Initial-Boundary Value Problem for Euler Equations with Incompatible Data *†

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Abstract

We study the initial-boundary value problem for the general 3-D Euler equations with data which are incompatible in the classical sense, but are "shock-compatible". We show that such data are also shock-compatible of infinite order and the initial-boundary value problem has a piece-wise smooth solution containing a shock.

1 Introduction

Initial-boundary value problem is one of the most important problems, both in theoretical research and in application, in the study of hyperbolic systems, in particular, Euler system of gas-dynamics. It is well known that the smoothness of both the initial and boundary data does not guarantee the existence of a classical solution. A necessary condition to the existence of a smooth solution is the compatibility of such data. In order the solution has higher differentiability, the higher order compatibility of the data is required, see, e.g., [3, 8, 13, 15]. Similarly, for certain free boundary value problems involving shock wave, rarefaction wave or contact discontinuity of Euler equations, the data are also required to be compatible, often of very high order, [1, 2, 4, 5, 9, 11, 12].

The compatibility is a set of conditions on the initial and boundary data at the points of intersection of the boundary with the initial manifold. They consist of the algebraic restrictions at the intersection on the values of data, together with their normal derivatives of high order (depending upon the order of compatibility). Usually, such conditions are complicated and very tedious to verify.

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In this paper, we study the initial-boundary value problem for the general 3-D Euler equation with data which are incompatible in the classical sense. The data may contain a jump discontinuity at the intersection of the initial and boundary manifolds. For such data, there exists no classical solution. Instead, we are looking for a piece-wise smooth solution containing a shock front under a simple general assumption on the data, see condition (A) in Theorem 1.1. We will call such data (satisfying (A)) as "shock-compatible". Similar to the situation for free boundary value problems studied in [4, 10], for such data, even though incompatible in the classical sense, permitting shock waves makes the compatibility issue much simpler. It turns out that the 0-order shock-compatible data (A) are automatically shock compatible of infinite order if both the initial and boundary data are smooth. Taking advantage of such fact, we are able to show the existence of the piece-wise smooth shock wave solution with the condition (A) only, without requiring any high-order compatibility as in [1, 2, 4, 5, 9, 11, 12].

As the most important example of quasi-linear hyperbolic system, the Euler equations for compressible non-viscous flow in 3-D space can be written as follows:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) + \partial_y(\rho v) + \partial_z(\rho w) = 0, \\ \partial_t(\rho u) + \partial_x(p + \rho u^2) + \partial_y(\rho u v) + \partial_z(\rho u w) = 0, \\ \partial_t(\rho v) + \partial_x(\rho u v) + \partial_y(p + \rho v^2) + \partial_z(\rho v w) = 0, \\ \partial_t(\rho w) + \partial_x(\rho u w) + \partial_y(\rho v w) + \partial_z(p + \rho w^2) = 0, \\ \partial_t(\rho E) + \partial_x(\rho E u + p u) + \partial_y(\rho E v + p v) + \partial_z(\rho E w + p w) = 0, \end{cases} \tag{1.1}$$

where (ρ, p, e) are the density, pressure, and the internal energy of the fluid, (u, v, w) is the velocity in the (x, y, z) direction, and the total energy $E = e + \frac{1}{2}(u^2 + v^2 + w^2)$. For convenience, we will consider the gas to be polytropic, with $p = A(S)\rho^\gamma$ with $\gamma > 1$.

One of the simplest and natural initial-boundary value problems for the Euler system (1.1) describes the gas flow bounded by a solid wall $x = 0$ with given initial status (ρ, u, v, w, e):

$$\begin{cases} (\rho, u, v, w, e)(0, x, y, z) = (\rho_0, u_0, v_0, w_0, e_0)(x, y, z) \ \text{ in } \ x \geq 0, \\ u(t, 0, y, z) = 0, \ \text{ on } \ t \geq 0. \end{cases} \tag{1.2}$$

In order to have a smooth solution for the problem (1.1)(1.2), it is necessary to require the initial data $(\rho_0, u_0, v_0, w_0, e_0)$ to be compatible, see e.g.,

[13]. The continuity of the solution requires the zero-order compatibility

$$u_0(0, y, z) = 0. \tag{1.3}$$

If one wants the solution belonging to C^k, the higher order compatible conditions are required, which consist of algebraic relations imposed upon u_0 and its derivatives $\partial_x^j u_0 (j \leq k)$ at $x = 0$.

If (1.3) is not satisfied, i.e., if the data is not compatible in the classical sense, then one cannot expect to have a continuous solution. However, there could be other solutions which are only piece-wise smooth. In this paper, we will study the initial-boundary value problem for (1.1) with data which is not compatible in the classical sense, but admits a piece-wise solution containing a shock wave.

In this paper, we will study a more general initial-boundary value problem, for which (1.2) is a special case. Denote

$$H_0 = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho E \end{pmatrix}, \ H_1 = \begin{pmatrix} \rho u \\ p + \rho u^2 \\ \rho uv \\ \rho uw \\ (\rho E + p)u \end{pmatrix}, \ H_2 = \begin{pmatrix} \rho v \\ \rho uv \\ p + \rho v^2 \\ \rho vw \\ (\rho E + p)v \end{pmatrix}, \ H_3 = \begin{pmatrix} \rho w \\ \rho uw \\ \rho vw \\ p + \rho w^2 \\ (\rho E + p)w \end{pmatrix}.$$

Then system (1.1) can be written briefly as

$$\partial_t H_0 + \partial_x H_1 + \partial_y H_2 + \partial_z H_3 = 0.$$

Introducing the unknown vector of functions $U = (p, u, v, w, S)$, it is well-known (see e.g., [6, 14]) that for smooth solutions, the system (1.1) is equivalent to the following system

$$\begin{cases} \dfrac{\partial p}{\partial t} + (u, v, w) \cdot \nabla p + \rho c^2 \nabla \cdot (u, v, w) = 0, \\ \rho \dfrac{\partial u}{\partial t} + \rho(u, v, w) \cdot \nabla u + \dfrac{\partial p}{\partial x} = 0, \\ \rho \dfrac{\partial v}{\partial t} + \rho(u, v, w) \cdot \nabla v + \dfrac{\partial p}{\partial y} = 0, \\ \rho \dfrac{\partial w}{\partial t} + \rho(u, v, w) \cdot \nabla w + \dfrac{\partial p}{\partial z} = 0, \\ \dfrac{\partial S}{\partial t} + (u, v, w) \cdot \nabla S = 0, \end{cases} \tag{1.4}$$

with $c^2 = p'_\rho(\rho, S) > 0$.

