(x) \end{aligned} \tag{11.10}$$

11.3.1 Recurrence Relation for $H_n(x)$

$$H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x), \qquad n \geq 1 \tag{11.11}$$

11.4 Additional Properties of $H_n(x)$

$$e^{-t^2 D_x^2} x^n = t^n H_n\left(\frac{x}{2t}\right) \tag{11.12}$$

$$(2x)^n = \sum_{k=0}^{[\frac{n}{2}]} \frac{n!}{k!(n-2k)!} H_{n-2k}(x) \tag{11.13}$$

$$D_x^2 H_n(x) - 2x D_x H_n(x) + 2n H_n(x) = 0 \tag{11.14}$$

$$\frac{1}{2^n n! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_k(x)\, dx = \binom{0}{n-k} \tag{11.15}$$

$$H_n(x) = (-1)^n \frac{e^{x^2} n!}{2\pi i} \int_C \frac{e^{-z^2}}{(z-x)^{n+1}}\, dz, \qquad \textit{C is a simple closed contour around } x \tag{11.16}$$

12 Bernstein Polynomials

Remark 12.1 *Consider the function f defined on $0 \le x \le 1$. The expansion*

$$\boldsymbol{B_n^f(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^k f\left(\frac{k}{n}\right), \qquad n \ge 1} \tag{12.1}$$

is the Bernstein Polynomial of order n for f. A reference for these polynomials is Bernstein's "Démonstration du théorème de Weierstrass, fondeé sur le calcul des probabilités", Commun. Soc. Math Kharkow (2), Vol. 13, 1912-13, pp. 1-2.

12.1 Connections with Difference Operators

Remark 12.2 *For this section, we assume r is a nonnegative integer. We also use the convention that $D_x^r f(x)$ represents the r^{th} derivative of $f(x)$. Finally, recall the definition of Δ_h^r, namely*

$$\Delta_h^r f(x) = \sum_{k=0}^{r} (-1)^{r+k} \binom{r}{k} \frac{f(x+kh)}{h^r}.$$

$$D_x^r B_n^f(x) = r! \binom{n}{r} \sum_{k=0}^{n-r} \binom{n-r}{k} x^k (1-x)^{n-r-k} \Delta_1^r f\left(\frac{k}{n}\right) \tag{12.2}$$

$$B_n^f(x) = \sum_{j=0}^{n} x^j \binom{n}{j} \Delta_1^j f\left(\frac{k}{n}\right) |_{k=0} \tag{12.3}$$

12.1.1 Examples of Equation (12.3)

Let $f(x) = x^r$. Then,

$$B_n^f(x) = \frac{1}{n^r}\sum_{k=0}^{n}\binom{n}{k}x^k\sum_{j=0}^{k}(-1)^{j+k}\binom{k}{j}j^r. \tag{12.4}$$

Let $f(x) = e^x$. Then,

$$B_n^f(x) = \sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}e^{\frac{k}{n}} = \sum_{k=0}^{n}\binom{n}{k}\left(e^{\frac{1}{n}}-1\right)^k x^k = \left(1+\left(e^{\frac{1}{n}}-1\right)x\right)^n. \tag{12.5}$$

13 A Property Catalog for Special Functions

Remark 13.1 *In this chapter, except for Section 13.7, we assume n and k are nonnegative integers. We always assume x, t and z are real or complex numbers. We also use the convention that D_x^n denotes the n^{th} derivative with respect to x.*

13.1 Laguerre Polynomials $L_n(x)$

Generating Function

$$\frac{e^{\frac{-xt}{1-t}}}{1-t} = \sum_{n=0}^{\infty}L_n(x)t^n, \qquad |t| < 1 \tag{13.1}$$

Operator Definition

$$L_n(x) = \frac{e^x}{n!}D_x^n(e^{-x}x^n) \tag{13.2}$$

Explicit Formula

$$L_n(x) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{x^k}{k!} \tag{13.3}$$

Orthogonality Property

$$\int_0^\infty e^{-x}L_n(x)L_k(x)\,dx = \binom{0}{n-k} \tag{13.4}$$

13.2 Generalized Laguerre Polynomials $L_n^{(\alpha)}(x)$

Remark 13.2 *Throughout this section, we assume α is a real or complex number independent of n. If $\alpha = 0$, we obtain $L_n(x)$, the Laguerre polynomial of the previous section.*

