Initial Boundary Value Problem for Euler Equations with Rarefaction Wave Solution *†

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Abstract

We study the initial-boundary value problem for the general non-isentropic 3-D Euler equations with data which are incompatible in the classical sense, but are "rarefaction-compatible". We show that such data are also rarefaction-compatible of infinite order and the initial-boundary value problem has a piece-wise smooth solution containing a rarefaction wave.

1 Introduction

In the study of the initial-boundary value problems for hyperbolic systems, in particular, for Euler system of equations in gas-dynamics, the smoothness of both the initial and boundary data does not guarantee the existence of a classical solution. A necessary condition to the existence of a smooth solution is the compatibility of such data. In order the solution to be differentiable of higher order, the higher order compatibility of the data is required, see, e.g., [3, 16, 18]. Similarly, for certain free boundary value problems involving shock wave, rarefaction wave or contact discontinuity of Euler equations, the data are also required to be compatible, often of very high order, [1, 2, 4, 6, 10, 13, 14].

The compatibility is a set of conditions on the initial and boundary data at the points of intersection of the boundary with the initial manifold. They consist of the algebraic restrictions at the intersection on the values of data, together with their normal derivatives of high order (depending upon the order of compatibility). Usually, such conditions are complicated and very tedious to verify explicitly.

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In this paper, we study the initial-boundary value problem for the general 3-D Euler equations with data which are incompatible in the classical sense. The data may contain a jump discontinuity at the intersection of the initial and boundary manifolds. For such data, there could exist no classical solution. It was established in [12] that a piece-wise solution containing a shock wave exists if the data are "shock-compatible". In this paper, we are looking for a piece-wise smooth solution containing a rarefaction wave under the simple assumption on the data, see conditions (A1-A3) in Theorem 1.1. We will call such data which satisfy (A1-A3) as "rarefaction-compatible". Similar to the situation for free boundary value problems studied in [4, 11], even though such data are incompatible in the classical sense, the compatibility issue involving rarefaction wave becomes much simpler. It turns out that the conditions (A1-A3) would automatically imply the rarefaction compatibility of infinite order for smooth initial and boundary data, similar to the case of shock waves in [12]. Taking advantage of such fact, we are able to show the existence of the piece-wise smooth solution containing a rarefaction wave under the conditions (A1-A3).

As the most important example of quasi-linear hyperbolic system, the Euler equations for compressible non-viscous flow in 3-D space can be written as follows:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) + \partial_y(\rho v) + \partial_z(\rho w) = 0, \\ \partial_t(\rho u) + \partial_x(p + \rho u^2) + \partial_y(\rho uv) + \partial_z(\rho uw) = 0, \\ \partial_t(\rho v) + \partial_x(\rho uv) + \partial_y(p + \rho v^2) + \partial_z(\rho vw) = 0, \\ \partial_t(\rho w) + \partial_x(\rho uw) + \partial_y(\rho vw) + \partial_z(p + \rho w^2) = 0, \\ \partial_t(\rho E) + \partial_x(\rho Eu + pu) + \partial_y(\rho Ev + pv) + \partial_z(\rho Ew + pw) = 0, \end{cases} \tag{1.1}$$

where (ρ, p, e) are the density, pressure, and the internal energy of the fluid, (u, v, w) is the velocity in the (x, y, z) direction, and $E = e + \frac{1}{2}(u^2 + v^2 + w^2)$. For convenience, we will consider the gas to be polytropic, with $p = A(S)\rho^\gamma$ with $\gamma > 1$.

One of the simplest and natural initial-boundary value problems for the Euler system (1.1) describes the gas flow bounded by a solid wall $x = 0$ with given initial status (ρ, u, v, w, e):

$$\begin{cases} (\rho, u, v, w, e)(0, x, y, z) = (\rho_0, u_0, v_0, w_0, e_0)(x, y, z) \ \text{ in } \ x \geq 0, \\ u(t, 0, y, z) = 0, \ \text{ on } \ t \geq 0. \end{cases} \tag{1.2}$$

In order to have a smooth solution for the problem (1.1)(1.2), one will obviously need the initial data $(\rho_0, u_0, v_0, w_0, e_0)$ to be compatible, see e.g.,

[16]. The continuity of the solution requires the zero-order compatibility

$$u_0(0, y, z) = 0. \tag{1.3}$$

If one wants the solution belonging to C^k, the higher order compatible conditions are required, which consist of algebraic relations imposed upon u_0 and its derivatives $\partial_x^j u_0 (j \le k)$ at $x = 0$.

If (1.3) is not satisfied, i.e., if the data is not compatible in the classical sense, then one cannot expect to have a continuous solution. However, there could be other solutions which are only piece-wise smooth. In this paper, we will study the initial-boundary value problem for (1.1) with data which is not compatible in the classical sense, but admits a piece-wise solution containing a rarefaction wave.

Let (H_0, H_1, H_2, H_3) be vectors defined as follows:

$$H_0 = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho E \end{pmatrix}, \ H_1 = \begin{pmatrix} \rho u \\ p + \rho u^2 \\ \rho u v \\ \rho u w \\ (\rho E + p) u \end{pmatrix}, \ H_2 = \begin{pmatrix} \rho v \\ \rho u v \\ p + \rho v^2 \\ \rho v w \\ (\rho E + p) v \end{pmatrix}, \ H_3 = \begin{pmatrix} \rho w \\ \rho u w \\ \rho v w \\ p + \rho w^2 \\ (\rho E + p) w \end{pmatrix}.$$

Then system (1.1) can be written briefly as

$$\partial_t H_0 + \partial_x H_1 + \partial_y H_2 + \partial_z H_3 = 0.$$

Introducing the unknown vector of functions $U = (p, u, v, w, S)$ where S is the entropy of the flow, it is well-known (see e.g., [7, 17]) that for smooth solutions, the system (1.1) is equivalent to the following system

$$\begin{cases} \dfrac{\partial p}{\partial t} + (u, v, w) \cdot \nabla p + \rho c^2 \nabla \cdot (u, v, w) = 0, \\ \rho \dfrac{\partial u}{\partial t} + \rho (u, v, w) \cdot \nabla u + \dfrac{\partial p}{\partial x} = 0, \\ \rho \dfrac{\partial v}{\partial t} + \rho (u, v, w) \cdot \nabla v + \dfrac{\partial p}{\partial y} = 0, \\ \rho \dfrac{\partial w}{\partial t} + \rho (u, v, w) \cdot \nabla w + \dfrac{\partial p}{\partial z} = 0, \\ \dfrac{\partial S}{\partial t} + (u, v, w) \cdot \nabla S = 0, \end{cases} \tag{1.4}$$

with $c^2 = p'_\rho(\rho, S) > 0$.

System (1.4) can be further rewritten into the following symmetric form

$$\mathscr{L} U \equiv A_0 \partial_t U + A_1(U) \partial_x U + A_2(U) \partial_y U + A_3(U) \partial_z U = 0, \tag{1.5}$$

where

$$A_0 = \begin{bmatrix} \frac{1}{\rho c^2} & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & \rho & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} \frac{u}{\rho c^2} & 1 & 0 & 0 & 0 \\ 1 & \rho u & 0 & 0 & 0 \\ 0 & 0 & \rho u & 0 & 0 \\ 0 & 0 & 0 & \rho u & 0 \\ 0 & 0 & 0 & 0 & u \end{bmatrix},$$

$$A_2 = \begin{bmatrix} \frac{v}{\rho c^2} & 0 & 1 & 0 & 0 \\ 0 & \rho v & 0 & 0 & 0 \\ 1 & 0 & \rho v & 0 & 0 \\ 0 & 0 & 0 & \rho v & 0 \\ 0 & 0 & 0 & 0 & v \end{bmatrix}, \quad A_3 = \begin{bmatrix} \frac{w}{\rho c^2} & 0 & 0 & 1 & 0 \\ 0 & \rho w & 0 & 0 & 0 \\ 0 & 0 & \rho w & 0 & 0 \\ 1 & 0 & 0 & \rho w & 0 \\ 0 & 0 & 0 & 0 & w \end{bmatrix}.$$

The matrix $A_0^{-1}[A_1(U) + A_2(U)\xi + A_3(U)\eta]$ has two simple eigenvalues $\lambda_\pm$ and one triple eigenvalue λ_0:

$$\begin{aligned} \lambda_- &= u - v\xi - w\eta - c\sqrt{1+\xi^2+\eta^2}, \\ \lambda_0 &= u - v\xi - w\eta, \\ \lambda_+ &= u - v\xi - w\eta + c\sqrt{1+\xi^2+\eta^2}, \end{aligned} \tag{1.6}$$

with $\lambda_- < \lambda_0 < \lambda_+$.

We will consider a more general initial-boundary value problem, for which (1.2) is a special case. Let $x = b(t, y, z)$ be a smooth surface in (t, x, y, z) space with $b(0,0,0) = b_y(0,0,0) = b_z(0,0,0) = 0$. Denote $b_0(y,z) = b(0, y, z)$. For the Euler system (1.1), consider the initial-boundary value problem in the domain bounded by the moving solid boundary $x = b(y, z, t)$ and the initial plane $t = 0$:

$$\begin{cases} \partial_t H_0(U) + \partial_x H_1(U) + \partial_y H_2(U) + \partial_z H_3(U) = 0, \\ U(0, x, y, z) = U_0(x, y, z) \ \text{ in } \ x \geq b_0(y, z), \\ u - b_t - b_y v - b_z w = 0, \ \text{ on } \ x = b(t, y, z), \ t \geq 0. \end{cases} \tag{1.7}$$

Obviously, (1.1)(1.2) is the special case of (1.7) with $b = 0$.

