SCIENCE CHINA
Mathematics

Progress of Projects Supported by NSFC
• REVIEWS •

April 2017 Vol. 60 No. 4: 581–592
doi: 10.1007/s11425-016-0437-x

Generalized Riemann problem for Euler system

Dedicated to Professor LI TaTsien on the Occasion of His 80th Birthday

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Received July 6, 2016; accepted December 26, 2016; published online February 10, 2017

Abstract This article is a survey on the progress in the study of the generalized Riemann problems for M-D Euler system. A new result on generalized Riemann problems for Euler systems containing all three main nonlinear waves (shock, rarefaction wave and contact discontinuity) is also introduced.

Keywords Euler system, Riemann problem, shock, rarefaction wave, contact discontinuity or vortex sheet

MSC(2010) 35F55, 35L45, 76N15, 76N10

Citation: Chen S X, Li D N. Generalized Riemann problem for Euler system. Sci China Math, 2017, 60: 581–592, doi: 10.1007/s11425-016-0437-x

1 Euler system and Riemann problem

Euler system of equations is the governing differential equations describing the state of the inviscid gas in hydrodynamics. It consists of the conservations of mass, momentum and total energy, and appears in many applications and has been widely studied both theoretically and numerically. The methods and techniques in its study are also extended and applied to the study of many other more general systems of conservation laws. In this article, we try to give a concise review of the achievements and the methods in the study of the generalized Riemann problem (GRP for short in the sequel) for the Euler system of equations, especially in the multiple dimensional spaces.

The Euler system for compressible non-viscous flow in 3-D space can be written as follows:

$$\begin{cases} \partial_t\rho + \partial_x(\rho u) + \partial_y(\rho v) + \partial_z(\rho w) = 0, \\ \partial_t(\rho u) + \partial_x(p + \rho u^2) + \partial_y(\rho uv) + \partial_z(\rho uw) = 0, \\ \partial_t(\rho v) + \partial_x(\rho uv) + \partial_y(p + \rho v^2) + \partial_z(\rho vw) = 0, \\ \partial_t(\rho w) + \partial_x(\rho uw) + \partial_y(\rho vw) + \partial_z(p + \rho w^2) = 0, \\ \partial_t(\rho E) + \partial_x(\rho Eu + pu) + \partial_y(\rho Ev + pv) + \partial_z(\rho Ew + pw) = 0, \end{cases} \tag{1.1}$$

where (ρ, p, e) are the density, pressure, and the internal energy of the fluid, (u, v, w) is the velocity in the (x, y, z) direction, and $E = e + \frac{1}{2}(u^2 + v^2 + w^2)$. Only two of the three thermodynamic variables (ρ, p, e) are independent. If the gas is polytropic, then we have the state function $p = A(S)\rho^\gamma$ with $\gamma > 1$.

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Denote

$$H_0 = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho E \end{pmatrix}, \quad H_1 = \begin{pmatrix} \rho u \\ p+\rho u^2 \\ \rho uv \\ \rho uw \\ (\rho E+p)u \end{pmatrix}, \quad H_2 = \begin{pmatrix} \rho v \\ \rho uv \\ p+\rho v^2 \\ \rho vw \\ (\rho E+p)v \end{pmatrix}, \quad H_3 = \begin{pmatrix} \rho w \\ \rho uw \\ \rho vw \\ p+\rho w^2 \\ (\rho E+p)w \end{pmatrix}.$$

Then system (1.1) can be written briefly as $\partial_t H_0 + \partial_x H_1 + \partial_y H_2 + \partial_z H_3 = 0$.

A piece-wise smooth function (ρ, p, u, v, w) which has a jump discontinuity along a smooth curve $\Phi(x, y, z, t) = 0$ is called a weak solution (or simply a solution) if

- (ρ, p, u, v, w) satisfies (1.1) in the two regions $\pm\Phi(x, y, z, t) > 0$;
- at $\Phi(x, y, z, t) = 0$, the following Rankine-Hugoniot conditions are satisfied:

$$\Phi_t[H_0]^+_- + \Phi_x[H_1]^+_- + \Phi_y[H_2]^+_- + \Phi_z[H_3]^+_- = 0, \tag{1.2}$$

where $[f]^+_- = f^+ - f_-$ denotes the jump difference of the function f along the curve $\Phi(x, y, z, t) = 0$.