System (1.4) can be further written into the following symmetric form

$$\mathscr{L}(U)U \equiv A_0\partial_t U + A_1(U)\partial_x U + A_2(U)\partial_y U + A_3(U)\partial_z U = 0, \tag{1.5}$$

where

$$A_0 = \begin{bmatrix} \frac{1}{\rho c^2} & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & \rho & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} \frac{u}{\rho c^2} & 1 & 0 & 0 & 0 \\ 1 & \rho u & 0 & 0 & 0 \\ 0 & 0 & \rho u & 0 & 0 \\ 0 & 0 & 0 & \rho u & 0 \\ 0 & 0 & 0 & 0 & u \end{bmatrix},$$

$$A_2 = \begin{bmatrix} \frac{v}{\rho c^2} & 0 & 1 & 0 & 0 \\ 0 & \rho v & 0 & 0 & 0 \\ 1 & 0 & \rho v & 0 & 0 \\ 0 & 0 & 0 & \rho v & 0 \\ 0 & 0 & 0 & 0 & v \end{bmatrix}, \quad A_3 = \begin{bmatrix} \frac{w}{\rho c^2} & 0 & 0 & 1 & 0 \\ 0 & \rho w & 0 & 0 & 0 \\ 0 & 0 & \rho w & 0 & 0 \\ 1 & 0 & 0 & \rho w & 0 \\ 0 & 0 & 0 & 0 & w \end{bmatrix}.$$

The matrix $A_0^{-1}[A_1(U) + A_2(U)\xi + A_3(U)\eta]$ has two simple eigenvalues $\lambda_\pm$ and one triple eigenvalue λ_0:

$$\begin{aligned} &\lambda_- = u - v\xi - w\eta - c\sqrt{1+\xi^2+\eta^2}, \\ &\lambda_0 = u - v\xi - w\eta, \\ &\lambda_+ = u - v\xi - w\eta + c\sqrt{1+\xi^2+\eta^2}, \end{aligned} \tag{1.6}$$

with $\lambda_- < \lambda_0 < \lambda_+$.

Let $x = b(t, y, z)$ be a smooth surface in (t, x, y, z) space with $b(0, 0, 0) = b_y(0, 0, 0) = b_z(0, 0, 0) = 0$. Denote $b_0(y, z) = b(0, y, z)$. For the Euler system (1.1), consider the initial-boundary value problem in the domain bounded by the moving solid boundary $x = b(y, z, t)$ and the initial plane $t = 0$:

$$\begin{cases} \partial_t H_0(U) + \partial_x H_1(U) + \partial_y H_2(U) + \partial_z H_3(U) = 0, \\ U(0, x, y, z) = U_0(x, y, z) \ \text{ in } \ x \geq 0, \\ u - b_t - b_y v - b_z w = 0, \ \text{ on } \ x = b(t, y, z), \ t \geq 0. \end{cases} \tag{1.7}$$

Obviously, (1.1)(1.2) is the special case of (1.7) with $b = 0$.

The main result of this paper is the following

Theorem 1.1 *For the initial-boundary value problem (1.7), assuming the following condition (A) is the satisfied:*

$$
\text{(A)} \quad \begin{cases} |b_t(0,0,0)| < c_0, \text{ with } c_0^2 = p_\rho \big|_{U_0(0,0,0)}, \\ u_0(0,0,0) < b_t(0,0,0). \end{cases}
$$

Then the problem (1.7) admits a piece-wise smooth solution near the origin in the domain $x > b(t,y,z)$, $t > 0$, containing one shock front $x = \phi(t,y,z)$ emanating from the initial curve $x = b_0(y,z)$.

Remark 1.1 *1. The assumption $|b_t(0,0,0)| < c$ in Theorem 1.1 is a necessary condition. Otherwise, the problem (1.7) is not well-posed. In particular, for the fixed boundary $x = 0$ in (1.2), the condition is trivially satisfied.*

2. The assumption $u_0(0,0,0) < b_t(0,0,0)$ in Theorem 1.1 ensures the existence of a shock wave. For the special case of fixed boundary $x = 0$ in (1.4), the condition becomes simply $u_0 < 0$.

3. It is worth mentioning here that if $u_0(0,0,0) > b_t(0,0,0)$, there would be a solution containing a rarefaction wave. Such case will be studied later in another paper. The degenerate case of $u_0(0,0,0) = b_t(0,0,0)$ would imply either a smooth solution or a solution with weak discontinuity such as sound wave.

In the following, Section 2 will be devoted to the set-up of the problem and the construction of an approximate solution of infinite order. The problem will be reformulated in Section 3 by introducing new coordinates in (1.7) to flatten the boundary $x = b(t,y,z)$ as well as the shock front $x = \phi(t,y,z)$. The linear stability of the transformed equivalent problem will be derived in Section 4 by combining the results from [3, 7, 12, 13]. Then the existence of a piece-wise smooth solution containing a shock front will be established in Section 5 by iteration.

2 Shock wave solution and its approximation

From the solid wall condition $u - b_t - b_y v - b_z w = 0$ on the moving boundary $x = b(t,y,z)$ in (1.7) and the condition $u_0(0,0,0) < b_t(0,0,0)$ in the assumption (A) of Theorem 1.1, it is obvious that the data for the initial-boundary value problem (1.7) is not compatible in the classical sense and hence (1.7) admits no classical smooth solution. Therefore we will look for a piece-wise smooth solution which contains, in the specific case of (A), a right propagating shock wave.

Specifically, a shock wave solution for (1.7) is a set of smooth functions (U, ϕ, U^*) near the origin $(0,0,0,0)$ such that

- The shock front $S : x = \phi(t, y, z)$ divides the domain $x > b(t, y, z), t > 0$ into two parts:

$$\Omega = \{(t, x, y, z) : b(t, y, z) < x < \phi(t, y, z), t > 0\}$$
$$\Omega^* = \{(t, x, y, z) : \phi(t, y, z) < x, t > 0\}$$

  with

$$\phi(0, y, z) = b(0, y, z), \text{ and } c_0 < \phi_t(0, 0, 0); \tag{2.1}$$

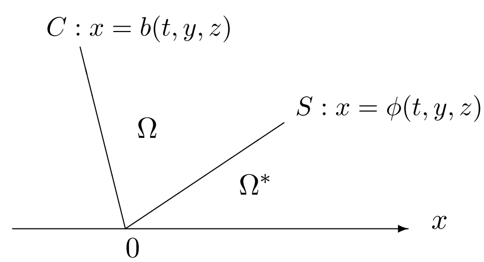

Figure 2.1: Shock wave solution for (1.7)

- U (or U^*) is defined and smooth in Ω (or Ω^*), and satisfies the equations

$$\partial_t H_0(U) + \partial_x H_1(U) + \partial_y H_2(U) + \partial_z H_3(U) = 0 \tag{2.2}$$

  in Ω (or Ω^*);

- (U, ϕ, U^*) satisfies the Rankine-Hugoniot condition on $x = \phi(t, y, z)$:

$$\phi_t[H_0]^+_- - [H_1]^+_- + \phi_y[H_2]^+_- + \phi_z[H_3]^+_- = 0 \tag{2.3}$$

  Here in (2.3), $[f]^+_-$ denotes the jump of the value of f across the shock front.