The main result of this paper is the following

Theorem 1.1 *For the initial-boundary value problem (1.7), assuming that there exists a constant state (ρ_l, p_l) such that the following condition (A1-A3) is satisfied at the origin $(0,0,0)$ on $\Gamma = \{x = b_0(y,z)\}$:*

$$
\begin{aligned}
&(\mathrm{A1}) \quad |\mathrm{b_t}(0,0,0)| < \mathrm{c_0},\\
&(\mathrm{A2}) \quad 0 < \mathrm{u_0}(0,0,0) - \mathrm{b_t}(0,0,0),\\
&(\mathrm{A3}) \quad \mathrm{u_0}(0,0,0) - \mathrm{b_t}(0,0,0) < \tfrac{2}{\gamma-1}(\mathrm{c_0}+\mathrm{c_l}),
\end{aligned}
$$

where

$$
c_l^2 = p_\rho\big|_{(\rho_l,p_l)}, \quad c_0^2 = p_\rho\big|_{(\rho_0,p_0)\ \text{at}\ (0,0,0)},
$$

then the problem (1.7) admits a piece-wise smooth solution near the origin in the domain $x > b(t,y,z)$, $t > 0$, containing a rarefaction wave emanating from the intersection manifold Γ.

Remark 1.1

1. *The assumption (A1) in Theorem 1.1 is a necessary condition. Otherwise, the problem (1.7) is not well-posed. In particular, for the fixed boundary $x = 0$ in (1.2), the condition is trivially satisfied.*

2. *The assumption $0 < u_0(0,0,0) - b_t(0,0,0) < \frac{2}{\gamma-1}(c_0 + c_l)$ in (A2-A3) ensures the existence of a rarefaction wave without a vacuum state. Here, c_l is sound speed for the state $(\rho_l, p_l, b_t(0,0,0))$ which can be connected to the given state U_0 by a rarefaction wave, see [17]. This is indeed nothing else but the 0-order rarefaction compatibility condition. For the special case of fixed boundary $x = 0$ in (1.4), the condition becomes simply $0 < u_0 < \frac{2}{\gamma-1}(c_0 + c_l)$.*

3. *The case of $u_0(0,0,0) - b_t(0,0,0) = 0$ is the 0-order compatibility condition for the existence of a classical solution. And the case of $u_0(0,0,0) - b_t(0,0,0) < 0$ is studied in [12], where a solution containing a shock wave is obtained.*

In the following, Section 2 will be devoted to the formulation of the problem and the construction of an approximate solution of infinite order. The problem will be tranformed in Section 3 by introducing new coordinates in (1.7) to to expand the rarefaction wave and to flatten both the boundary $x = b(t,y,z)$ and the rarefaction wave surface $x = \psi(t,y,z)$. The linear stability of the transformed equivalent problem will be derived in Section 4 by combining the results from [1, 3, 16]. Then the existence of a piece-wise smooth solution containing a rarefaction will be established in Section 5 by iteration.

2 Rarefaction wave solution and its approximation

From the solid wall condition $u - b_t - b_y v - b_z w = 0$ on the moving boundary $x = b(t, y, z)$ in (1.7) and the condition $u_0(0,0,0) > b_t(0,0,0)$ in the assumption (A2) in Theorem 1.1, it is obvious that the data for the initial-boundary value problem (1.7) is not compatible in the classical sense and hence (1.7) admits no classical smooth solution. Therefore we have to look for a piecewise smooth solution which contains, in the specific case of (A2-A3), a right propagating rarefaction wave.

Specifically, a rarefaction wave solution for (1.7) is formulated [1] as a set of smooth functions (U, U_r, χ, U_*) near the origin $(0,0,0,0)$ such that

- (U, U_r, U_*) satisfies (1.7) separately in each of the corner domains $(\Omega, \Omega_r, \Omega_*)$ defined by (see Figure 2.1):

$$\begin{cases} \Omega = \{b(t,y,z) < x < \chi^-(t,y,z), t > 0\}, \\ \Omega_r = \{\chi^-(t,y,z) < x < \chi^+(t,y,z), t > 0\}, \\ \Omega_* = \{\chi^+(t,y,z) < x, t > 0\}; \end{cases} \tag{2.1}$$

  with $b(0,y,z) = b_0(y,z) = \chi^-(0,y,z) = \chi^+(0,y,z)$.

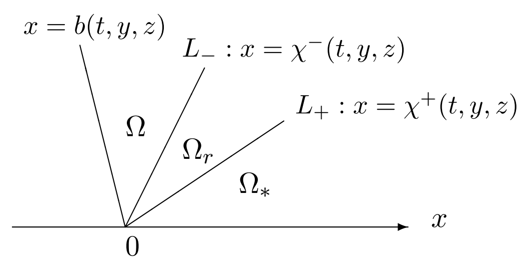

Figure 2.1: Shock wave solution for (1.7)

- $x = \chi(t,s,y,z), (1 \le s \le 2)$ is a parametrization of the domain Ω_r with

$$\chi(t,1,y,z) = \chi^-(t,y,z), \quad \chi(t,2,y,z) = \chi^+(t,y,z);$$

  In addition, $x = \chi(t,s,y,z), (1 \le s \le 2)$ is a family of characteristics issuing from Γ for each $s \in [1,2]$, such that

$$\det |A_1 - \chi_t - \chi_y A_2 - \chi_z A_3| = 0, \tag{2.2}$$

  or more specifically,

$$\chi_t = \lambda_+(U_r; \nabla\chi), \tag{2.3}$$

  where $\lambda_+(U;\phi)$ is the maximal eigenvalue in (1.6).

- Let the function $W(t,s,y,z)$ be defined by:

$$W(t,s,y,z)=U_r(t,\chi(t,s,y,z),y,z), \tag{2.4}$$

then $W(t,s,y,z)$ satisfies

$$\begin{aligned}\tilde{\mathscr{L}}W \equiv \chi_s(\frac{\partial W}{\partial t}+A_2\frac{\partial W}{\partial y}+A_3\frac{\partial W}{\partial z})\\ +(A_1-\chi_t-\chi_y A_2-\chi_z A_3)\frac{\partial W}{\partial s}=0.\end{aligned} \tag{2.5}$$

Since the surface $x=\chi^+(t,y,z)$ is characteristic for (1.7), the function U_* is uniquely determined in Ω_* by the initial data $U_0(x,y,z)$. To find the rarefaction wave solution, one needs only to determine the functions (U,W,χ).

In summary, the rarefaction wave solution is the set of functions $U(t,x,y,z), W(t,s,y,z), \chi(t,s,y,z)$ satisfying, in addition to (2.3),

$$\begin{cases}\mathscr{L}U=0 \quad \text{in } \Omega,\\ \tilde{\mathscr{L}}W=0 \quad \text{in } 1<s<2.\end{cases} \tag{2.6}$$

$$b_t-u-vb_y-wb_z=0, \text{ at } x=b(t,y,z); \tag{2.7}$$

$$\begin{cases}\chi(t,1,y,z)=\chi^-(t,y,z),\\ \chi(t,2,y,z)=\chi^+(t,y,z),\\ W(t,1,y,z)=U(t,\chi(t,1,y,z),y,z),\\ W(t,2,y,z)=U_*(t,\chi(t,2,y,z),y,z).\end{cases} \tag{2.8}$$

Finally, we have the initial conditions at Γ:

$$b_0(y,z)=\chi(0,s,y,z)=0, \tag{2.9}$$

with the assumption as in [1]

$$\chi_s=\gamma(t,s,y,z)t \quad \text{with } \gamma(t,s,y,z)\geq\delta>0. \tag{2.10}$$

The set of functions (U^a,χ^a,W^a) is called an approximate solution of order k if near $t=0$,

$$\begin{cases}\mathscr{L}U^a=O(t^k), \text{ in } \Omega\\ \tilde{\mathscr{L}}W^a=O(t^k) \text{ in } 1<s<2\\ \chi^a_t-\lambda_+(U^a_r;\nabla\chi^a)=O(t^k), \text{ in } \Omega_r,\\ b_t-u^a-v^ab_y-w^ab_z=O(t^k), \text{ on } x=b(t,y,z),\ t\geq 0.\end{cases} \tag{2.11}$$

Obviously, the existence of the k-th order approximate solution is equivalent to the fact that all the derivatives at $t = 0$ up to the order of k for (U^a, χ^a, W^a) can be uniquely determined by the equations in (2.11) along the initial sub-surface $x = b(0, y, z)$. For this, we have the following

Theorem 2.1 Under the assumptions (A1)-(A3) in Theorem 1.1 and let $b(t, y, z), U_0(x, y, z) \in C^\infty$, then for the initial-boundary value problem (1.7) with rarefaction wave configurations (2.6)-(2.9), all the derivatives of (U, χ, W) at $t = 0$ can be uniquely determined from (2.11) at the intersection $\Gamma : x = b(0, y, z)$, and consequently, there exists an infinite order approximate solution (U^a, χ^a, W^a).

To prove Theorem 2.1, we need to show that $\forall k \geq 0$, all the derivatives up to the order k of (U, χ_t, W) can be uniquely determined at $x = b(0, y, z)$. We prove this inductively.