Introducing the unknown vector of functions $U = (p, u, v, w, S)$, it is well known (see [15, 49]) that for the smooth solutions, (1.1) is equivalent to the following system:

$$\begin{cases} \dfrac{\partial p}{\partial t} + (u, v, w) \cdot \nabla p + \rho c^2 \nabla \cdot (u, v, w) = 0, \\ \rho \dfrac{\partial u}{\partial t} + \rho(u, v, w) \cdot \nabla u + \dfrac{\partial p}{\partial x} = 0, \\ \rho \dfrac{\partial v}{\partial t} + \rho(u, v, w) \cdot \nabla v + \dfrac{\partial p}{\partial y} = 0, \\ \rho \dfrac{\partial w}{\partial t} + \rho(u, v, w) \cdot \nabla w + \dfrac{\partial p}{\partial z} = 0, \\ \dfrac{\partial S}{\partial t} + (u, v, w) \cdot \nabla S = 0, \end{cases} \tag{1.3}$$

with $c^2 = p'_\rho(\rho, S) > 0$.

System (1.3) can be further written into the following symmetric form:

$$\mathscr{L}U \equiv A_0 \partial_t U + A_1(U)\partial_x U + A_2(U)\partial_y U + A_3(U)\partial_z U = 0, \tag{1.4}$$

where

$$A_0 = \begin{bmatrix} \frac{1}{\rho c^2} & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & \rho & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} \frac{u}{\rho c^2} & 1 & 0 & 0 & 0 \\ 1 & \rho u & 0 & 0 & 0 \\ 0 & 0 & \rho u & 0 & 0 \\ 0 & 0 & 0 & \rho u & 0 \\ 0 & 0 & 0 & 0 & u \end{bmatrix},$$

$$A_2 = \begin{bmatrix} \frac{v}{\rho c^2} & 0 & 1 & 0 & 0 \\ 0 & \rho v & 0 & 0 & 0 \\ 1 & 0 & \rho v & 0 & 0 \\ 0 & 0 & 0 & \rho v & 0 \\ 0 & 0 & 0 & 0 & v \end{bmatrix}, \quad A_3 = \begin{bmatrix} \frac{w}{\rho c^2} & 0 & 0 & 1 & 0 \\ 0 & \rho w & 0 & 0 & 0 \\ 0 & 0 & \rho w & 0 & 0 \\ 1 & 0 & 0 & \rho w & 0 \\ 0 & 0 & 0 & 0 & w \end{bmatrix}.$$

More generally, consider an $m \times m$ system of differential equations for $U \in \mathbb{R}^m$, i.e.,

$$\sum_{k=0}^{n} A_k(U)\partial_{x_k} U = f. \tag{1.5}$$

The system (1.5) is called a symmetric quasi-linear hyperbolic system if all the matrices $A_k(U)$ ($k = 0, 1, \dots, n$) are symmetric and the matrix $A_0(U)$ is positively definite.

In system (1.4), A_0, A_1, A_2 and A_3 are all symmetric and $A_0 > 0$ for $\rho > 0$. Therefore, Euler system of (1.3) is a special case and the most important example of the symmetric hyperbolic system.

The symmetric hyperbolic system has been studied in [20, 29, 30]. [20, 30] also studied a more general form of positive symmetric system which includes the symmetric hyperbolic system (1.5) as a special case, and may further include other type of systems, including some mixed type equations.

For (1.5), one may associate an initial condition for U satisfying on $x_0 = t = 0$,

$$U(x, x_0) = U_0(x). \tag{1.6}$$

The combination of (1.5) and (1.6) is called a Cauchy problem. Cauchy problem is one of the basic problems in the study of partial differential equations.

The well-posedness for the Cauchy problem (as well as initial-boundary value problem, and some other problems) of the linear symmetric hyperbolic system (1.5) has been well established [20, 21, 30, 48]. It is also well known that for smooth initial data, the system admits a smooth solution local in time. However, for the quasi-linear hyperbolic system (1.5), the smooth initial data may develop singularities in finite time, no matter how smooth the initial data are.