The Lax's shock condition [14] implies that the shock front $x = \phi(t, y, z)$ is space-like in front of the shock front, thus the condition in (2.1). So the flow status U^* is uniquely determined in Ω^* by the initial data $U_0(x, y, z)$. In order to find the solution, one needs only to determine the functions (U, ϕ).

The set of functions $(\tilde{U}, \tilde{\phi}, \tilde{U}^*)$ is called an approximate solution of order k for (1.7) and (2.3), if the following is satisfied near $t = 0$

$$\begin{cases} \partial_t H_0(\tilde{U}) + \partial_x H_1(\tilde{U}) + \partial_y H_2(\tilde{U}) + \partial_z H_3(\tilde{U}) = O(t^k), \ \ in \ \ \Omega \\ \partial_t H_0(\tilde{U}^*) + \partial_x H_1(\tilde{U}^*) + \partial_y H_2(\tilde{U}^*) + \partial_z H_3(\tilde{U}^*) = O(t^k), \ \ in \ \ \Omega^* \\ \tilde{\phi}_t [H_0]_-^+ - [H_1]_-^+ + \tilde{\phi}_y [H_2]_-^+ + \tilde{\phi}_z [H_3]_-^+ = O(t^k), \ \ \text{on} \ \ x = \tilde{\phi}; \\ U^*(0, x, y, z) = U_0(x, y, z) \ \ \text{in} \ \ x \geq 0, \\ \tilde{u} - b_t - b_y \tilde{v} - b_z \tilde{w} = O(t^k), \ \ \text{on} \ \ x = b(t, y, z), \ t \geq 0. \end{cases} \tag{2.4}$$

Since U^* can be uniquely determined by $U_0(x, y, z)$, one can simply take $\tilde{U}^* = U^*$, the conditions in (2.4) for $\tilde{U}^*$ can be dropped. (2.4) can be simplified into the following conditions containing only $(\tilde{U}, \tilde{\phi})$:

$$\begin{cases} \partial_t H_0(\tilde{U}) + \partial_x H_1(\tilde{U}) + \partial_y H_2(\tilde{U}) + \partial_z H_3(\tilde{U}) = O(t^k), \ \ in \ \ \Omega \\ \tilde{\phi}_t [H_0]_-^+ - [H_1]_-^+ + \tilde{\phi}_y [H_2]_-^+ + \tilde{\phi}_z [H_3]_-^+ = O(t^k), \ \ \text{on} \ \ x = \tilde{\phi}; \\ \tilde{u} - b_t - b_y \tilde{v} - b_z \tilde{w} = O(t^k), \ \ \text{on} \ \ x = b(t, y, z), \ t \geq 0. \end{cases} \tag{2.5}$$

Obviously, the existence of the k-th order approximate solution is equivalent to the fact that all the derivatives at $t = 0$ up to the order of k for (U, ϕ) can be uniquely determined by the equations in (2.5) along the initial sub-surface $x = b(0, y, z)$. For the existence of an infinite order approximate solution, we have the following theorem

Theorem 2.1 Under the condition (A) in Theorem 1.1, and also assuming that $b(t, y, z) \in C^\infty$, $U_0(x, y, z) \in C^\infty$ in the initial-boundary value problem (1.7) and (2.3), then all the derivatives of (U, ϕ) at $t = 0$ can be uniquely determined by the equations in (2.5) at the intersection $x = b(0, y, z)$, and consequently, there exists an infinite order approximate solution $(\tilde{U}, \tilde{\phi})$.

To prove Theorem 2.1, we need to show that $\forall k \geq 0$, all the derivatives up to the order k of (U, ϕ) can be uniquely determined at $x = b(0, y, z)$. First we prove the case $k = 0$.

The 0-order compatibility does not include any derivatives of U, and we have 6 variables

$$U(0, \phi_0(y, z), y, z), \ \partial_t \phi(0, y, z).$$

to satisfy 6 equations in the boundary conditions of (2.5).

Due to the continuity in the variables y and z and by the implicit function theorem, we need only to show that at the origin (0,0,0,0), the system consisting of six boundary conditions in (2.5) has one solution

$$U(0,0,0,0),\ \partial_t\phi(0,0,0).$$

and the corresponding Jacobian matrix is non-degenerate.

At (0,0,0,0) the six boundary equations in (2.5) become

$$\phi_t(0)\begin{pmatrix}\rho-\rho_0\\ \rho u-\rho_0u_0\\ \rho v-\rho_0v_0\\ \rho w-\rho_0w_0\\ \rho E-\rho_0E_0\end{pmatrix}=\begin{pmatrix}\rho u-\rho_0u_0\\ p+\rho u^2-p_0-\rho_0u_0^2\\ \rho uv-\rho_0u_0v_0\\ \rho uw-\rho_0u_0w_0\\ (\rho E+p)u-(\rho_0E_0+p_0)u_0\end{pmatrix}; \tag{2.6}$$

$$u=b_t. \tag{2.7}$$

The variable u is obviously uniquely determined by (2.7). The two variables (v,w), each appears only in one equation of (2.6), and they all have non-zero coefficient $\rho(\phi_t-u)$ because $\phi_t>b_t=u$.

Eliminating these three variables from (2.6) and (2.7) and replacing the energy conservation by the equivalent thermodynamic Hugoniot relation [6], we obtain a 3×3 system for (ϕ_t,ρ,p):

$$\begin{cases}\phi_t(0)\begin{pmatrix}\rho-\rho_0\\ \rho u-\rho_0u_0\end{pmatrix}-\begin{pmatrix}\rho u-\rho_0u_0\\ p-p_0+\rho u^2-\rho_0u_0^2\end{pmatrix}=0;\\ (\rho_0-\mu^2\rho)p-(\rho-\mu^2\rho_0)p_0=0;\end{cases} \tag{2.8}$$

Here $\tau=1/\rho$ as usual and $\mu^2=(\gamma-1)/(\gamma+1)$.

Because $u=b_t>u_0$ in (2.8), there exists a unique solution (ρ,p,ϕ_t), by the shock curve [14] with $\rho>\rho_0$, $p>p_0$ and ϕ_t-u_0 supersonic and ϕ_t-u subsonic.