The 0-order compatibility contains no derivative of (U, W). The existence of 1-D plane rarefaction wave (U, χ_t, W) follows directly from the conditions (A2-A3), see e.g., [17].

For **the first order compatibility**, we need to show that the first order derivatives of (U, χ_t, W) can be uniquely determined at Γ.

Following the approach in [1], see also [11], let $H(v, \eta)$ be the matrix satisfying

$$H^{-1}(A_1 - \chi_y A_2 - \chi_z A_3)H = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^b \end{pmatrix} \ \left(\triangleq d\right), \tag{2.12}$$

where the superscript b denotes the last four rows.

From (2.5) we have

$$\begin{aligned} & H^{-1}(W_t + A_2 W_y + A_3 W_z) \\ & \quad = -H^{-1}(A_1 - \chi_t I - A_2 \chi_y - A_3 \chi_z) W_s / \chi_s \\ & \quad = \begin{pmatrix} \chi_t - \lambda & 0 \\ * & * \end{pmatrix} W_s / \chi_s = \begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix} W_s / \chi_s. \end{aligned} \tag{2.13}$$

Then the first row becomes

$$(H^{-1}(W_t + A_2 W_y + A_3 W_z))^1 = 0. \tag{2.14}$$

Multiplying (2.13) by χ_s and then differentiating with respect to t, we have

$$\begin{aligned} & \chi_{ts} H^{-1}(W_t + A_2 W_y + A_3 W_z) + \chi_s (H^{-1}(W_t + A_2 W_y + A_3 W_z))_t \\ & \quad + (d - \chi_t)_t H^{-1} W_s + (d - \chi_t)(H^{-1} W_s)_t = 0. \end{aligned} \tag{2.15}$$

Since $\chi_s = 0$ at $t = 0$, consequently

$$\chi_{ts}\left(H^{-1}(W_t + A_2W_y + A_3W_z)\right)^b \\ +(\lambda^b - \chi_t)_t\left(H^{-1}W_s\right)^b + (\lambda^b - \chi_t)\left(H^{-1}W_s\right)^b_t = 0,$$

or

$$\chi_{ts}\left(H^{-1}(W_t + A_2W_y + A_3W_z)\right)^b \\ +(\lambda^b - \chi_t)_t((H^{-1}W_t)^b)_s + (H^{-1}W_t)^b \cdot * = *, \tag{2.16}$$

here $*$ stands again for the known terms. Therefore, the value of $(H^{-1}W_t)^b$ at any $s \in [1,2]$ can be uniquely determined by its value at $s = 2$. On the other hand, the value $(H^{-1}W_t)^1$ is determined from (2.14). Hence the value of all the components of $H^{-1}W_t$ are uniquely determined, and so are all the components of W_t.

Once W_t is known, we can obtain χ_{tt} by differentiating (2.3) with respect to t:

$$\chi_{tt} = \lambda_W W_t - \lambda_\eta \chi_{yt} - \lambda_\zeta \chi_{zt}. \tag{2.17}$$

Since the tangential derivatives of U_r and U coincide on $x = \chi(t,1,y,z) \equiv \chi_1(t,y,z)$, the value of the tangential derivative $D_rU \equiv (\partial_t + \chi_{1t}\partial_x)U$ is also known. Therefore

$$D_r\rho = *, \quad D_ru = *, \quad D_rp = *, \quad D_rv = *, \quad D_rw = *, \tag{2.18}$$

with

$$D_r = \partial_t + (u+c)\partial_x = D_c + c\partial_x,$$

since at the origin $\chi_1(0,0,0) = u + c$.

Because $\chi_{1t} = \lambda_+$ is an eigenvalue for the system (1.5), only two of the first three relations in (2.18) are independent. Hence (2.18) consists of only four independent relations for (ρ, u, v, w, p) which we denote as

$$D_r\rho = *, \quad D_rp = *, \quad D_rv = *, \quad D_rw = *. \tag{2.19}$$

At the origin (0,0,0,0), the interior equations in (1.1) become

$$\begin{cases} D_u\rho + \rho\partial_x u = *, \\ D_u u + \frac{1}{\rho}\partial_x p = *, \\ D_u v = *, \\ D_u w = *, \\ D_u p + \rho c^2 \partial_x u = *. \end{cases} \quad \text{in } \Omega \tag{2.20}$$

From the boundary condition in (1.7) on $x = b$ we have

$$D_u u = *, \tag{2.21}$$

The linear system (2.17), (2.19)-(2.21) consists of eleven equations for the eleven variables (χ_{tt}, U_t, U_x), where $U = (\rho, u, v, w, p)$. They can be simplified as follows.

Since vectors (D_r, D_u) span (∂_t, ∂_x), hence (v_t, v_x, w_t, w_x) can be eliminated from (2.19) and (2.20). Also from (2.17), χ_{1tt} can be eliminated.

From (2.21), $D_u u$ is known, and hence $\partial_x p$ is given by the second equation in (2.20). Since $D_r p$ is given from (2.19), so (p_t, p_x) can be eliminated.

Then $\partial_x u$ is known from the last equation in (2.20). Combining this with (2.21), (u_t, u_x) are uniquely determined. And finally, (ρ_t, ρ_x) can be uniquely determined from (2.19) and (2.20). This finishes the proof of the first order compatibility.

For **the k-th order compatibility**, we apply ∂_t^{k-1} to (2.17), apply the tangential derivatives D_r^{k-1} to (2.19), and apply the tangential derivatives D_u^{k-1} to (2.20) and (2.21). Evaluating the resulting equations at the origin yields 11 linear equations

$$\partial_t^{k+1}\chi = \partial_t^{k-1}(\lambda_W W_t - \lambda_\eta \chi_{yt}). \tag{2.22}$$

$$D_r^k \rho = *, \quad D_r^k p = *, \quad D_r^k v = *, \quad D_r^k w = *. \tag{2.23}$$

$$\begin{cases} D_u^k \rho + \rho D_u^{k-1}\partial_x u = *, \\ D_u^k u + \frac{1}{\rho} D_u^{k-1}\partial_x p = *, \\ D_u^k v = *, \\ D_u^k w = *, \\ D_u^k p + \rho c^2 D_u^{k-1}\partial_x u = *. \end{cases} \tag{2.24}$$

$$D_u^k u = *. \tag{2.25}$$

For (2.22)-(2.25), there are 11 independent variables

$$\partial_t^{k+1}\chi, \;\; D_u^k U, \;\; D_u^{k-1}\partial_x U.$$

By the same argument as in the first order compatibility, the 5 variables

$$\partial_t^{k+1}\chi, D_u^k v, D_u^{k-1}\partial_x v, D_u^k w, D_u^{k-1}\partial_x w$$

can be eliminated immediately from (2.22)-(2.24). Straightforward computations further eliminate the two variables $D_u^k \rho$ and $D_u^{k-1}\partial_x \rho$ from (2.23) and (2.24).

$D_u^k u$ is given from (2.25), and $D_u^{k-1}\partial_x u$ is known from (2.24). Hence $(D_u^k p, D_u^{k-1}\partial_x p)$ are also uniquely determined.

Once $(D_u^k U, D_u^{k-1}\partial_x U)$ are given, by the interior equation (2.20) and induction, we can determine all the derivatives $(D_c^{k-j}\partial_x^j U, D_c^{k-j}\partial_x^j U)$ for $j = 2, 3, \ldots, k$.

This concludes the proof of the k-th order compatibility and therefore Theorem 2.1.

3 Transformation and Reformulation

To establish the existence of the piece-wise smooth solution containing a rarefaction wave in Theorem 1.1, we first perform a singular coordinates transformation to reformulate the problem as in [1]. The purpose of the transformation is to change the angular domains Ω, Ω_r into standard cylindrical domains with fixed boundary, see also [1, 2, 4, 10, 12].

Denote

$$\tilde{\Omega}_j = \{(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z}) : \tilde{t} > 0,\ j < \tilde{x} < j+1\} \ \ (j = 0, 1). \tag{3.1}$$

Let $\phi^{(j)}(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z})$ be defined on $\tilde{\Omega}_j$ as

$$\begin{aligned} &\phi^{(0)}(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z}) = (1-\tilde{x})b(t, y, z) + \tilde{x}\chi^-(\tilde{t}, \tilde{y}, \tilde{z}), \\ &\phi^{(1)}(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z}) = \chi(\tilde{t}, 2-\tilde{x}, \tilde{y}, \tilde{z}). \end{aligned} \tag{3.2}$$

Then we have

$$\begin{aligned} &\phi^{(0)}(t, 0, y, z) = b(t, y, z), \\ &\phi^{(0)}(t, 1, y, z) = \phi^{(1)}(t, 1, y, z) = \chi^-(t, y, z), \\ &\phi^{(1)}(t, 2, y, z) = \chi^+(t, y, z). \end{aligned} \tag{3.3}$$

For $\tilde{t} > 0$, the transformations

$$x = \phi^{(j)}(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z}),\ y = \tilde{y},\ z = \tilde{z},\ t = \tilde{t} \ \ (j = 0, 1, 2) \tag{3.4}$$

are bijections from $\tilde{\Omega}_0$ to Ω, and from $\tilde{\Omega}_1$ to Ω_r. See Fig. 3.1.