Therefore, it is necessary and important, both in theory and in application, to study the Cauchy problem with discontinuous initial data. The simplest case with such discontinuous initial data is the Riemann problem. For the Riemann problem, the initial data $U_0(x)$ consist of two constant states on the two sides of a hyperplane.

Because Euler system of (1.1) is both rotation- and translation-invariant, without loss of generality, the Riemann problem for the Euler system of (1.1) can be written as

$$\begin{cases} \partial_t H_0 + \partial_x H_1 + \partial_y H_2 + \partial_z H_3 = 0, \\ U(x, y, z, 0) = \begin{cases} U_- & \text{for} \quad x < 0, \\ U_+ & \text{for} \quad x > 0. \end{cases} \end{cases} \tag{1.7}$$

Here in (1.7), (U_-, U_+) are two constant states for U.

Since the initial data (U_-, U_+) are independent of the space variables (y, z), the solution U for (1.7) should also be independent of the space variables (y, z). Therefore, the Riemann problem (1.7) actually becomes a problem in one space dimension x. Since the (v, w) components of the velocity are all constants, (1.7) (or (1.1)) becomes

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(p + \rho u^2), \\ \partial_t(\rho E) + \partial_x(\rho E u + p u) = 0, \end{cases} \qquad (\rho, u, E)(0, x) = \begin{cases} (\rho_-, u_-, E_-), & \text{for} \ \ x < 0, \\ (\rho_+, u_+, E_+), & \text{for} \ \ x > 0. \end{cases} \tag{1.8}$$

The Riemann problem (1.8) has been extensively studied by many authors. In general, a solution for (1.9) may contain three different waves: shock wave, rarefaction wave and contact discontinuity.

A shock wave is a solution which contains a jump discontinuity in (ρ, p) as well as the normal velocity along a curve (shock front) $x = \phi(t)$ and satisfies the Rankine-Hugoniot conditions (1.2).

A rarefaction wave is a simple wave where one Riemann invariant is a constant.

A contact is a characteristic curve along which the velocity u and the pressure p are continuous, but the density ρ may have a jump discontinuity.

For the Riemann problem (1.8), or more generally, for the Riemann problem of a hyperbolic conservation laws, the following result is well-known [15, 16, 49].

Theorem 1.1. *For a system of hyperbolic conservation laws, if each characteristic field is either genuinely nonlinear or linearly degenerate, the two initial states are sufficiently close to each other, then the Riemann problem has a solution, which consists of shocks, centered simple waves and contact discontinuity.*

2 GRP for 1-D Euler system of equations

The generalized Riemann problem (or GRP) [40] for the Euler system of equations (1.1) is the following:

$$\begin{cases} \partial_t H_0 + \partial_x H_1 + \partial_y H_2 + \partial_z H_3 = 0, \\ U(x,y,z,0) = \begin{cases} U_-(x,y,z) & \text{for} \quad x < \phi_0(y,z), \\ U_+(x,y,z) & \text{for} \quad x > \phi_0(y,z). \end{cases} \end{cases} \tag{2.1}$$

Here in (2.1), $\phi_0(y,z)$ is a smooth surface with (without loss of generality) $\phi_0(0,0) = 0$ and $\nabla\phi_0(0,0) = 0$. (U_-, U_+) are two smooth functions in $x < \phi_0(y,z)$ and $x > \phi_0(y,z)$, respectively.

The simplest form of GRP (2.1) is the case when $\phi_0(y,z) = 0$ and the initial data (U_-, U_+) are functions of the variable x only. Then the GRP (2.1) can be written as:

$$\begin{cases} \begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(p + \rho u^2) = 0, \\ \partial_t(\rho E) + \partial_x(\rho E u + pu) = 0, \end{cases} \\ U(x,0) = \begin{cases} U_-(x) & \text{for} \quad x < 0, \\ U_+(x) & \text{for} \quad x > 0. \end{cases} \end{cases} \tag{2.2}$$

Indeed, the initial value problem (2.2) can be reduced to the case of one space dimension and the solution $U(x,y,z,t) = U(x,t)$.

For the nonlinear hyperbolic system with two independent variables, there have long been many research works. The local existence of C^1 solutions to the Cauchy problem of 1-D Euler equations (or more general nonlinear hyperbolic systems) was established by Friedrichs [19] and Douglis [17].