Denote by F the left hand sides of the boundary conditions in (2.5), and J the coefficient matrix of their linearization. We need to show

$$\det J=\det\ \frac{\partial F}{\partial(U,\phi_t)}\neq 0,\ \ \text{at}\ \ (0,0,0,0). \tag{2.9}$$

Similar as in (2.6) and (2.7), the last row of J contains only variable u, and the variables (v,w) appear only in the third and the fourth rows and with non-zero coefficients $(=\rho_0(\phi_t(0)-u_0))$. We need to consider only the first two rows and the fifth row for the variables (ρ,p,ϕ_t). Also, we may replace the energy conservation by the equivalent thermodynamic Hugoniot relation as in (2.8). The corresponding coefficient matrix is

$$\begin{pmatrix} \phi_t(0) - u & 0 & \rho - \rho_0 \\ (\phi_t(0) - u)u & -1 & \rho u - \rho_0 u_0 \\ -\mu^2 p - p_0 & \rho_0 - \mu^2 \rho & 0 \end{pmatrix} \tag{2.10}$$

Using the relation $\phi_t(\rho - \rho_0) = \rho u - \rho_0 u_0$ to simplify the last column, the determinant of (2.10) is equal to

$$(\rho - \rho_0) \det \begin{pmatrix} (\phi_t(0) - u)^2 & 1 \\ \mu^2 p + p_0 & \mu^2 \rho - \rho_0 \end{pmatrix} \tag{2.11}$$

Obviously, (2.11) is non-zero if $\mu^2 \rho - \rho_0 < 0$, which follows readily from the restrictions on the compression ratio (see [6], p. 148)

$$\mu^2 < \rho/\rho_0 < \mu^{-2}.$$

The first order compatibility consists of eleven linear equations for the eleven variables

$$U_t(0, \phi_0(y,z), y, z),\ U_n(0, \phi_0(y,z), y, z),\ \phi_{tt}(0, y, z).$$

Here U_n denotes the normal derivative to the shock front $x = \phi(t, y, z)$.

Again by the continuity in (y, z) and because the equations for these variables are linear, we need only to show the Jacobian of these 11 equations is non-degenerate at (0,0,0,0). In particular at the origin, $U_n = U_x$, $\phi_{ly} = \phi_{lz} = \phi_{ry} = \phi_{rz} = \theta_y = \theta_z = 0$ and $\theta_t = u_a = u_b = u$.

Let (D_ϕ, D_u) be the tangential vectors at the origin in the directions of the curves $x = \phi(t, 0, 0)$ and $x = b(t, 0, 0)$ (noticing $b_t = u$ at the origin):

$$D_\phi = \partial_t + \phi_t \partial_x, \quad D_u = \partial_t + u \partial_x.$$

As usual, we will replace the energy conservation by the thermodynamic Hugoniot relation. Denote by (H_{04}, H_{14}) the first four components of (H_0, H_1) and then take tangential derivatives D_ϕ along $x = \phi$ of thus modified boundary equations in (2.5) in the t-x plane. Evaluating them at (0,0,0,0), we obtain (here and in the following in this paper, $*$ stands for terms already determined by lower order compatibility):

$$\begin{cases} \phi_{tt}[H_{04}]_-^+ + (\phi_t H'_{04} - H'_{14}) D_\phi U = *, \\ (p_0 + \mu^2 p) D_\phi \rho + (\mu^2 \rho - \rho_0) D_\phi p = *. \end{cases} \tag{2.12}$$

where the 4×5 matrix $(\phi_t H'_{04} - H'_{14})$ is

$$\phi_t H'_{04} - H'_{14} = \begin{bmatrix} \phi_t - u & -\rho & 0 & 0 & 0 \\ u(\phi_t - u) & \rho(\phi_t - 2u) & 0 & 0 & -1 \\ 0 & 0 & \rho(\phi_t - u) & 0 & 0 \\ 0 & 0 & 0 & \rho(\phi_t - u) & 0 \end{bmatrix}.$$

At the origin (0,0,0,0), the interior equations in (2.5) becomes

$$\begin{cases} D_u\rho + \rho\partial_x u = *, \\ D_u u + \frac{1}{\rho}\partial_x p = *, \\ D_u v = *, & \text{in } \Omega \\ D_u w = *, \\ D_u p + \rho c^2 \partial_x u = *. \end{cases} \tag{2.13}$$

Obviously, from the last boundary condition in (2.5) on $x = b$ we have

$$D_u u = *, \tag{2.14}$$

The linear system (2.12)-(2.14) consists of eleven equations for the eleven variables (ϕ_{tt}, U_t, U_x), where $U = (\rho, u, v, w, p)$. They can be simplified as follows.

Because D_ϕ, D_u are not parallel, (v_t, v_x) are uniquely determined by $(D_u v, D_\phi v,)$. Since the derivatives of v appears only in one equation in (2.13) in the form $D_u v$, and also appears only in one equation in (2.12) in the form $D_\phi v$, both with non-zero coefficients, hence $(D_u v, D_\phi v_)$ can be uniquely determined. Therefore (v_t, v_x) can be eliminated.

Same argument also applies to (w_t, w_x). Thus, we can eliminate the four variables (v_t, v_x, w_t, w_x) from (2.12)-(2.14) and obtain seven equations for the remaining seven variables

$$(\phi_{tt}, \rho_t, \rho_x, u_t, u_x, p_t, p_x).$$

Eliminating ϕ_{tt} from (2.12) yields

$$\begin{bmatrix} (u - \phi_t)^2 & 2\rho(u - \phi_t) & 1 \\ p_0 + \mu^2 p & 0 & \mu^2\rho - \rho_0 \end{bmatrix} D_\phi \begin{bmatrix} \rho \\ u \\ p \end{bmatrix} = *. \tag{2.15}$$

Eliminating $D_\phi \rho$ from (2.15) yields

$$(m_1, m_2) D_\phi \begin{bmatrix} u \\ p \end{bmatrix} = * \tag{2.16}$$

with

$$\begin{cases} m_1 = 2\rho(u-\phi_t)(p_0+\mu^2 p) < 0, \\ m_2 = (p_0+\mu^2 p) - (\mu^2\rho-\rho_0)(u-\phi_t)^2 > 0. \end{cases} \tag{2.17}$$

Dropping the two equations containing $D_u v$ and $D_u w$ in (2.13), the remaining last two equations contain no term of $D_u\rho$:

$$\begin{cases} D_u u + \frac{1}{\rho}\partial_x p = *, \\ D_u p + \rho c^2 \partial_x u = *. \end{cases} \tag{2.18}$$

Or equivalently

$$D_u \begin{bmatrix} u \\ p \end{bmatrix} + c\mathscr{E}\partial_x \begin{bmatrix} u \\ p \end{bmatrix} = *, \tag{2.19}$$

with the operator $\mathscr{E}$ defined as (see also [10])

$$\mathscr{E} \equiv \begin{bmatrix} 0 & (c\rho)^{-1} \\ c\rho & 0 \end{bmatrix} = \mathscr{E}^{-1}. \tag{2.20}$$

Since $D_\phi = D_u + (\phi_t - u)\partial_x$, we have by (2.19)

$$\left\{ D_\phi \begin{bmatrix} u \\ p \end{bmatrix} = (I - \beta\mathscr{E})\, D_u \begin{bmatrix} u \\ p \end{bmatrix} + *, \right. \tag{2.21}$$

where

$$\beta \equiv \frac{\phi_t - u}{c} > 0, \ \text{ with } \ |\beta| < 1 \tag{2.22}$$

by the Lax' shock inequality.