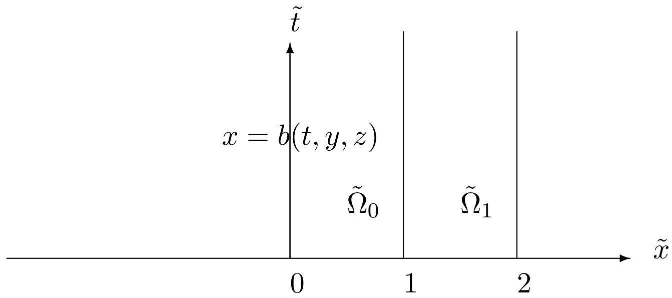

Figure 3.1: rarefaction wave configuration on $(\tilde t, \tilde x)$ plane

With the transformation (3.4), the system (1.5) of interior differential equations becomes in the new coordinates $(\tilde t, \tilde x, \tilde y, \tilde z)$

$$A_0(U^{(j)})\partial_{\tilde t}U^{(j)}+\tilde A_1(U^{(j)})\partial_{\tilde x}U^{(j)}+A_2(U^{(j)})\partial_{\tilde y}U^{(j)}+A_3(U^{(j)})\partial_{\tilde z}U^{(j)}=0, \tag{3.5}$$

with

$$\tilde A_1(U^{(j)}) = \frac{1}{\partial_{\tilde x}\phi}\left[A_1(U^{(j)}) - A_0(U^{(j)})\phi_{\tilde t} - A_2(U^{(j)})\phi_{\tilde y} - A_3(U^{(j)})\phi_{\tilde z}\right].$$

Because $\partial_{\tilde x}\phi = O(\tilde t)$, the system (3.5) is singular at $\tilde t = 0$ with order $\tilde t$. To formally remove this singularity, let (see also [2, 4, 12])

$$\tilde t = \tau, \quad \text{with} \quad \partial_{\tilde t} = e^{-\tau}\partial_\tau. \tag{3.6}$$

The transform (3.6) changes the domain $\tilde\Omega_j$ $(j = 0, 1)$ into ω_j:

$$\omega_j = \{(\tau, \tilde x, \tilde y, \tilde z) : j < \tilde x < j + 1,\ \tau > -\infty,\ (\tilde y, \tilde z) \in \mathbb{R}^2\}.$$

In the coordinates $(\tau, \tilde x, \tilde y, \tilde z)$, the system (3.5) becomes

$$\begin{aligned}\mathscr{L}^{(j)}(U^{(j)}, \phi^{(j)}) \equiv \partial_\tau U^{(j)} + \tilde{\tilde A}_1(U^{(j)})\partial_{\tilde x}U^{(j)}\\ +e^\tau A_2(U^{(j)})\partial_{\tilde y}U^{(j)} + e^\tau A_3(U^{(j)})\partial_{\tilde z}U^{(j)} = 0,\end{aligned} \tag{3.7}$$

with

$$\tilde{\tilde A}_1(U^{(j)}) = \frac{e^\tau}{\xi_{\tilde x}}(A_1(U^{(j)})-e^{-\tau}\phi_\tau A_0(U^{(j)})-\phi_{\tilde y}A_2(U^{(j)}))-\phi_{\tilde z}A_3(U^{(j)})). \tag{3.8}$$

We notice that with the coordinates transform (3.6), the $\tilde{t}^\eta$-weighted integration in the domain $\tilde{\Omega}_j$ becomes the hyperbolic $(\eta+1)$-weighted integration in ω_j:

$$\int_{\tilde{\Omega}_j} \tilde{t}^\eta |U^{(j)}(\tilde{t},\tilde{x},\tilde{y},\tilde{z})|^2 d\tilde{t}d\tilde{x}d\tilde{y}d\tilde{z} = \int_{\omega_j} e^{(\eta+1)\tau}|U^{(j)}(\tau,\tilde{x},\tilde{y},\tilde{z})|^2 d\tau d\tilde{x}d\tilde{y}d\tilde{z}.$$

To simplify the notation, we will drop the tilde in the new coordinates in the following and replace τ by t. Therefore, the initial-boundary value problem containing a rarefaction wave can be formulated as the following boundary value problem in the domains ω_j $(j = 0, 1)$ for the unknown functions $(U^{(j)}(t,x,y,z), \phi^{(j)}(t,x,y,z))$ $(j = 0,1)$ satisfying:

- Interior equations:

$$\mathscr{L}^{(j)}(U^{(j)}, \phi^{(j)}) = 0 \qquad \text{in } \omega_j \ (j = 0, 1); \tag{3.9}$$

- On the solid boundary $x = 0$:

$$\mathscr{B}^{(0)}(U^{(0)}) \equiv e^{-t}b_t - u^{(0)} + v^{(0)}b_{ey} + w^{(0)}b_z; \tag{3.10}$$

- Continuous boundary conditions for rarefaction waves at $x = 1, 2$:

$$\begin{aligned} &U^{(0)}(t,1,y,z) = U^{(1)}(t,1,y,z),\\ &U^{(1)}(t,2,y,z) = U^{(r)}(t,2,y,z),\\ &\phi^{(0)}(t,1,y,z) = \phi^{(1)}(t,1,y,z); \end{aligned} \tag{3.11}$$

  with $U^{(r)}(t,2,y,z)$ given;

- Rarefaction wave structure:

$$\begin{aligned} &\phi_t^{(1)}(t,x,y,z) = \lambda_+(U^{(1)}; -\phi_y^{(1)}, -\phi_z^{(1)}),\\ &\partial_x\phi^{(1)}(t,x,y,z) = \gamma(t,x,y,z)e^t \ \text{ with } \ \gamma \geq \delta > 0; \end{aligned} \tag{3.12}$$

- "Initial" condition:

$$\begin{aligned} &(U^{(j)} - U^{a(j)}, \phi^{(j)} - \phi^{a(j)}) = O(e^{\eta t}) \ \text{ at } \ t = -\infty,\\ &\phi^{(j)}(-\infty,x,y,z) = b_0(y,z) \ (j = 0,1). \end{aligned} \tag{3.13}$$

Therefore, in order to prove Theorem 1.1, we need only to prove the following

Theorem 3.1 *There exists a C^∞ solution $(U^{(j)}, \phi^{(j)})$ to the boundary value problem (3.9)-(3.13) near $t = -\infty$.*

Theorem 3.1 will be proved by linear iteration of (3.9)-(3.13) near the approximate solution $(U^{a(j)}, \phi^{a(j)})$.

4 Linearization and Energy Estimate

Because of the loss the regularity in the estimates for the linearized rarefaction wave, the Nash-Moser iteration will be used to prove the existence of the rarefaction wave solution. The proof proceeds here along the same approach as in [1, 4].

Let $U = (U^{(0)}, U^{(1)})$ and $\phi = (\phi^{(0)}, \phi^{(1)})$. We will construct a sequence of smooth approximate solutions $(U^a + U_n, \phi^a + \phi_n)$, $(n = 0, 1, 2, \dots)$ near the C^∞ approximate solution (U^a, ϕ^a) established in Theorem 2.1, and show its convergence in an appropriate space to the required solution of the problem (3.9)-(3.13).

Let $\{\theta_n\}$ be the sequence defined by

$$\theta_0 \gg 1, \ \ \theta_n = \sqrt{\theta_0^2 + n}, \ \ \Delta_n = \theta_{n+1} - \theta_n. \tag{4.1}$$

The sequence $\{\Delta_n\}$ is decreasing with

$$\frac{1}{3\theta_n} \leq \Delta_n = \sqrt{\theta_n^2 + 1} - \theta_n \leq \frac{1}{2\theta_n}. \tag{4.2}$$

Let $(U_0, \phi_0) = (0, 0)$ and

$$U_{n+1} = U_n + \Delta_n \dot{U}_n, \ \phi_{n+1} = \phi_n + \Delta_n \dot{\phi}_n \qquad (n = 0, 1, 2, \dots), \tag{4.3}$$

where $\dot{U}_n$ and $\dot{\phi}_n$ will be the solution of an appropriate boundary value problem for a linear hyperbolic system specified as follows.

4.1 Interior Equation

Denote $\mathscr{L}(U, \phi) = (\mathscr{L}^{(0)}(U^{(0)}, \phi^{(0)}), \mathscr{L}^{(1)}(U^{(1)}, \phi^{(1)})))$.