When the initial data are not piece-wise constant, the 1-D GRP (2.2) has been extensively studied in [22–24, 31–34], as well as the monograph [40]. A complete theory about the piece-wise smooth solutions for (2.2) has been established in their work. In particular, the approach in [34, 40] was to fold the different regions separated by the wave boundaries into a standard corner domain, coupled with the functions defining the free boundary. For the resulting boundary value problem, there is no characteristic curve entering into the region from the origin. It was called the typical boundary value problem in [34, 40], and was thoroughly studied. By using the technique and estimate of integration along the characteristics, they then obtain the existence of the piece-wise smooth solution.

Specifically, consider the following special Riemann problem obtained from (2.2) by freezing the initial data at the origin $(x,t) = (0,0)$:

$$\begin{cases} \begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(p + \rho u^2) = 0, \\ \partial_t(\rho E) + \partial_x(\rho E u + pu) = 0, \end{cases} \\ U(x,0) = \begin{cases} U_-(0) & \text{for} \quad x < 0, \\ U_+(0) & \text{for} \quad x > 0. \end{cases} \end{cases} \tag{2.3}$$

Then we have the following [40].

Theorem 2.1. *For the generalized Riemann problem* (2.2) *for the Euler equations, if* $|U_-(0) - U_+(0)|$ *is sufficiently small, then there is a piece-wise smooth solution near the origin which contains exactly the same wave patterns* (*including shocks, centered simple waves and contact discontinuities*) *as the corresponding Riemann problem* (2.3).

In particular, they showed that for the GRP (1.1) and (2.2) with non-constant initial data $(U_-(x), U_+(x))$, one still has the similar wave structure as Riemann problem, i.e., the solution may contain shock, rarefaction wave and contact discontinuity.

The literature on the 1-D GRP is very rich and numerous. Readers are referred to [16, 40] for more references.

3 GRP for 3-D Euler equations: Shock waves

For the genuine 3-D GRP (2.1), one has to deal with some new difficulties.

(1) First, the characteristic curves in 1-D case become characteristic varieties and the usual integration along characteristic curves and the estimate in C^1 norms do not work anymore.

(2) The function $x = \phi(y, z, t)$ describing a 3-D wave boundary (such as shock front, simple wave or contact discontinuity) cannot be estimated without loss of regularity from a single relation in the Rankine-Hugoniot conditions, and has to be dealt with more delicate analysis.

(3) There are new wave structures for the contact discontinuity, i.e., instead of only discontinuity in the density, there could also be a jump in the tangential flow velocity, or vortex sheet, which is known to be linearly instable in general.

The first result for the genuine 3-D GRP for the Euler equation (1.1) was obtained by Majda [41, 42] for the 3-D shock waves.

A 3-D shock wave for (1.1) consists of a piece-wise smooth solution U of (1.1) and a smooth surface $\phi(x, y, z, t) = 0$ along which U has a jump discontinuity and satisfies the Rankine-Hugoniot conditions

$$\phi_t[H_0]_-^+ + \phi_x[H_1]_-^+ + \phi_y[H_2]_-^+ + \phi_z[H_3]_-^+ = 0. \tag{3.1}$$

For the stability of the shock wave, we will always assume that the shock is compressive so that Lax's shock inequality is satisfied.

In [42], the stability of the linearized 3-D shock wave problem is established, and the existence of a solution containing a shock issuing from a prescribed smooth curve $x = \phi_0(y, z)$ is obtained.

The key issue in his success is the derivation of an a priori energy estimate for the solution (including both U and ϕ) of the linearized shock front problem.