Replacing $(D_\phi u, D_\phi p)$ in (2.16) by (2.21), we obtain:

$$\begin{bmatrix} m_1 - m_2\beta c\rho & \quad -m_1\beta(c\rho)^{-1} + m_2 \end{bmatrix} D_u \begin{bmatrix} u \\ p \end{bmatrix} = *. \tag{2.23}$$

By (2.17) and (2.22), we have

$$m_1 - m_2\beta c\rho < 0, \ \ -m_1\beta(c\rho)^{-1} + m_2 > 0.$$

Since $D_u u$ is already uniquely determined by (2.14), $D_u p$ is also uniquely determined. Consequently (u_x, p_x) are uniquely determined by (2.19). Also, $(D_\phi\rho, D_u\rho)$ are uniquely determined by (2.12) and (2.13). This finishes the proof of the 1st order compatibility.

For the **k-th order compatibility**, apply D_ϕ^k to the modified boundary conditions (2.5) and evaluate them at the origin (0,0,0,0). Also apply D_u^{k-1}

to the interior equations and evaluate them at (0,0,0,0). Similar to the first order compatibility, the nine variables $(D^k_\phi v, D^{k-1}_\phi \partial_x v, D^k_\phi w, D^{k-1}_\phi \partial_x w)$ can be determined independently and thus be eliminated.

For the remaining seven variables

$$\partial_t^{k+1}\phi, D^k_u\rho, D^{k-1}_u\partial_x\rho, D^k_u u, D^{k-1}_u\partial_x u, D^k_u p, D^{k-1}_u\partial_x p,$$

we have seven equations:

$$\begin{cases} \begin{bmatrix} \rho \\ \rho u \end{bmatrix}^+_- \partial_t^{k+1}\phi + \begin{bmatrix} \phi_t - u & -\rho & 0 \\ u(\phi_t - u) & \rho(\phi_t - 2u) & -1 \end{bmatrix} D^k_\phi \begin{bmatrix} \rho \\ u \\ p \end{bmatrix} = *; \\ (p_0 + \mu^2 p)D^k_\phi\rho + (\mu^2\rho - \rho_0)D^k_\phi p = *. \end{cases} \tag{2.24}$$

$$\begin{cases} D^k_u\rho + \rho D^{k-1}_u\partial_x u = *, \\ D^k_u u + \frac{1}{\rho}D^{k-1}_u\partial_x p = *, \\ D^k_u p + \rho c^2 D^{k-1}_u\partial_x u = *. \end{cases} \tag{2.25}$$

$$D^k_u u = *. \tag{2.26}$$

As usual, eliminating $\partial_t^{k+1}\phi$ and $D^k\rho$ from (2.24) and (2.25), we obtain

$$(m_1, m_2)D^k_\phi \begin{bmatrix} u \\ p \end{bmatrix} = *. \tag{2.27}$$

Furthermore, we can use (2.25) to replace $(D^k_\phi u, D^k_\phi p)$ with $(D^k_u u, D^k_u p)$ by applying the following lemma, see [4, 10]

Lemma 2.1 $(D^k_\phi u, D^k_\phi p)$ in (2.25) can be expressed by $(D^k_u u, D^k_u p)$ as

$$D^k_\phi \begin{bmatrix} u \\ p \end{bmatrix} = \delta(\alpha_k - \beta\mathscr{E})D^k_u \begin{bmatrix} u \\ p \end{bmatrix} = *, \tag{2.28}$$

where $0 < |\beta| < \alpha_k \leq 1$, and δ is a positive constant which may depend on k and the explicit form of which is of no consequence in our discussion.

We omit the proof here.

Applying the lemma, we can rewrite (2.27) as follows

$$(m_1, m_2)\delta(\alpha_k - \beta\mathscr{E})D^k_u \begin{bmatrix} u \\ p \end{bmatrix} = *, \tag{2.29}$$

or

$$[m_1\alpha_k - m_2\beta(c\rho)]D^k_u u + [-m_1\beta(c\rho)^{-1} + m_2\alpha_k]D^k_u p = * \tag{2.30}$$

From (2.17) and (2.22), we have

$$m_1\alpha_k - m_2\beta(c\rho) < 0, \ \ -m_1\beta(c\rho)^{-1} + m_2\alpha_k > 0.$$

Combining (2.26) and (2.30), $(D_u^k u, D_u^k p)$ are uniquely determined. Then $(D_u^{k-1}\partial_x u, D_u^{k-1}\partial_x p)$ can also be determined by (2.25), as well as $D_u^k\rho$. $D_\phi^k\rho$ and consequently $D_u^{k-1}\partial_x\rho$ can be determined from (2.24). An induction on the index j would give all k-th order derivatives for $D_u^{k-j}\partial_x^j\rho, D_u^{k-j}\partial_x^j u, D_u^{k-j}\partial_x^j p$ $(j = 2, \cdots, k)$. This finishes the proof of k-th order compatibility. Since k is arbitrary, this implies infinite order compatibility.

Once all the derivatives of (U, ϕ) can be uniquely determined at $t = 0, x = 0$, we can construct explicitly an infinite order approximate solution by the usual Borel technique, see [4]. In particular, for the case considered in this paper, we also have the following

Remark 2.1 *The infinite order approximate solution in Lemma 2.1 can be constructed such that the condition on the boundary $x = b(t, y, z)$ in (1.7) and (2.5) be satisfied accurately, i.e.,*

$$\tilde{u} - b_t - b_y\tilde{v} - b_z\tilde{w} = 0, \ \text{ on } \ x = b(t, y, z), \ t \geq 0. \tag{2.31}$$

Indeed, given the approximate solution $\tilde{U}$, one can construct $\hat{U}$ by solving a linear boundary value problem for the linearized equation of (1.7) at the approximate solution $\tilde{U}$ with the linear boundary condition

$$\hat{u} - b_t - b_y\hat{v} - b_z\hat{w} = -[\tilde{u} - b_t - b_y\tilde{v} - b_z\tilde{w}], \ \text{ on } \ x = b(t, y, z), \ t \geq 0.$$

Then $(\hat{U} + \tilde{U}, \tilde{\phi})$ will be the desired approximate solution.

3 Transformation and Reformulation

To prove the existence of the piece-wise smooth shock wave solution in Theorem 1.1, we first perform a singular transformation to reformulate the problem. The purpose of the transformation is to fix the shock front and straighten the solid boundary $x = b(t, y, z)$, see also [1, 2, 4, 9].