For the linearization of $\mathscr{L}(U, \phi)$ at (U, ϕ), introduce a new variable $\dot{V}$ (see [1]):

$$\dot{V} = \dot{U} - \frac{U_x}{\phi_x}\dot{\phi}. \tag{4.4}$$

The linearized operator $\ell(U, \phi)(\dot{U}, \dot{\phi})$ of $\mathscr{L}(U, \phi)$ at (U, ϕ) can be written as

$$\ell(U, \phi)(\dot{U}, \dot{\phi}) = \mathscr{L}'(U, \phi)\dot{V} + B(U, \phi)\dot{V} + \frac{\dot{\phi}}{\phi_x}\partial_x \mathscr{L}(U, \phi), \tag{4.5}$$

where the operators $\mathscr{L}'(U,\phi)$ and $B(U,\phi)$ are defined as

$$\begin{aligned}\mathscr{L}'(U,\phi) \equiv \partial_t + e^t A_2(U)\partial_y + e^t A_3(U)\partial_z \\ \frac{e^t}{\phi_x}(A_1(U) - e^{-t}\phi_t - \phi_y A_2(U) - \phi_z A_3(U))\partial_x.\end{aligned} \tag{4.6}$$

$$B(U,\phi) \equiv \frac{e^t}{\phi_x} B_1(U,\phi) + e^t B_2(U,\phi) + e^t B_3(U,\phi) \tag{4.7}$$

with

$$\begin{aligned}B_1(U,\phi) = (A_1'(U) - \phi_y A_2'(U) - \phi_z A_3'(U))U_x \\ B_2(U,\phi) = A_2'(U)U_y, \ \ B_3(U,\phi) = A_3'(U)U_z.\end{aligned}$$

For simplicity, let

$$\begin{aligned}&\mathscr{L}_a(U_n,\phi_n) \equiv \mathscr{L}(U^a + U_n, \phi^a + \phi_n)' \\ &\ell_a(U_n,\phi_n) \equiv \ell(U^a + U_n, \phi^a + \phi_n), \\ &B_a(U_n,\phi_n), \ \ \mathscr{B}_a(U_n,\phi_n), \cdots .\end{aligned}$$

Then we have

$$\mathscr{L}(U_{n+1},\phi_{n+1}) - \mathscr{L}(U_n,\phi_n) = \ell_a(U_n,\phi_n)(\dot{U}_n,\dot{\phi}_n)\Delta_n + \Delta_n e_{n1}', \tag{4.8}$$

where e_{n1}' is the standard quadratic error in the Newtonian iteration.

For the linearized rarefaction wave, there exists only the tame estimate, we need to apply a regularizing operator $\mathscr{S}_n$ to (U_n,ϕ_n) before the next step in the Nash-Moser iteration. Hence

$$\mathscr{L}_a(U_{n+1},\phi_{n+1}) - \mathscr{L}_a(U_n,\phi_n) = \ell_a(\mathscr{S}_n U_n, \mathscr{S}_n \phi_n)(\dot{U}_n,\dot{\phi}_n)\Delta_n + \Delta_n e_{n1} \tag{4.9}$$

where $e_{n1} \equiv e_{n1}' + e_{n1}''$ with e_{n1}'' being the smoothing error.

In the new variable $\dot{V}_n$ in (4.4) and introducing the operator $\tilde{\ell}_a(U,\phi)\dot{V} = \mathscr{L}_a'(U,\phi)\dot{V} + B(U,\phi)\dot{V}$, we have

$$\begin{aligned}&\mathscr{L}_a(U_{n+1},\phi_{n+1}) - \mathscr{L}_a(U_n,\phi_n) = \\ &\quad = \Delta_n \tilde{\ell}_a(\mathscr{S}_n U_n, \mathscr{S}_n \phi_n)\dot{V}_n + \Delta_n(e_{n1} + e_{n2}),\end{aligned} \tag{4.10}$$

where $e_{n2} \equiv \dfrac{\dot{\phi}_n}{\phi_x^a + \bar{\phi}_{nx}} \partial_x[\mathscr{L}_a(\mathscr{S}_n U_n, \mathscr{S}_n \phi_n)]$.

In order to apply the estimates established for the linearized system in [1] and [5] where the boundaries $x = 0$ as well as $x = \alpha$ with $1 \le \alpha \le 2$ are required to be uniformly characteristic, the values $(\mathscr{S}_n U_n, \mathscr{S}_n \phi_n)$ are further adjusted and the error term e_{n3} is introduced. We denote the linear operator obtained from $\tilde{\ell}_a(U_n, \phi_n)$ through this adjustment as $\mathbb{L}_a$:

$$\mathbb{L}_a(U_n, \phi_n) \equiv \tilde{\ell}_a(\bar{U}_n, \bar{\phi}_n). \tag{4.11}$$

Then we obtain

$$\mathscr{L}_a(U_{n+1}, \phi_{n+1}) - \mathscr{L}_a(U_n, \phi_n) = \Delta_n \mathbb{L}_a(U_n, \phi_n)\dot{V}_n + \Delta_n e_n, \tag{4.12}$$

where

$$e_n \equiv e_{n1} + e_{n2} + e_{n3}.$$

To obtain the combined estimate including both the rarefaction wave in [1] and the usual solid wall boundary value problem, the operator (4.11) needs to be constructed in the rarefaction wave domain and near the solid boundary separately, and then be patched together by the usual localization technique, see also [4].

Let $\dot{F}_n$ be chosen such that

$$\sum_{k=0}^{n} \Delta_k \dot{F}_k = -\mathscr{S}_n \mathscr{F}_T \mathscr{L}(U^a, \phi^a) - \mathscr{S}_n \sum_{k=0}^{n-1} \Delta_k e_k. \tag{4.13}$$

Here the operator $\mathscr{F}_T$ is the extension operator from $(-\infty, T)$ to $(-\infty, \infty)$ as in [1]. The final iteration scheme for $(\dot{V}_n, \dot{\phi}_n)$ in the interior domain is the following hyperbolic system

$$\mathbb{L}_a(U_n, \phi_n)\dot{V}_n = \dot{F}_n. \tag{4.14}$$

4.2 Boundary Conditions

At the solid wall boundary $x = 0$, $\phi^{(0)}(t, 0, y, z) = b(t, y, z)$ by (3.3), hence $\dot{\phi}^{(0)} = 0$. The boundary condition at $x = 0$ for $U^{(0)}$ is linear and is satisfied accurately by the approximate solution (U^a, ϕ^a). Therefore

$$\mathscr{B}_a^{(0)}(U_{n+1}^{(0)}, \phi_{n+1}^{(0)}) - \mathscr{B}_a^{(0)}(U_n^{(0)}, \phi_n^{(0)}) = \mathscr{B}_a^{(0)} \dot{V}_n \Delta_n. \tag{4.15}$$

It can be formally rewritten as

$$\mathscr{B}_a^{(0)} \dot{V}_n \equiv \mathbb{B}_a^{(0)}(U_n^{(0)}, \phi_n^{(0)})(\dot{V}_n^{(0)}, \dot{\phi}_n^{(0)}) = \dot{G}_n^{(0)} = 0. \tag{4.16}$$

At the rarefaction wave boundaries, the solution is continuous: $U^{(0)} = U^{(1)}$ on $x = 1$ and $U^{(1)} = U^{(2)}$ on $x = 2$ (here $U^{(2)}$ and $V^{(2)}, \phi^{(2)}$ are already uniquely determined from the initial data).

The boundary iteration scheme at $x = 1$ becomes :

$$U_{n+1}^{(0)} - U_{n+1}^{(1)} = U_n^{(0)} - U_n^{(1)} + \Delta \dot{G}_n^{(1)} + \Delta d_n^{(1)}, \tag{4.17}$$

where $d_n^{(1)}$ is the error and $\dot{G}_n^{(1)}$ is chosen to secure the convergence of the iteration as in [1]. The iteration scheme at $x = 2$ is constructed in exactly the same way.

Since the boundary $L^{\pm}$ is characteristic and the matrix $A_1(U^{(1)}) - \partial_t \phi^{(1)} - A_2(U^{(1)})\partial_y \phi^{(1)} - A_3(U^{(1)})\partial_z \phi^{(1)}$ is degenerate, we need to further adjust the approximate solution U_n, ϕ_n to $\bar{U}_n, \bar{\phi}_n$, such that the adjusted boundary matrix

$$A_1(\bar{U}_n^{(1)}) - \partial_t \bar{\phi}_n^{(1)} - A_2(\bar{U}_n^{(1)})\partial_y \bar{\phi}_n^{(1)} - A_3(\bar{U}_n^{(1)})\partial_z \bar{\phi}_n^{(1)}. \tag{4.18}$$

is uniformly degenerate with rank 2. Its eigenvectors form an orthogonal basis in $\mathbb{R}^3$. Let $\Pi_n^{(1)} \equiv \Pi(\bar{U}_n^{(1)}, \bar{\phi}_n^{(1)})$ be the matrix formed by these three unit column eigenvectors, we can perform an orthogonal transformation such that the first column vector corresponds to the right-propagation rarefaction wave. This vector spans an one-dimensional subspace in which the matrix (4.18) is degenerate, and non-degenerate in its orthogonal complement.

Let P_n and $I - P_n$ be the projectors corresponding to the non-degenerate and degenerate subspaces, we can write the boundary conditions on $x = 1$ as

$$P_n^{(0)} \dot{V}_n^{(0)} - P_n^{(1)} \dot{V}_n^{(1)} = P_n^{(0)} \dot{G}_n^{(0)} \qquad \text{on } \ x = 1. \tag{4.19}$$

$$\begin{aligned} &(1 - P_n^{(0)})\dot{V}_n^{(0)} - (1 - P_n^{(1)})\dot{V}_n^{(1)} \\ &\quad = Z_n^{(0)} \dot{\phi}_n^{(0)} + (1 - P_n^{(0)})\dot{G}_n^{(0)} \end{aligned} \qquad \text{on } \ x = 1. \tag{4.20}$$

The relation (4.19) is the boundary condition coupled with the interior differential equations while (4.20) is used to determine $\dot{\phi}_n^{(0)}$.