After some coordinates transformation to fix and flatten the free boundary, the linearized shock front problem of (1.1) can be reduced to an initial-boundary value problem for the perturbation unknown function $(\dot{U}, \dot{\varphi})$ in the domain $\Omega = \{(x, y, z, t) : x > 0, t > 0\}$, i.e.,

$$\begin{cases} A_0(U,\phi)\partial_t\dot{U} + A_1(U,\phi)\partial_x\dot{U} + A_2(U,\phi)\partial_y\dot{U} + A_3(U,\phi)\partial_z\dot{U} \\ \quad + C(U,\phi)(\dot{U}, \dot{\varphi}_t, \nabla\dot{\varphi}) = f, \quad \text{in } x > 0, \quad t > 0, \\ [H_0]_-^+\partial_t\dot{\varphi} + [H_2]_-^+\partial_y\dot{\varphi} + [H_3]_-^+\partial_z\dot{\varphi} \\ \quad + B(U,\phi)\dot{U} = g, \quad \text{on } x = 0, \quad t > 0, \\ \dot{U}(x, y, z, 0) = 0, \quad \dot{\varphi} = 0, \quad x > 0, \quad t \leqslant 0. \end{cases} \tag{3.2}$$

Here in (3.2), $C(U, \phi)$ is a zero-order operator on $(\dot{U}, \dot{\varphi}_t, \nabla\dot{\varphi})$, and $B(U, \phi)$ is a 5×5 matrix, derived from the linearization with respect to U only of the Rankine-Hugoniot condition.

Since the first order derivatives of $\dot{\varphi}$ appear in the form of $C(U, \phi)(\dot{U}, \dot{\varphi}_t, \nabla\dot{\varphi})$ in the interior equations of (3.1), one has to derive the estimate for $(\dot{\varphi}_t, \nabla\dot{\varphi})$ from the boundary condition in (3.1). The key in the linear stability analysis in [42] is to notice that the boundary operator for $\dot{\varphi}$, $[H_0]_-^+\partial_t + [H_2]_-^+\partial_y + [H_3]_-^+\partial_z$ is an over-determined 1st order elliptic system. One can obtain the estimate for the first-order derivative norm of $\dot{\varphi}$ only by deriving it micro-locally on the unit circle in the dual space of (t, y, z). This necessitates considering the boundary value problem for a hyperbolic system with boundary conditions given by zero order pseudo-differential equations, instead of the usual algebraic equations. It happens that the uniform Lopatinsky condition for the initial-boundary value problem of hyperbolic system in [28] has exactly the micro-local form in the dual space which provides the necessary and sufficient conditions for the linear stability of multi-D shock front.

The main result of [42] is the derivation of the a priori energy estimate, assuming Lax's shock inequality, for the solution (U, ϕ) of (1.1) and (3.1),

$$\eta\|\dot{U}\|^2_{H^s_\eta(\Omega)} + \|\dot{U}\|^2_{H^s_\eta(\partial\Omega)} + \|\dot{\phi}\|^2_{H^{s+1}_\eta(\partial\Omega)} \leqslant C_s\left(\frac{1}{\eta}\|f\|^2_{H^s_\eta(\Omega)} + \|g\|^2_{H^s_\eta(\partial\Omega)}\right). \tag{3.3}$$

Here $\|\cdot\|_{H^s_\eta}$ is the usual η-weighted norm of Sobolev space H^s_η.

Using the energy estimate (3.3) and assuming some necessary but complicated compatibility conditions on the initial data (U_0, ϕ_0), Majda [41] proved the existence of shock wave solution (U, ϕ) for (1.1) and (3.1), using the Newtonian iteration which, however, is not necessary here and can be replaced by a simpler modified iteration, see [4].

Notice that the success of the iteration in [4,41] depends upon the nature of the energy estimate (3.3), where the order of the estimated norms for $(\dot{U}, \dot{\phi})$ matches exactly with the order of (U, ϕ) appearing in the coefficients in the linearized problem (3.2). This will not be the case for other nonlinear waves in three dimensional case.

Applying similar technique to the second order quasi-linear equations, the 3-D shock wave for an isentropic irrotational flow is studied in [44]. The combination of two shock waves was studied in [45] for a 2-D 2×2 conservation laws, and was obtained for 3-D Euler equations using the result obtained in [42], coupled with a blow-up at the origin, and by a simplified iteration procedure. Also the sonic wave was studied in [46] as the limit of a shock wave as shock strength goes to zero.

4 GRP for 3-D Euler equations: Rarefaction waves

The genuine 3-D rarefaction wave solution for the Euler equations (1.1) was first obtained by Alinhac [1]. To simplify our presentation, we will assume in the following that (1.1) has already been reduced to the form $\frac{\partial U}{\partial t} + A_1\frac{\partial U}{\partial x} + A_2\frac{\partial U}{\partial y} + A_3\frac{\partial U}{\partial z} = 0$.