Generating Function

$$\frac{e^{\frac{-xt}{1-t}}}{(1-t)^{\alpha+1}} = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n, \qquad |t| < 1 \tag{13.5}$$

Operator Definition

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha}e^x}{n!} D_x^n(e^{-x}x^{\alpha+n}) \tag{13.6}$$

Explicit Formula

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{x^k}{k!} \tag{13.7}$$

Remark 13.3 *Recall that, for nonnegative integers p and q, we define*

$${}_pF_q[a_1, a_2, ..., a_p; b_1, b_2, ...b_q; z] = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k, ..., (a_p)_k}{k!(b_1)_k(b_2)_k...(b_q)_k} z^k, \tag{13.8}$$

where $(a)_k = (a)(a+1)(a+2)...(a+k-1)$ whenever k is a positive integer. Otherwise $(a)_0 \equiv 1$. Note that

$$(a)_k = (-1)^k \binom{-a}{k} k!. \tag{13.9}$$

Hypergeometric Series

$$L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!} \, {}_1F_1[-n; 1+\alpha; x] \tag{13.10}$$

Recurrence Formula

$$(n+1)L_{n+1}^{(\alpha)}(x) = (-x+2n+\alpha+1)L_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x), \;\; n \geq 1, \tag{13.11}$$
$$L_0^{(\alpha)}(x) = 1, \;\; L_1^{(\alpha)}(x) = -x+\alpha+1$$

Contour Integral

$$L_n^{(\alpha)}(x) = \frac{e^x x^{-\alpha}}{2\pi i} \int_C \frac{z^{n+\alpha}e^{-z}}{(z-x)^{n+1}} \, dz, \;\; x \neq 0, \tag{13.12}$$

C a simple closed curve around $z = x$, not containing $z = 0$

13.3 Tschebyscheff Polynomials

13.3.1 Tschebyscheff Polynomials of the First Kind $T_n(x)$

Explicit Formula

$$T_n(x) = \frac{1}{2^{n-1}} \cos n\theta, \ \textit{where } x = \cos\theta \tag{13.13}$$

Orthogonality Property

$$\int_{-1}^{1} \frac{T_n(x)T_k(x)}{\sqrt{1-x^2}}\, dx = \begin{cases} 0, & k \neq n \\ \frac{\pi}{2^{2n-1}}, & k = n \end{cases} \tag{13.14}$$

Operator Definition

$$T_n(x) = \frac{(-1)^n}{1 \cdot 3 \cdot 5 \cdot \ldots (2n-1)} (1-x^2)^{\frac{1}{2}} D_x^n (1-x^2)^{n-\frac{1}{2}} \tag{13.15}$$

13.3.2 Tschebyscheff Polynomials of the Second Kind $U_n(x)$

Explicit Formula

$$U_n(x) = \frac{1}{n+1} D_x T_{n+1}(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \ \textit{where } x = \cos\theta \tag{13.16}$$

Orthogonality Property

$$\int_{-1}^{1} U_n(x)U_k(x)\sqrt{1-x^2}\, dx = \begin{cases} 0, & k \neq n \\ \frac{\pi}{2}, & k = n \end{cases} \tag{13.17}$$

13.4 Gegenbauer Polynomials $P_n^{(\gamma)}(x)$

Remark 13.4 *Throughout this section, we assume γ is a real or complex number independent of n. We also let, whenever x is a real number, $[x]$ denote the floor of x. Finally, we use the Γ function to evaluate noninteger factorials.*

Generating Function

$$(1-2xt+t^2)^{-\gamma} = \sum_{n=0}^{\infty} P_n^{(\gamma)}(x) t^n \tag{13.18}$$

Explicit Formulas

$$P_n^{(\gamma)}(x) = \frac{2^{-n}(\gamma-\frac{1}{2})!(n+2\gamma-1)!}{(2\gamma-1)!(n+\gamma-\frac{1}{2})!}\sum_{k=0}^{n}\binom{n+\gamma-\frac{1}{2}}{n-k}\binom{n+\gamma-\frac{1}{2}}{k}(x-1)^k(x+1)^{n-k} \tag{13.19}$$

$$P_n^{(\gamma)}(x) = \sum_{k=0}^{[\frac{n}{2}]}(-1)^k\binom{\gamma+n-k-1}{n-k}\binom{n-k}{k}(2x)^{n-2k} \tag{13.20}$$