The term $\dot{G}_n^{(j)}$ in (4.19)-(4.20) is the modified error as shown later in (4.23), and

$$Z_n^{(0)} \equiv (1 - P_n^{(1)}) \frac{U_x^{a(1)} + \bar{U}_{nx}^{(1)}}{\phi_x^{a(1)} + \bar{\phi}_{nx}^{(1)}} - (1 - P_n^{(0)}) \frac{U_x^{a(0)} + \bar{U}_{nx}^{(0)}}{\phi_x^{a(0)} + \bar{\phi}_{nx}^{(0)}}, \tag{4.21}$$

with (see [1], or [3, 10])

$$e^t Z_n^{(1)} \neq 0. \tag{4.22}$$

In summary, we will denote the boundary iteration scheme (4.19)-(4.20) on $x = 1$ as

$$\mathbb{B}_a^{(1)}(U_n^{(1)}, \phi_n^{(1)})(\dot{V}_n^{(1)}, \dot{\phi}_n^{(1)}) = \dot{G}_n^{(1)} \tag{4.23}$$

with $\dot{G}_n^{(0)}$ chosen according to the following

$$\sum_{k=0}^{n} \dot{G}_k^{(1)}\Delta_k = -\mathscr{S}_n\mathscr{F}_T\mathscr{B}^{(1)}(U^a,\phi^a) - \mathscr{S}_n\sum_{k=0}^{n-1} d_k^{(1)}\Delta_k. \tag{4.24}$$

Similarly on $x=2$, we have

$$\mathbb{B}_a^{(1)}(U_n^{(1)},\phi_n^{(1)})(\dot{V}_n^{(1)},\dot{\phi}_n^{(1)}) = \dot{G}_n^{(2)} \tag{4.25}$$

4.3 Estimate for linearized problem

Let s be a non-negative integer, and $k=(k_0,k_1,k_2,k_3)$ be the multiple index with $|k|=k_0+k_1+k_2+k_3$. Let $\omega_j^T=\omega_j\bigcap\{t;t<T\}$ $(j=0,1)$ and let $H_\eta^s(\omega_j^T)$ be the η-weighted Sobolev space with the norm

$$\|U\|^2_{H^s_\eta(\omega^T)} = \sum_{0\le|k|+2m\le s}\int_{\omega^T}|\partial_t^{k_0}D_x^{k_1}\partial_y^{k_2}\partial_z^{k_3}\partial_x^m(e^{-\eta t}U(x,y,z,t))|^2dydzdxdt,$$

where η being sufficiently large, $D_x=x(x-1)(x-2)\partial_x$ is an operator tangential to the boundary $x=0,1,2$. The space $H_\eta^s(\omega_j^T)$ is the usual η-weighted Sobolev space H^s away from the boundary $x=0,1,2$. At the boundaries, the regularity in the x-derivatives is reduced, see [1, 8].

Let Γ_j^T $(j=0,1,2)$ be the boundary

$$\Gamma_j^T=\{(t,x,y,z);-\infty<t<T,x=j,(y,z)\in\mathbb{R}^2\}.$$

And the Sobolev space $H_\eta^s(\Gamma_j^T)$ on the boundary Γ_j^T is defined by

$$|U|^2_{H^s_\eta(\Gamma_j^T)} = \sum_{0\le|k|\le s}\int_{\omega_j^T}|\partial_t^{k_0}\partial_y^{k_2}(e^{-\eta t}U(y,z,t))_{x=j}|^2dydzdt.$$

For the Sobolev spaces $H_\eta^s(\omega_j^T)$, we have the imbedding and trace theorems (see [1])

$$H_\eta^s(\omega_j^T)\subset C^m, \qquad \text{for } s>2+2m.$$

$$s>1,\ u\in H_\eta^s(\omega_j^T) \implies u\mid_{x=j-1}\in H_\eta^{s-1}(\Gamma_{j-1}),\ u\mid_{x=j}\in H_\eta^{s-1}(\Gamma_j).$$

The linearized boundary value problem for the system (4.14) with boundary conditions (4.16), (4.19) and (4.20) at $x=0,1,2$ can be briefly written as follows

$$\begin{cases} \mathbb{L}_a(U_n,\phi_n)(\dot{V}_n,\dot{\phi}_n)=\dot{F}_n, \\ \mathbb{B}_a(U_n,\phi_n)(\dot{V}_n,\dot{\phi}_n)=\dot{G}_n, \end{cases} \tag{4.26}$$

where

$$\mathbb{B}_a \equiv (\mathbb{B}_a^{(0)}, \mathbb{B}_a^{(1)}, \mathbb{B}_a^{(2)}), \;\; U_n \equiv (U_n^{(0)}, U_n^{(1)}), \;\; \phi_n \equiv (\phi_n^{(0)}, \phi_n^{(1)}),$$
$$\dot{U}_n \equiv (\dot{U}_n^{(0)}, \dot{U}_n^{(1)}), \;\; \dot{V}_n \equiv (\dot{V}_n^{(0)}, \dot{V}_n^{(1)}), \;\; \dot{\phi}_n \equiv (\dot{\phi}_n^{(0)}, \dot{\phi}^{(1)}),$$

Combining the estimates for the solid boundary problem in [16] and for the linearized rarefaction wave in [1] by the usual localization technique, we have the following estimate for the linearized problem (4.26)

Theorem 4.1 *For the complete linearized solid wall - rarefaction wave problem (4.26), assume*

- *for a sufficiently small* $\gamma_0 > 0$,

$$\|U_n\|_{H^6_\eta(\omega^T)} + \|\phi_n\|_{H^6_\eta(\Gamma^T)} = \gamma \le \gamma_0; \tag{4.27}$$

- *Integer* $s_0 \ge 6$ *and even integer* $s \ge s_0$*;*
- $-T \gg 1$*;*
- $\dot{F}_n \in H^s_\eta(\omega^T)$ *and* $\dot{G}_n \in H^{s+1}_\eta(\Gamma^T)$.

Then the boundary value problem (4.26) has a unique solution $(\dot{V}_n, \dot{\phi}_n)$ *satisfying the following estimate*

$$\begin{aligned} \|\dot{V}_n\|_{H^s_\eta(\omega^T)} + \|\dot{\phi}_n\|_{H^{s-1}_\eta(\Gamma^T)} \le C_s \Big[& \|\dot{F}_n\|_{H^s_\eta(\omega^T)} + \|\dot{G}_n\|_{H^{s+1}_\eta(\Gamma^T)} \\ & +(\|\dot{F}_n\|_{H^{s_0}_\eta(\omega^T)} + \|\dot{G}_n\|_{H^{s_0}_\eta(\Gamma^T)})(1 + \|\mathrm{coeff}\|_{s+3})\Big]. \end{aligned} \tag{4.28}$$

5 Convergence of Nash-Moser iteration

It remains to prove the convergence of function sequence $(\dot{V}_n, \dot{\phi}_n)$ (or equivalently $(\dot{U}_n, \dot{\phi}_n)$ in (4.3)(4.4)) constructed by solving the linear boundary value problem (4.26) in Nash-Moser iteration. The procedure is standard but tedious. For completeness, we give a concise proof in the following. The interested reader can refer to [1, 4] for more details.

Let $(\mathcal{H}_n)$ be the following recurrence hypotheses:

$$\|(\dot{U}_k, \dot{\phi}_k)\|_{H^s_\eta(\omega^T)} + \|\dot{\phi}_k\|_{H^{s+1}_\eta(\Gamma^T)} \le \delta\theta_k^{s-\alpha-1}, \; 0 \le k \le n, \; s_0 \le s \le s_+, \tag{5.1}$$

$$\|\mathscr{L}_a(U_k, \phi_k)\|_{H^s_\eta(\omega^T)} \le \delta\theta_k^{s-\alpha}, \; 0 \le k \le n, \; s_0 \le s \le s_+ - 2, \tag{5.2}$$

$$\|\mathscr{B}_a(U_k, \phi_k)\|_{H^s_\eta(\Gamma^T)} \le \delta\theta_k^{s-\alpha}, \; 0 \le k \le n, \; s_0 \le s \le s_+ - 1. \tag{5.3}$$

Then we have the following:

Theorem 5.1 *There exist constants* $(\delta, \alpha, s_0, s_+)$*, satisfying*

$$\delta \ll 1; \quad s_0 = 6(> \frac{3+1}{2} + 2); \quad \alpha > s_0 + 6 = 12; \quad s_+ = 2\alpha - s_0 \geq \alpha + 6', \tag{5.4}$$

such that $(\mathcal{H}_n)$ *are true for all* $n \geq 0$.

Remark: Theorem 5.1 implies that (U_n, ϕ_n) converges in the space $H^s_\eta(\omega^T) \times H^s_\eta(\Gamma^T)$ with $s < \alpha$, since $\theta_n \sim \sqrt{n}$, and hence the existence of the solution $(U, \phi) \in H^{\alpha-1}_\eta(\omega^T) \times H^{\alpha-1}_\eta(\Gamma^T)$ for (3.9)-(3.13).

Also, since s_+ can be arbitrarily large by Theorem 2.1, the index α can be larger than any given integer k. This implies the existence of C^∞ solution.

Theorem 5.1 will be proved in two steps

- $(\mathcal{H}_{n-1}) \Rightarrow (\mathcal{H}_n)$;
- $(\mathcal{H}_0)$.