Let

$$t = \tilde{t}, \ \ x = \xi(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z}), \ \ y = \tilde{y}, \ \ z = \tilde{z}. \tag{3.1}$$

where

$$\xi(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z}) = (1 - \tilde{x})b(\tilde{t}, \tilde{y}, \tilde{z}) + \tilde{x}\phi(\tilde{t}, \tilde{y}, \tilde{z})$$

With the transformation (3.1), the domain Ω in the (t, x, y, z) coordinates becomes a rectanglar region $\tilde{\Omega}$ in the $(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z})$ coordinates:

$$\tilde{\Omega} = \{(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z}) : 0 < \tilde{x} < 1,\ \tilde{t} > 0,\ (\tilde{y}, \tilde{z}) \in \mathbb{R}^2\}.$$

The system (1.5) of interior differential equations in the new coordinates $(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z})$ becomes

$$A_0(U)\partial_{\tilde{t}}U + \tilde{A}_1(U)\partial_{\tilde{x}}U \ + A_2(U)\partial_{\tilde{y}}U + +A_3(U)\partial_{\tilde{z}}U = 0 \tag{3.2}$$

with

$$\tilde{A}_1(U) = \frac{1}{\partial_{\tilde{x}}\xi}(A_1(U) - A_0(U)\xi_{\tilde{t}} - A_2(U)\xi_{\tilde{y}} - A_3(U)\xi_{\tilde{z}}).$$

Because $\partial_{\tilde{x}}\xi = O(\tilde{t})$, the system (3.2) is singular at $\tilde{t} = 0$ with order $O(\tilde{t})$. To formally remove this singularity, let (see also [2, 4])

$$\tilde{t} = \tau, \ \text{ with } \ \partial_{\tilde{t}} = e^{-\tau}\partial_\tau. \tag{3.3}$$

The transform (3.3) changes the domain $\tilde{\Omega}$ into ω:

$$\omega = \{(\tau, \tilde{x}, \tilde{y}, \tilde{z}) : 0 < \tilde{x} < 1,\ \tau > -\infty,\ (\tilde{y}, \tilde{z}) \in \mathbb{R}^2\}.$$

In the coordinates $(\tau, \tilde{x}, \tilde{y}, \tilde{z})$, the system (3.2) becomes

$$\mathscr{L}(U, \phi) \equiv \partial_\tau U + \tilde{\tilde{A}}_1(U)\partial_{\tilde{x}}U + e^\tau A_2(U)\partial_{\tilde{y}}U + e^\tau A_3(U)\partial_{\tilde{z}}U = 0 \tag{3.4}$$

with

$$\tilde{\tilde{A}}_1(U) = \frac{e^\tau}{\xi_{\tilde{x}}}(A_1(U) - e^{-\tau}\xi_\tau A_0(U) - \xi_{\tilde{y}}A_2(U)) - \xi_{\tilde{z}}A_3(U)). \tag{3.5}$$

In addition, we also notice that under the coordinates transform (3.3), the $\tilde{t}^\eta$-weighted integration in the domain $\tilde{\Omega}$ becomes the hyperbolic $(\eta+1)$-weighted integration in ω:

$$\int_{\tilde{\Omega}} \tilde{t}^\eta |U(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z})|^2 d\tilde{t}d\tilde{x}d\tilde{y}d\tilde{z} = \int_\omega e^{(\eta+1)\tau}|U^{(j)}(\tau, \tilde{x}, \tilde{y}, \tilde{z})|^2 d\tau d\tilde{x}d\tilde{y}d\tilde{z}.$$

Denote the boundary operators on $\tilde{x} = 0, 1$ in the coordinates $(\tau, \tilde{x}, \tilde{y}, \tilde{z})$ as follows:

- On $\tilde{x} = 0$:
$$\mathscr{B}^{(0)}(U) \equiv e^{-\tau}b_\tau - u + vb_{\tilde{y}} + wb_{\tilde{z}};$$

- On $\tilde{x} = 1$:

$$\mathscr{B}^{(1)}(U, \phi) \equiv \partial_\tau \phi[H_0] - e^\tau [H_1] + e^\tau \partial_{\tilde{y}} \phi[H_2] + e^\tau \partial_{\tilde{z}} \phi[H_3].$$

To simplify the notation, we will drop the $\tilde{}$ in the coordinates $(\tau, \tilde{x}, \tilde{y}, \tilde{z})$ in the following, and replace τ by t. In summary, the proof of Theorem 1.1 is reduced to finding the unknown functions (U, ϕ) near $t = -\infty$ in the domain $\omega = \{(t, x, y, z) : 0 < x < 1,\ t > -\infty\}$, satisfying:

- Interior equations:

$$\mathscr{L}(U, \phi) = 0 \qquad \text{in } \omega; \tag{3.6}$$

- Boundary conditions:

$$\mathscr{B}^{(0)}(U) = 0 \qquad \text{on } x = 0, \tag{3.7}$$

$$\mathscr{B}^{(1)}(U, \phi) = 0 \qquad \text{on } x = 1; \tag{3.8}$$

- "Initial" condition:

$$(U - \tilde{U}, \phi - \tilde{\phi}) = O(e^{\eta t}) \ \text{ at } \ t = -\infty. \tag{3.9}$$

Remark 3.1 *Through these transformations, the fixed boundary function $b(t, y, z)$ is incorporated into the coefficients of the operators $\mathscr{L}$ and $\mathscr{B}^{(0)}$, in the form of ∇b, i.e., containing the derivatives of order one, the same order as for the variable ϕ. This fact will not be used in this paper, but will be useful in future studies.*

By the coordinates transformations introduced above, the original initial-boundary value problem (1.7) with solution containing a shock front is now transformed into an equivalent problem (3.6)-(3.9). Therefore, in order to prove Theorem 1.1, we need only to prove the following

Theorem 3.1 *There exists a C^∞ solution (U, ϕ) to the boundary value problem (3.6)-(3.9).*

Theorem 3.1 will be proved by linear iteration of (3.6)-(3.9) near the approximate solution $(\tilde{U}, \tilde{\phi})$.

4 Linearization and energy estimate

First of all, we rewrite the problem (3.6)-(3.9). Let $(V, \psi) = (U - \tilde{U}, \phi - \tilde{\phi})$.