Without loss of generality, consider the right-propagating rarefaction wave. A right-propagating rarefaction wave solution for (1.1) can be formulated as a set of smooth functions (U_-, U_r, χ, U_+) near the origin $(0, 0, 0, 0)$ such that

• (U_-, U_r, U_+) satisfies (1.1) separately in each of the corner domains $(\Omega_-, \Omega_r, \Omega_+)$ defined by (see Figure 1)

$$\begin{cases} \Omega_- = \{x < \chi^-(t, y, z), t > 0\}, \\ \Omega_r = \{\chi^-(t, y, z) < x < \chi^+(t, y, z), t > 0\}, \\ \Omega_+ = \{\chi^+(t, y, z) < x, t > 0\}, \end{cases} \tag{4.1}$$

with $\phi_0(y, z) = \chi^-(0, y, z) = \chi^+(0, y, z)$.

• $x = \chi(t, s, y, z), (0 \leqslant s \leqslant 1)$ is a parametrization of the domain Ω_r with $\chi(t, 0, y, z) = \chi^-(t, y, z)$, $\chi(t, 1, y, z) = \chi^+(t, y, z)$ and $\chi_s = \gamma(t, s, y, z)t$ with $\gamma(t, s, y, z) \geqslant \delta > 0$. Indeed, $x = \chi(t, s, y, z)$ $(0 \leqslant s \leqslant 1)$ is a family of characteristics issuing from Γ for each $s \in [0, 1]$, such that

$$\det|A_1 - \chi_t - \chi_y A_2 - \chi_z A_3| = 0, \tag{4.2}$$

or more specifically,

$$\chi_t = \lambda_+(U_r; \nabla\chi), \tag{4.3}$$

where $\lambda_+(U; \phi)$ is the maximal eigenvalue of $A_1 - \chi_y A_2 - \chi_z A_3$.

• Let the function $W(t, s, y, z)$ be defined by

$$W(t, s, y, z) = U_r(t, \chi(t, s, y, z), y, z). \tag{4.4}$$

Then $W(t, s, y, z)$ satisfies

$$\tilde{\mathscr{L}}W \equiv \chi_s\left(\frac{\partial W}{\partial t} + A_2\frac{\partial W}{\partial y} + A_3\frac{\partial W}{\partial z}\right) + (A_1 - \chi_t - \chi_y A_2 - \chi_z A_3)\frac{\partial W}{\partial s} = 0. \tag{4.5}$$

Since the surface $x = \chi^+(t, y, z)$ is characteristic for (1.1), the function U_+ is uniquely determined in Ω_+ by the initial data $U_{0+}(x, y, z)$. To find the rarefaction wave solution, one needs only to determine the functions (U_-, W, χ).

The existence of the rarefaction wave is obtained by the energy estimate for the linearized problem and then by iteration.

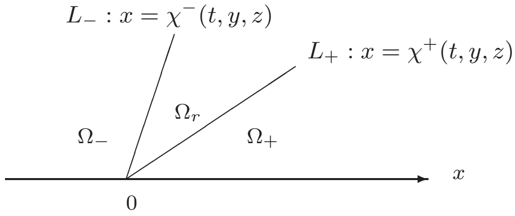

Figure 1 Rarefaction wave solution for (1.1)

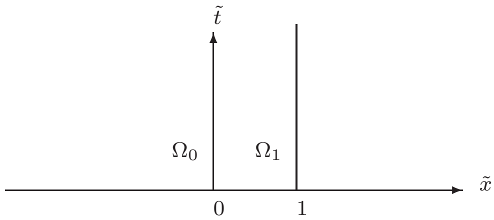

Figure 2 Rarefaction wave configuration on $(\tilde{t}, \tilde{x})$ plane

First of all a singular coordinates transformation is performed to change the angular domains (Ω_-, Ω_r) into standard cylindrical domains (Ω_0, Ω_1) with fixed boundary (see Figure 2)

$$\Omega_j = \{(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z}) : \tilde{t} > 0,\ j - 1 < \tilde{x} < j\}, \quad j = 0, 1.$$

We drop the tilde in the coordinates $(\tilde{t}, \tilde{x})$ to simplify the notation.