5.1 $(\mathcal{H}_{n-1}) \Longrightarrow (\mathcal{H}_n)$: Estimates for (U_n, ϕ_n) and $(\mathscr{S}_n U_n, \mathscr{S}_n \phi_n)$

By the definition (U_n, ϕ_n) in (4.3) and from the property of mollifier $\mathscr{S}_k$ [1], we have

$$\begin{aligned}
\|(U_n, \phi_n)\|_{H^s_\eta(\omega^T)} &\leq \sum_{k=0}^{n-1} \|(\dot{U}_k, \dot{\phi}_k)\|_{H^s_\eta(\omega^T)} \Delta_k \\
&\leq C\delta \sum_{k=1}^{n-1} \theta_k^{s-\alpha-1} \frac{1}{\theta_k} = C\delta \sum_{k=1}^{n-1} \theta_k^{s-\alpha-2},
\end{aligned}$$

so

$$\begin{cases}
\|(U_n, \phi_n)\|_{H^s_\eta(\omega^T)} \leq \delta \theta_n^{(s-\alpha)_+}, \ s_0 \leq s \leq s_+, \quad s \neq \alpha \\
\|(U_n, \phi_n)\|_{H^\alpha_\eta(\omega^T)} \leq \delta \log \theta_n;
\end{cases} \tag{5.5}$$

For the mollification $(\mathscr{S}_n U_n, \mathscr{S}_n \phi_n)$:

$$\begin{aligned}
&\|(\mathscr{S}_n U_n, \mathscr{S}_n \phi_n)\|_{H^s_\eta(\omega^T)} \leq C\delta \theta_n^{\epsilon + (s-\alpha)_+}, \ s \geq s_0, \quad (\epsilon = 0 \ \text{ if } \ s \neq \alpha); \\
&\|(U_n - \mathscr{S}_n U_n, \phi_n - \mathscr{S}_n \phi_n)\|_{H^s_\eta(\omega^T)} \leq C\delta \theta_n^{s-\alpha}, \ s_0 \leq s \leq s_+.
\end{aligned} \tag{5.6}$$

5.2 $(\mathcal{H}_{n-1}) \Longrightarrow (\mathcal{H}_n)$: Estimate for error (e_k, d_k) $(k \leq n-1)$

The error estimate for (e_k, d_k) for the rarefaction wave $1 \leq x \leq 2$ are already obtained in [1] in the form of equivalent t-weighted norms. We have

$$\|e_k\|_{H^s_\eta(\omega^T)} + \|d_k\|_{H^s_\eta(\Gamma^T)} \leq C\delta^2 \theta_k^{s+s_0+3-2\alpha} \tag{5.7}$$

with $s_0 \le s \le s_+ - 4$.

Near the solid wall boundary $x = 0$, the estimates for e_k is the same, and $d_k = 0$ since the boundary condition is linear.

5.3 $(\mathcal{H}_{n-1}) \Longrightarrow (\mathcal{H}_n)$: Estimate for $(\dot{F}_n, \dot{G}_n)$

From (4.13) and (4.24), we have

$$\Delta_n \dot{F}_n = -(\mathscr{S}_n - \mathscr{S}_{n-1})\left(\mathscr{F}_T \mathscr{L}(U^a, \phi^a) + \sum_{k=0}^{n-2} \Delta_k e_k\right) - \mathscr{S}_n \Delta_{n-1} e_{n-1}$$
$$\Delta_n \dot{G}_n = -(\mathscr{S}_n - \mathscr{S}_{n-1})\left(\mathscr{F}_T \mathscr{B}(U^a, \phi^a) + \sum_{k=0}^{n-2} \Delta_k d_k\right) - \mathscr{S}_n \Delta_{n-1} d_{n-1}.$$

Since $\Delta_{n-1}/\Delta_n \sim 1$, then from (5.7) we have for $s_0 \le s \le s_+ - 4$:

$$\|\frac{\Delta_{n-1}}{\Delta_n} \mathscr{S}_n e_{n-1}\|_{H^s_\eta(\omega^T)} \le C\delta^2 \theta_n^{s+s_0+3-2\alpha}, \tag{5.8}$$

For $s \ge s_+ - 4 \ge s_0$, we have

$$\begin{aligned} \|\mathscr{S}_n e_{n-1}\|_{H^s_\eta(\omega^T)} &\le C\|\mathscr{S}_n e_{n-1}\|_{H^{s_+-4}} \theta_n^{s-(s_+-4)} \\ &\le C\delta^2 \theta_n^{s_+-4+s_0+3-2\alpha} \theta_n^{s-(s_+-4)} \le C\delta^2 \theta_n^{s+s_0+3-2\alpha}. \end{aligned} \tag{5.9}$$

From (5.7)

$$\|\sum_{k=0}^{n-2} \Delta_k e_k\|_{H^{s_+-4}_\eta(\omega^T)} \le C\delta^2 \sum_{k=0}^{n-2} \Delta_k \theta_k^{(s_+-4)+s_0+3-2\alpha} \le C\delta^2 \theta_n^{s_++s_0-2\alpha},$$

and by the property of $\mathscr{S}_n$ [1], we have for all $s \ge s_0$,

$$\begin{aligned} &\frac{1}{\Delta_n} \|(\mathscr{S}_n - \mathscr{S}_{n-1}) \sum_{k=0}^{n-2} \Delta_k e_k\|_{H^s_\eta(\omega^T)} \\ &\quad \le C\theta_n^{s-(s_+-4)-1} \|\sum_{k=0}^{n-2} \Delta_k e_k\|_{s_+-4} \\ &\quad \le C\delta^2 \theta_n^{s-(s_+-4)-1} \theta_n^{s_++s_0-2\alpha} \le C\delta^2 \theta_n^{s+s_0+3-2\alpha}. \end{aligned} \tag{5.10}$$

On the other hand, we have

$$\begin{aligned} &\frac{1}{\Delta_n} \|(\mathscr{S}_n - \mathscr{S}_{n-1}) \mathscr{F}_T \mathscr{L}(U^a, \phi^a)\|_{H^s_\eta(\omega^T)} \\ &\quad \le C\theta_n^{s-\beta-1} \|\mathscr{L}(U^a, \phi^a)\|_{H^\beta_\eta(\omega^T)}. \end{aligned} \tag{5.11}$$

Taking $\beta = 2\alpha - s_0 - 4$ in (5.11) and noticing that (U^a, ϕ^a) is the C^∞ approximate solution, then we have, for $-T \gg 1$,

$$C\|\mathscr{L}(U^a, \phi^a)\|_{H^\beta_\eta(\omega^T)} \le \delta^2, \tag{5.12}$$

and

$$\frac{1}{\Delta_n}\|(\mathscr{S}_n - \mathscr{S}_{n-1})\mathscr{F}_T\mathscr{L}(U^a, \phi^a)\|_{H^s_\eta(\omega^T)} \le C\delta^2\theta_n^{s+s_0+3-2\alpha}. \tag{5.13}$$

Combining (5.8)-(5.13), we obtain, for all $s \ge s_0$,

$$\|\dot{F}_n\|_{H^s_\eta(\omega^T)} \le C\delta^2\theta_n^{s+s_0+3-2\alpha}. \tag{5.14}$$

Similarly, we have

$$\|\dot{G}_n\|_{H^s_\eta(\Gamma^T)} \le C\delta^2\theta_n^{s+s_0+3-2\alpha}. \tag{5.15}$$

5.4 $(\mathcal{H}_{n-1}) \Longrightarrow (\mathcal{H}_n)$: Estimate for $(\dot{U}_n, \dot{\phi}_n)$, $\mathscr{L}(U_k, \phi_k)$ and $\mathscr{B}(U_k, \phi_k)$

From (4.28) in Theorem 4.1, we have for all $s_0 \le s \le s_+$

$$\begin{aligned}\|(\dot{U}_n, \dot{\phi}_n)\|_{H^s_\eta(\omega^T)} \le C_{s_+}\Big[&\|\dot{F}_n\|_{H^{s+1}_\eta(\omega^T)} + \|\dot{G}_n\|_{H^{s+2}_\eta(\Gamma^T)} + \\ &+(\|\dot{F}_n\|_{H^4_\eta(\omega^T)} + \|\dot{G}_n\|_{H^5_\eta(\Gamma^T)})(1 + \|(\bar{U}_n, \bar{\phi}_n)\|_{H^{s+4}_\eta(\omega^T)})\Big].\end{aligned} \tag{5.16}$$

By (5.6), (5.14) and (5.15), we obtain

$$\|(\dot{U}_n, \dot{\phi}_n)\|_{H^s_\eta(\omega^T)} \le C_{s_+}\delta^2\left[\theta_n^{s+s_0+5-2\alpha} + \theta_n^{8+s_0-2\alpha}\theta_n^{\epsilon+(s+4-\alpha)_+}\right]. \tag{5.17}$$

By the choice of s_0 and α in (5.4), we have

$$s + s_0 + 5 - 2\alpha = s - \alpha - 1 + (s_0 + 6 - \alpha) \le s - \alpha - 1$$

and

$$\begin{cases} \text{if } \ s + 4 - \alpha \ge 0: \\ 8 + s_0 - 2\alpha + \epsilon + (s + 4 - \alpha)_+ = s - \alpha - 1 + (s_0 + 12 - 2\alpha) \le s - \alpha - 1; \\ \text{if } \ s + 4 - \alpha < 0: \\ 8 + s_0 - 2\alpha + (s + 4 - \alpha)_+ = s - \alpha - 1 + (9 - \alpha) \le s - \alpha - 1. \end{cases}$$

Choosing $\delta \ll 1$ such that $\delta C_{s_+} \le 1$, we obtain (5.1) for the interior norms $\|(\dot{U}_k, \dot{\phi}_k)\|_{H^s_\eta(\omega^T)}$ for $k = n$.