By the Taylor formula with Cauchy integral remainder, we have

$$f(x) = f(x_0) + \left[\int_0^1 f'(x_0 + \theta(x - x_0))d\theta\right](x - x_0).$$

Therefore, the operators $\mathscr{L}(U, \phi)$ and $\mathscr{B}^{(1)}(U, \phi)$ in (3.6) and (3.8) can be written as

$$\begin{cases} \mathscr{L}(U, \phi) = \mathscr{L}(\tilde{U}, \tilde{\phi}) + \mathbb{L}(V, \psi)(V, \psi), \\ \mathscr{B}^{(1)}(U, \phi) = \mathscr{B}^{(1)}(\tilde{U}, \tilde{\phi}) + \mathbb{B}^{(1)}(V, \psi)(V, \psi). \end{cases}$$

Then the problem (3.6)-(3.9) can be rewritten for (V, ψ) as follows

$$\begin{cases} \mathbb{L}(V, \psi)(V, \psi) = f \equiv -\mathscr{L}(\tilde{U}, \tilde{\phi}), \;\; 0 < x < 1, \\ \mathbb{B}^{(0)}V = \mathscr{B}^{(0)}(V) = 0, \;\; x = 0, \\ \mathbb{B}^{(1)}(V, \psi)(V, \psi) = g \equiv -\mathscr{B}^{(1)}(\tilde{U}, \tilde{\phi}), \;\; x = 1, \\ (V, \psi) = O(e^{\eta t}) \;\text{ at }\; t = -\infty. \end{cases} \tag{4.1}$$

Here in (4.1), the boundary condition on $x = 0$ remains unchanged since $\mathscr{B}^{(0)}$ is linear and $\tilde{U}$ satisfies $\mathscr{B}^{(0)}(\tilde{U}) = 0$ accurately.

Consider the following linearization of (4.1):

$$\begin{cases} \mathbb{L}(V, \psi)(\dot{V}, \dot{\psi}) = f, \;\; 0 < x < 1, \\ \mathbb{B}^{(0)}\dot{V} = 0, \;\; x = 0, \\ \mathbb{B}^{(1)}(V, \psi)(\dot{V}, \dot{\psi}) = g, \;\; x = 1, \\ (\dot{V}, \dot{\psi}) = O(e^{\eta t}) \;\text{ at }\; t = -\infty. \end{cases} \tag{4.2}$$

For fixed (V, ψ), (4.2) is a initial-boundary value problem for $(\dot{V}, \dot{\psi})$ in the domain ω. The linear stability of the linearized shock front problem (4.2) is discussed as follows, by combining the known results for solid boundary and for shock front.

For Euler system of gas-dynamics, the linearized solid boundary problem and the linearized shock wave front problem, both have already been well-studied see, e.g., [12, 13]. Near the solid boundary $x = 0$ and the shock front boundary $x = 1$, one can obtain the energy estimates and the existence of solutions.

Let $\omega^T = \omega \bigcap \{t; t < T\}$ and $k = (k_0, k_1, k_2, k_3)$ be the multiple index with $|k| = k_0 + k_1 + k_2 + k_3$. For any non-negative integer s, let $H^s_\eta(\omega^T)$ be the η-weighted Sobolev space with the norm

$$\|U\|^2_{H^s_\eta(\omega^T)} = \sum_{0\le|k|+2m\le s} \int_{\omega^T} |\partial_t^{k_0} D_x^{k_1} \partial_y^{k_2} \partial_z^{k_3} \partial_x^m (e^{-\eta t} U(x,y,z,t))|^2 dydzdxdt, \quad (4.3)$$

where η is a fixed sufficiently large constant, $D_x = x\partial_x$ is an operator tangential to the boundary $x = 0$. The space $H^s_\eta(\omega^T)$ is the usual η-weighted Sobolev space H^s away from the boundary $x = 0$. At $x = 0$, the regularity in the x-derivatives is reduced, see [1, 7].

The boundary spaces $H^s_\eta(\Gamma^T_j)$ can be similarly defined with

$$\Gamma^T_j = \{x; x = j\} \bigcap \{t; t < T\}, \ (j = 0, 1).$$

As with the standard Sobolev space, there are also similar embedding and trace results for the space $H^s_\eta(\omega^T)$, see also [1].

It is well known from [1, 13] that near the boundary $x = 0$, the solution $\dot{V}$ of (4.2) satisfies the energy estimate

$$\eta \|\varphi_0 \dot{V}\|^2_{H^s_\eta(\omega^T)} \le \frac{C_s}{\eta} \|f\|^2_{H^s_\eta(\omega^T)}, \quad (4.4)$$

where $\varphi_0(x)$ is a smooth function with $\varphi_0(x) = 1$ near $x = 0$ and $\varphi_0(x) = 0$ near $x = 1$.

Near the boundary $x = 1$, the solution $(\dot{V}, \dot{\psi})$ of (4.2) satisfies the energy estimate [12]

$$\begin{aligned} &\eta \|\varphi_1 \dot{V}\|^2_{H^s_\eta(\omega^T)} + \|\dot{V}\|^2_{H^s_\eta(\Gamma^T_1)} + \|\dot{\psi}\|^2_{H^{s+1}_\eta(\omega^T)} \\ &\quad \le C_s \left(\frac{1}{\eta} \|f\|^2_{H^s_\eta(\omega^T)} + \|g\|^2_{H^s_\eta(\Gamma^T_1)} \right). \end{aligned} \quad (4.5)$$

where $\varphi_1(x)$ is a smooth function with $\varphi_1(x) = 0$ near $x = 0$ and $\varphi_1(x) = 1$ near $x = 1$.

Using the usual localization technique, we can combine the two energy estimates in (4.4) and (4.5) to obtain an energy estimate for the solution $(\dot{V}, \dot{\psi})$ of (4.2) throughout the domain ω^T:

$$\begin{aligned} &\eta \|\dot{V}\|^2_{H^s_\eta(\omega^T)} + \|\dot{V}\|^2_{H^s_\eta(\Gamma^T_1)} + \|\dot{\psi}\|^2_{H^{s+1}_\eta(\omega^T)} \\ &\quad \le C_s \left(\frac{1}{\eta} \|f\|^2_{H^s_\eta(\omega^T)} + \|g\|^2_{H^s_\eta(\Gamma^T_1)} \right). \end{aligned} \quad (4.6)$$

With the energy estimate (4.6), we can use the standard dual argument to establish the existence of the solution $(\dot{V}, \dot{\psi})$ of (4.2) with specified regularity.

In summary, we obtained the following

Lemma 4.1 For $s \geq 5$, there exists a constant $\delta > 0$ such that $\forall (V, \psi) \in H^s_\eta(\omega^T) \times H^s_\eta(\Gamma^T_1)$ and

$$\eta \|\dot{V}\|^2_{H^s_\eta(\omega^T)} + \|\dot{V}\|^2_{H^s_\eta(\Gamma^T_1)} + \|\dot{\psi}\|^2_{H^{s+1}_\eta(\omega^T)} \leq \delta,$$

the linear problem (4.2), for every $(f, g) \in H^s_\eta(\omega^T) \times H^s_\eta(\Gamma^T_1)$, there exists a unique solution $(\dot{V}, \dot{\psi}) \in H^s_\eta(\omega^T) \times H^s_\eta(\Gamma^T_1)$, satisfying the energy estimate (4.6).