Hypergeometric Series

$$P_n^{(\gamma)}(x) = \binom{2\gamma+n-1}{n}\,{}_2F_1[-n, 2\gamma+n; \gamma+\frac{1}{2}; \frac{1-x}{2}] \tag{13.21}$$

13.4.1 Connections to Other Special Functions

Legendre Polynomials $P_n(x)$

$$P_n(x) = P_n^{(\frac{1}{2})}(x) \tag{13.22}$$

Tschehyscheff Polynomials $U_n(x)$

$$U_n(x) = P_n^{(1)}(x) \tag{13.23}$$

13.5 Jacobi Polynomials $P_n^{(\alpha,\beta)}(x)$

Remark 13.5 *Throughtout this section, we assume α and β are real or complex numbers independent of n.*

Operator Definition

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}D_x^n\left((1-x)^{\alpha+n}(1+x)^{\beta+n}\right) \tag{13.24}$$

Explicit Formulas

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n}\binom{n+\alpha}{k}\binom{n+\beta}{n-k}\left(\frac{x-1}{2}\right)^{n-k}\left(\frac{x+1}{2}\right)^k \tag{13.25}$$

$$(z-1)^n P_n^{(\alpha,\beta)}\left(\frac{z+1}{z-1}\right) = \sum_{k=0}^{n}\binom{n+\alpha}{k}\binom{n+\beta}{n-k}z^k \tag{13.26}$$

13.5.1 Connections to Other Special Functions

Legendre Polynomials $P_n(x)$

$$P_n(x) = P_n^{(0,0)}(x) \tag{13.27}$$

Ultraspherical Polynomials: The Ultraspherical Polynomials are $P_n^{(\alpha,\alpha)}(x) \equiv P_n^{(\alpha)}$.

Tschebyscheff Polynomials

$$T_n(x) = \frac{n!}{(\frac{1}{2})_n} P_n^{(\frac{1}{2},\frac{1}{2})}(x) \tag{13.28}$$

$$U_n(x) = \frac{(n+1)!}{(\frac{3}{2})_n} P_n^{(-\frac{1}{2},-\frac{1}{2})}(x) \tag{13.29}$$

13.6 Bessel Polynomials $y_n(x)$

Generating Function

$$e^{\frac{1-(1-2xt)^{\frac{1}{2}}}{x}} = \sum_{n=0}^{\infty} y_{n-1}(x) \frac{t^n}{n!} \tag{13.30}$$

Operator Definition

$$y_n(x) = 2^{-n} e^{\frac{2}{x}} D_x^n \left(x^{2n} e^{-\frac{2}{x}} \right) \tag{13.31}$$

Explicit Formula

$$y_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(n+k)!}{n!} \left(\frac{x}{2}\right)^k \tag{13.32}$$

Recurrence Relation

$$y_{n+1}(x) = (2n+1)xy_n(x) + y_{n-1}(x), \qquad n \geq 1 \tag{13.33}$$

Orthogonality Property

$$\frac{1}{2\pi i} \int_{|z|=1} y_n(z) y_k(z) e^{-\frac{2}{z}}\, dz = \begin{cases} 0, & n \neq k \\ \frac{2(-1)^{n+1}}{2n+1}, & k = n \end{cases} \tag{13.34}$$

Expansion Properties

$$x^n = \sum_{k=0}^{n} (-1)^k \frac{n!(2n-2k+1)}{k!(2n-k+1)!} 2^n y_{n-k}(x) \tag{13.35}$$

$$e^z = \sum_{n=0}^{\infty} y_{n-1}(x) \frac{(z - \frac{xz^2}{2})^n}{n!} \tag{13.36}$$

13.6.1 Generalized Bessel Polynomials $y_n(x, a, b)$

Explicit Formula

$$y_n(x, a, b) = \sum_{k=0}^{n} \binom{n}{k} \frac{(n+k+a-2)!}{(n+a-2)!} \left(\frac{x}{b}\right)^k, \tag{13.37}$$

a and b are nonzero real or complex numbers

$$y_n(x) = y_n(x, 2, 2) \tag{13.38}$$

13.7 Ordinary Bessel Functions $J_n(x)$

Remark 13.6 *For this section, the subscript of the Bessel Function can be an arbitrary real or complex number. We will specify any restrictions on the subscripts as necessary. We evaluate any noninteger factorials by the Γ function.*