The estimate for $\|\dot{\phi}_n\|_{H^{s+1}_\eta(\Gamma^T)}$ can be obtained similarly.

For $\mathscr{L}_a(U_n,\phi_n)$ in (5.2), we have, by (4.12) and (4.13)

$$\begin{aligned}\mathscr{L}_a(U_n,\phi_n) &= \mathscr{F}_T\mathscr{L}(U^a,\phi^a) + \sum_{k=0}^{n-1}\dot{F}_k\Delta_k + \sum_{k=0}^{n-1} e_k\Delta_k \\ &= (1-\mathscr{S}_{n-1})\left[\mathscr{F}_T\mathscr{L}(U^a,\phi^a) + \sum_{k=0}^{n-2} e_k\Delta_k\right] + e_{n-1}\Delta_{n-1}.\end{aligned} \tag{5.18}$$

Therefore

$$\|(1-\mathscr{S}_n)\mathscr{F}_T\mathscr{L}(U^a,\phi^a)\|_{H^s_\eta(\omega^T)} \le C\theta_n^{s-\alpha}\|\mathscr{L}(U^a,\phi^a)\|_{H^\alpha_\eta(\omega^T)}. \tag{5.19}$$

Together with (5.7), we have for $s_0 \le s \le s_+ - 4$

$$\begin{aligned}\|(1-\mathscr{S}_n)\sum_{k=0}^{n-2} e_k\Delta_k\|_s &\le C\theta_n^{s-(s_+-4)}\delta^2\sum_{k=0}^{n-2}\theta_k^{(s_+-4)+s_0+3-2\alpha} \\ &\le C\delta^2\theta_n^{s+s_0+3-2\alpha} \le C\delta^2\theta_n^{s-\alpha}.\end{aligned} \tag{5.20}$$

In (5.19) and (5.20), choosing first $-T \gg 1$ such that $C\|\mathscr{L}(U^a,\phi^a)\|_{H^\alpha_\eta(\omega^T)} \le \frac{1}{2}\delta$, and then choosing $\delta \ll 1$ such that $C\delta \le \frac{1}{2}$, we obtain (5.2) for $k=n$.

The estimate for $\mathscr{B}_a(U_n,\phi_n)$ in (5.3) can be proven exactly in the same way.

5.5 Proof for $(\mathcal{H}_0)$

For $n=0$

$$\mathscr{L}_a(U_0,\phi_0) = \mathscr{L}(U^a,\phi^a), \quad \mathscr{B}_a(U_0,\phi_0) = \mathscr{B}(U^a,\phi^a).$$

If $\alpha + 4 \le s \le s_+ + 2$, we choose $\theta_0 \gg 1$ such that

$$\|\mathscr{L}(U^a,\phi^a)\|_{H^{s_++2}_\eta(\omega^T)} + \|\mathscr{B}(U^a,\phi^a)\|_{H^{s_++2}_\eta(\Gamma^T)} \le \frac{\delta}{2(1+C_{s_+})}\theta_0, \tag{5.21}$$

and therefore

$$\begin{aligned}&\|\mathscr{L}(U^a,\phi^a)\|_{H^s_\eta(\omega^T)} + \|\mathscr{B}(U^a,\phi^a)\|_{H^s_\eta(\Gamma^T)} \\ &\quad\le \|\mathscr{L}(U^a,\phi^a)\|_{H^{s_++2}_\eta(\omega^T)} + \|\mathscr{B}(U^a,\phi^a)\|_{H^{s_++2}_\eta(\Gamma^T)} \\ &\quad\le \frac{\delta}{(1+C_{s_+})}\theta_0^{s-\alpha-3} \le \delta\theta_0^{s-\alpha}.\end{aligned} \tag{5.22}$$

If $s_0 \le s < \alpha + 4$, then we choose $-T \gg 1$ such that

$$\|\mathscr{L}(U^a, \phi^a)\|_{H_\eta^{\alpha+4}(\omega^T)} + \|\mathscr{B}(U^a, \phi^a)\|_{H_\eta^{\alpha+4}(\Gamma^T)} \le \frac{\delta}{2(1 + C_{s_+})}\theta_0^{s_0-\alpha-3}, \tag{5.23}$$

and therefore

$$\begin{aligned} &\|\mathscr{L}(U^a, \phi^a)\|_{H_\eta^s(\omega^T)} + \|\mathscr{B}(U^a, \phi^a)\|_{H_\eta^s(\Gamma^T)} \\ &\quad \le \|\mathscr{L}(U^a, \phi^a)\|_{H_\eta^{\alpha+4}(\omega^T)} + \|\mathscr{B}(U^a, \phi^a)\|_{H_\eta^{\alpha+4}(\Gamma^T)} \\ &\quad \le \frac{\delta}{(1 + C_{s_+})}\theta_0^{s_0-\alpha-3} \le \delta\theta_0^{s-\alpha}. \end{aligned} \tag{5.24}$$

These are (5.2) and (5.3) for $(\mathcal{H}_0)$.

From the expressions for $(\dot{F}_0, \dot{G}_0)$,

$$\Delta_0 \dot{F}_0 = -\mathscr{S}_0 \mathscr{F}_T \mathscr{L}(U^a, \phi^a), \qquad \Delta_0 \dot{G}_0 = -\mathscr{S}_0 \mathscr{F}_T \mathscr{B}(U^a, \phi^a),$$

and the estimate (4.28) for solutions of linearized problem, we obtain similarly as (5.16)

$$\begin{aligned} \|(\dot{U}_0, \dot{\phi}_0)\|_{H_\eta^s(\omega^T)} &\le C_{s_+}\left[\|\dot{F}_0\|_{H_\eta^{s+1}(\omega^T)} + \|\dot{G}_0\|_{H_\eta^{s+2}(\Gamma^T)}\right] \\ &\le C_{s_+}\left[\|\mathscr{L}(U^a, \phi^a)\|_{H_\eta^{s+2}(\omega^T)} + \|\mathscr{B}(U^a, \phi^a)\|_{H_\eta^{s+2}(\Gamma^T)}\right] \end{aligned} \tag{5.25}$$

From (6.49)-(6.52), we have for $s_0 \le s \le s_+ + 2$

$$\begin{aligned} &\|\mathscr{L}(U^a, \phi^a)\|_{H_\eta^s(\omega^T)} + \|\mathscr{B}(U^a, \phi^a)\|_{H_\eta^s(\Gamma^T)} \\ &\quad \le \frac{\delta}{(1 + C_{s_+})}\theta_0^{s-\alpha-3} \end{aligned} \tag{5.26}$$

Combining (5.25) and (5.26) gives (5.1) for $n = 0$.

References

[1] S. Alinhac, *Existence d'ondes de rarefaction pour des systèmes quasi-linéaires hyperboliques multidimensionnels*, Comm. P.D.E.,14(1989),173-230.

[2] A. T. Bui and D. Li, *The double shock front solutions for hyperbolic conservation laws in multi-dimensional space*, Trans. Amer. Math. Soc. 316(1989), 233-250.

[3] S. X. Chen, *Initial boundary value problems for quasilinear symmetric hyperbolic systems with characteristic boundary*, Front. Math. China, 2(2007), 87-102.

[4] S. X. Chen and D. Li, *Cauchy problem with general discontinuous initial data along a smooth curve for 2-D Euler system*, J. D. E. 257(2015), 1939-1988.

[5] J-F. Coulombel and P. Secchi, *The stability of compressible vortex sheets in two-space dimension*, Indiana Univ. Math. Jour.,53(4)(2004),941-1012.

[6] J-F. Coulombel and P. Secchi, *Nonlinear compressible vortex sheets in two space dimensions*, Ann. Sci. Ec. Norm. Super.,41(2008),85-139.

[7] R. Courant and K. O. Friedrichs, *Supersonic Flow and Shock Waves*, Springer-Verlag, New York, 1949.

[8] K. O. Friedrichs, *Symmetric positive linear differential equations.* Comm. Pure Appl. Math. 11(1958), 333-418.

[9] R.S. Hamilton, *The inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc., 7(1982), 65-222.

[10] D. Li, *Rarefaction and shock waves for multi-dimensional hyperbolic conservation laws*, Comm. in PDEs,16(1991),425-450.

[11] D. Li, *Compatibility of jump Cauchy data for non-isentropic Euler equations*, J. Math. Anal. Appl. 425(2015), 565-587.. JMAA (2015).

[12] D. Li, *Initial-boundary value problem for Euler equations with incompatible data*, Quarterly Appl. Math. 76(2018), 47-64.

[13] T. T. Li and W. C. Yu, *Boundary value problem for quasi-linear hyperbolic systems.* Duke Univ. Math., Ser.5 (1985).

[14] A. Majda *The existence of multi-dimensional shock front*, Memoirs Amer. Math. Soc., 281(1983).

[15] A. Majda and S. Osher, *Initial-Boundary value problems for hyperbolic equations with uniformly characteristic boundary*, Comm. Pure Appl. Math., 28(5), 1975, 607-675.

[16] J. Rauch and F. Massey, *Differentiability of solutions to hyperbolic initial-boundary value problems*, Trans. Amer. Math. Soc. 189(1974), 303-318.

[17] J. Smoller, *Shock waves and reaction-diffusion equations*, Springer-Verlag, 1983.

[18] D. S. Tartakoff, *Regularity of solutions to boundary value problems for first order systems*, Indiana Univ. Math. J. 21(1972), 1113-1129.