The linearization of the transformed rarefaction wave problem yields a boundary value problem for a symmetric hyperbolic system, which has first order degeneracy at $t = 0$ in Ω_1, with characteristic boundary conditions on $x = 0$ and $x = 1$. Because of the degeneracy at $t = 0$, a t^γ weighted norm is used in deriving the energy estimate $\|U\|^2_{H^0_\gamma(\Omega)} = \int_\Omega t^{2\gamma}|U(t, x, y, z)|^2 dtdxdydz$.

Briefly writing the linearized rarefaction wave problem as follows:

$$\begin{cases} \mathbb{L}_a(U, \phi)(\dot{V}, \dot{\phi}) = \dot{F}, \\ \mathbb{B}_a(U, \phi)(\dot{V}, \dot{\phi}) = \dot{G}, \end{cases} \tag{4.6}$$

the solution $(\dot{V}, \dot{\phi})$ of (4.6) then satisfies the following estimate:

$$\|\dot{V}\|_{H^s_\gamma(\Omega)} + \|\dot{\phi}\|_{H^{s-1}_\gamma(\partial\Omega)} \leqslant C[\|\dot{F}\|_{H^s_\gamma(\Omega)} + \|\dot{G}\|_{H^{s+1}_\gamma(\partial\Omega)}]. \tag{4.7}$$

The outstanding feature in (4.7) is the loss of regularity in the solution $(\dot{V}, \dot{\phi})$ compared with the data $(\dot{F}, \dot{G})$. Such kind of energy estimates are also called tame estimates as in [25] which are present for a wide range of problems where various kinds of degeneracy are involved. This necessitates the use of the Nash-Moser type of iteration [25, 26, 47]. Nash-Moser iteration (or Nash-Moser implicit function theorem) is a powerful tool to establish the existence of solution for some nonlinear problems of which the linearized problems have an energy estimate with loss of regularity order. It was first applied to the nonlinear elliptic problem and later adapted to the evolutionary equations [27]. It is a modified Newtonian iteration, consisting of a mollification at every step of iteration and is coupled with further minor modifications according to the nature of the problem. In [1], the iteration process has to make more adjustment to accommodate the errors induced from the requirement of uniformly characteristic boundary, see [48]. Finally, the local existence of the solution with a rarefaction wave for the Cauchy problem of (1.1) was successfully proved in [1] by using Nash-Moser iteration, provided the corresponding compatibility conditions on the initial data are satisfied.

Incorporating the techniques for both the 3-D shock waves in [41, 42] and 3-D rarefaction waves in [1], a combination of shock and rarefaction waves for 3-D Euler equations was also obtained in [35].

5 GRP for Euler equations: Contact discontinuity

A vortex sheet solution (or contact discontinuity) of (1.1) is a set of functions (U_-, U_+, ϕ) such that

(1) (U_-, U_+) satisfy (1.1) in the domains $x \pm \phi(y, z, t) > 0$, respectively.

(2) On the surface $x = \phi(y, z, t)$, a special set of Rankine-Hugoniot conditions is satisfied, i.e.,

$$u_\pm - \phi_y v_\pm - \phi_z w_\pm = 0, \quad \text{and} \quad p_- = p_+. \tag{5.1}$$

Of all the three types of waves for the 3-D Euler equations, i.e., shock, rarefaction wave and contact discontinuity, the solution containing a contact discontinuity (or vortex sheet) is the most elusive to obtain. Indeed, it is shown in [2, 3] that a genuine 3-D vortex sheet is highly unstable and various kinds of kinks would develop with minor perturbation.

Nevertheless, Coulombel and Secchi [12] showed that in the two-dimensional case, if the tangential velocity jump is suitably large, the linearized vortex sheet is stable in a weak sense. In addition, based upon this fact, they also obtained in [13, 14] the existence of such vortex sheet solution.

In [12], a special parametrization and hence coordinate transformation, inspired in [18], is chosen to satisfy the eikonal equations $\partial_t \phi_\pm + v_\pm \partial_y \phi_\pm - u_\pm = 0$ so that the contact discontinuity is uniformly characteristic in the whole concerned domain.