13.7.1 Generating Functions

$$e^{\frac{x}{2}(t - \frac{1}{t})} = \sum_{n=-\infty}^{\infty} J_n(x) t^n \tag{13.39}$$

$$J_0(\sqrt{x^2 - 2xt}) = \sum_{n=0}^{\infty} J_n(x) \frac{t^n}{n!} \tag{13.40}$$

13.7.2 Explicit Formulas

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left(\frac{x}{2}\right)^{n+2k} \tag{13.41}$$

$$J_{-n}(x) = (-1)^n J_n(x),\ n \textit{ an integer} \tag{13.42}$$

$$J_n(x) = \frac{1}{2\pi} \int_{\alpha}^{2\pi+\alpha} \cos(nt - x\sin t)\, dt = \frac{1}{\pi} \int_0^{\pi} \cos(nt - x\sin t)\, dt,\ n \textit{ an integer} \tag{13.43}$$

$$J_n(z) = \frac{(\frac{z}{2})^n}{\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})} \int_0^{\pi} e^{iz\cos x} \sin^{2n} x\, dx = \frac{(\frac{z}{2})^n}{\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})} \int_{-1}^{1} e^{izt}(1-t^2)^{n-\frac{1}{2}}\, dt \tag{13.44}$$

13.7.3 Recurrence Formulas

$$2nJ_n(x) = xJ_{n-1}(x) + xJ_{n+1}(x) \tag{13.45}$$

$$xD_xJ_n(x) = nJ_n(x) - xJ_{n+1}(x) \tag{13.46}$$

$$xD_xJ_n(x) = -nJ_n(x) + xJ_{n-1}(x) \tag{13.47}$$

13.7.4 Operator Properties

$$D_x\left(x^n J_n(x)\right) = x^n J_{n-1}(x) \tag{13.48}$$

$$D_x\left(x^{-n} J_n(x)\right) = -x^{-n} J_{n+1}(x) \tag{13.49}$$

$$x^{-n-\alpha} J_{n+\alpha}(x) = (-1)^{\alpha} \left(\frac{1}{x} D_x\right)^{\alpha} \left(x^{-n} J_n(x)\right),\ \ \alpha \textit{ a nonnegative integer} \tag{13.50}$$

13.7.5 Orthogonality Properties

$$\int_0^{\infty} \frac{J_n(x)J_k(x)}{x}\, dx = \begin{cases} 0, & k \neq n,\ k+n > 0 \\ \frac{1}{2n}, & k = n,\ n > 0 \end{cases} \tag{13.51}$$

Let γ_i be the positive roots of $J_n(x) = 0$. Then,

$$\int_0^1 \sqrt{x} J_n(\gamma_i x) \sqrt{x} J_n(\gamma_j x)\, dx = \begin{cases} 0, & i \neq j \\ \frac{1}{2}\left(J_n'(\gamma_i)\right)^2, & i = j. \end{cases} \tag{13.52}$$

13.7.6 Convolution Properties

$$J_n(z)J_m(z) = \sum_{k=0}^{\infty}(-1)^k \binom{n+m+2k}{k} \frac{(\frac{z}{2})^{n+m+2k}}{(n+k)!(m+k)!} \tag{13.53}$$

$$J_n(z+x) = \sum_{k=-\infty}^{\infty} J_k(z)J_{n-k}(x) \tag{13.54}$$

Remark 13.7 *The following identity is from Harry Bateman. Please see Proc. of the London Math. Society, (2), III (1905), p. 120.*

$$\frac{J_{m+n}(z)}{m} = \int_0^z \frac{J_m(t)J_n(z-t)}{t}\,dt, \;\; \textit{for } Re(m) > 0 \textit{ and } Re(n) > -1 \tag{13.55}$$

13.7.7 Connections to Other Special Functions

Legendre Polynomials $P_n(x)$

$$e^{tx}J_0(t\sqrt{1-x^2}) = \sum_{n=0}^{\infty} P_n(x)\frac{t^n}{n!} \tag{13.56}$$

Laguerre Polynomials $L_n(x)$

$$e^t J_0(2\sqrt{xt}) = \sum_{n=0}^{\infty} L_n(x)\frac{t^n}{n!} \tag{13.57}$$