In addition, the constant C_s in (4.6) depends only upon δ and is independent of specific (V, ψ).

5 Linear iteration and existence of solution

We are now ready to use linear iteration to establish the existence of solution for the problem (4.1). In particular, one notices that in the energy estimate (4.6), the order of regularity is the same for $(\dot{V}, \dot{\psi})$ and for (f, g). So the standard iteration can be used instead of the more sophisticated Nash-Moser iteration.

Let $(V_0, \phi_0) = (0, 0)$ and (V_n, ϕ_n) $(n = 1, 2, \cdots)$ be the solution of the following linear boundary value problem

$$\begin{cases} \mathbb{L}(V_{n-1}, \psi_{n-1})(V_n, \psi_n) = f, \quad 0 < x < 1, \\ \mathbb{B}^{(0)} V_n = 0, \quad x = 0, \\ \mathbb{B}^{(1)}(V_{n-1}, \psi_{n-1})(V_n, \psi_n) = g, \quad x = 1. \end{cases} \tag{5.1}$$

Here, we drop the initial condition in (5.1) because it is automatically satisfied for solutions $(V_n, \psi_n) \in H^s_\eta(\omega^T) \times H^s_\eta(\Gamma^T_1)$.

From Lemma 4.1, we have the following

Lemma 5.1 $\forall s \geq 5$, assume that

$$\|V_{n-1}\|^2_{H^s_\eta(\omega^T)} + \|V_{n-1}\|^2_{H^s_\eta(\Gamma^T_1)} + \|\psi_{n-1}\|^2_{H^{s+1}_\eta(\omega^T)} \leq \delta, \tag{5.2}$$

there exists a unique solution (V_n, ψ_n) of (5.1), satisfying

$$\begin{aligned} &\eta\|V_n\|^2_{H^s_\eta(\omega^T)} + \|V_n\|^2_{H^s_\eta(\Gamma_1^T)} + \|\psi_n\|^2_{H^{s+1}_\eta(\omega^T)} \\ &\quad \leq C_s\left(\frac{1}{\eta}\|f\|^2_{H^s_\eta(\omega^T)} + \|g\|^2_{H^s_\eta(\Gamma_1^T)}\right). \end{aligned} \tag{5.3}$$

Here, the constant C_s depends upon s and δ, but independent of the (V_{n-1}, ψ_{n-1}).

Now we need only to show that the sequence (V_n, ψ_n) is well-defined for some fixed T and it is also convergent in an appropriate norm.

Lemma 5.2 $\exists T$ with $-T \gg 1$ such that for all $n = 0, 1, 2, \cdots$, we have

$$\|V_n\|^2_{H^s_\eta(\omega^T)} + \|V_n\|^2_{H^s_\eta(\Gamma_1^T)} + \|\psi_n\|^2_{H^{s+1}_\eta(\omega^T)} \leq \delta, \tag{5.4}$$

Proof: By (4.1), $f = -\mathscr{L}(\tilde{U}, \tilde{\phi})$ and $g = -\mathscr{B}^{(1)}(\tilde{U}, \tilde{\phi})$. Because $(\tilde{U}, \tilde{\phi})$ is an approximate solution of infinite order in Theorem 2.1, this implies that for any fixed s,

$$\lim_{T\to-\infty}\left(\|f\|^2_{H^s_\eta(\omega^T)} + \|g\|^2_{H^s_\eta(\Gamma_1^T)}\right) = 0.$$

Choose $-T \gg 1$ such that

$$\left(\|f\|^2_{H^s_\eta(\omega^T)} + \|g\|^2_{H^s_\eta(\Gamma_1^T)}\right) \leq \frac{\delta}{C_s}. \tag{5.5}$$

Combining (5.5) and (5.3) in Lemma 5.1 yields that (5.4) is satisfied for n, hence the sequence (V_n, ψ_n) is well-defined for such T.

Lemma 5.3 The sequence (V_n, ψ_n) is convergent in the space $H^s_\eta(\omega^T) \times H^s_\eta(\Gamma_1^T)$ to the solution (V, ψ) of (4.1).

Proof: From Lemma 5.2, it is already known that the sequence (V_n, ψ_n) is uniformly bounded in the space $H^s_\eta(\omega^T) \times H^s_\eta(\Gamma_1^T)$. By Banach-Saks Theorem, we need only to show that (V_n, ψ_n) is convergent in the space $H^0_\eta(\omega^T) \times H^0_\eta(\Gamma_1^T)$.

Let

$$(\dot{V}_n, \dot{\psi}_n) = (V_{n+1} - V_n, \psi_{n+1} - \psi_n), \ \ n = 0, 1, 2, \cdots.$$

Then $(\dot{V}_n, \dot{\psi}_n)$ satisfies

$$\begin{cases} \mathbb{L}(V_n, \psi_n)(\dot{V}_n, \dot{\psi}_n) \\ \qquad = [\mathbb{L}(V_{n-1}, \psi_{n-1}) - \mathbb{L}(V_n, \psi_n)](V_n, \psi_n), \quad 0 < x < 1, \\ \mathbb{B}^{(0)} \dot{V}_n = 0, \quad x = 0, \\ \mathbb{B}^{(1)}(V_n, \psi_n)(\dot{V}_n, \dot{\psi}_n) \\ \qquad = [\mathbb{B}^{(1)}(V_{n-1}, \psi_{n-1}) - \mathbb{B}^{(1)}(V_n, \psi_n)](V_n, \psi_n), \ x = 1. \end{cases} \tag{5.6}$$

Applying the estimate (4.6) in Lemma 4.1 with $s = 0$ to the problem (5.6), we have

$$\begin{aligned} &\eta \|\dot{V}_n\|^2_{H^0_\eta(\omega^T)} + \|\dot{V}_n\|^2_{H^0_\eta(\Gamma^T_1)} + \|\dot{\psi}_n\|^2_{H^1_\eta(\omega^T)} \\ &\quad \leq C'_s |\!|\!| (\dot{V}_{n-1}, \dot{\psi}_{n-1}) |\!|\!|^2_0 \ |\!|\!| (V_n, \psi_n) |\!|\!|^2_s. \end{aligned} \tag{5.7}$$

Here,

$$|\!|\!| (V_n, \psi_n) |\!|\!|^2_s \equiv \|V_n\|^2_{H^s_\eta(\omega^T)} + \|V_n\|^2_{H^s_\eta(\Gamma^T_1)} + \|\psi_n\|^2_{H^s_\eta(\Gamma^T_1)}.$$

For sufficiently small δ, (5.7) implies that the sequence $(\dot{V}_n, \dot{\psi}_n)$ is contracting and hence the sequence (V_n, ψ_n) converges in $H^0_\eta(\omega^T) \times H^0_\eta(\Gamma^T_1)$. This concludes the proof of the existence of solution.

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