Consequently, after folding the domain $x < 0$ into $x > 0$, the linearized vortex sheet problem for $(\dot{U}, \dot{\phi})$ can be written as the following boundary value problem:

$$\begin{cases} \partial_t \dot{U} + A_1(U)\partial_x \dot{U} + A_2(U)\partial_y \dot{U} = f, \quad \text{in } x > 0, \\ B(\dot{U}, \dot{\phi}) \equiv \begin{pmatrix} (v_+ - v_-)\partial_y \dot{\phi} - (\dot{u}_+ - \dot{u}_-) \\ \partial_t \dot{\phi} + v_r \partial_y \dot{\phi} - \dot{u}_+ \\ \dot{\rho}_+ - \dot{\rho}_- \end{pmatrix} = g, \quad \text{on } x = 0. \end{cases} \tag{5.2}$$

For the linearized vortex sheet problem (5.2), the following is proved in [12].

Theorem 5.1. *If the tangential velocity difference satisfies "supersonic" condition*

$$|v_+ - v_-| > 2\sqrt{2}c, \tag{5.3}$$

where $v_\pm$ is the tangential velocity in the both sides of the curve carrying initial discontinuity, c is the corresponding sound speed, then a weakly Lopatinsky condition in the sense of [43] *is satisfied, and hence the result obtained in* [43] *is applicable. An energy estimate of the following form is obtained*:

$$\eta \|U\|^2_{H^0_\gamma(\Omega)} + \|U\|^2_{H^0_\gamma(\partial\Omega)} + \|\phi\|^2_{H^1_\gamma(\partial\Omega)} \leqslant C\left[\frac{1}{\eta}\|f\|^2_{H^1_\gamma(\Omega)} + \|g\|^2_{H^1_\gamma(\partial\Omega)}\right], \tag{5.4}$$

which shows the loss of one order regularity and it is the so-called "tame" estimate in [25].

A k-th order version of the energy estimate (5.4) is also available.

The two important features in the proof of Theorem 5.1 are the following:

• Notice that the operator on $\dot{\phi}$ in $B(\dot{U}, \dot{\phi})$,

$$\begin{cases} (v_+ - v_-)\partial_y \dot{\phi}, \\ \partial_t \dot{\phi} + v_r \partial_y \dot{\phi}, \end{cases}$$

is an overdetermined first order elliptic system, similar to the operator $[H_0]^+_- \partial_t + [H_2]^+_- \partial_y + [H_3]^+_- \partial_z$ in (3.2) for the linearized shock wave problem. It is also obvious from here that the "2-D" condition is necessary and cannot be generalized into the "3-D" case.

• Eliminating microlocally $\dot{\phi}$ from the boundary conditions in (5.2) yields a boundary value problem with microlocal boundary conditions for $\dot{U}$ and with uniformly characteristic boundary. One can apply the result in [43]. Indeed, the requirement (5.3) in Theorem 5.1 is equivalent to the weakly Lopatinski condition in [43], which is nothing but the uniformly Lopatinski condition applied to the non-degenerate components of the variable $\dot{U}$.

Using the "tame" energy estimates of (5.4) and all its high-order version, the Nash-Moser type of iteration is used in [13]. To accommodate the uniform characteristic requirement in the weakly Lopatinski condition [43], extra modification errors were introduced, in addition to the usual quadratic and mollification errors. Such errors also need to be taken into account when performing the Nash-Moser iteration. Specifically, for the following GRP of 2-D Euler equations:

$$\begin{cases} \partial_t H_0 + \partial_x H_1 + \partial_y H_2 = 0, \\ U(x,y,0) = \begin{cases} U_-(x,y) & \text{for} \quad x < \phi_0(y), \\ U_+(x,y) & \text{for} \quad x > \phi_0(y), \end{cases} \end{cases} \tag{5.5}$$

it is obtained in [13].

Theorem 5.2. *Let*

- $\phi_0(y)$ *be sufficiently smooth with* $\phi_0(0) = \phi_{0y}(0) = 0$;
- (U_-, U_+) *be sufficiently smooth in their respective domain and satisfy on* $x = \phi_0(y)$:

$$p_- = p_+, \quad u_\pm - v_\pm \phi_{0y} = 0;$$

- $|v_+ - v_-| > 2\sqrt{2}c$.

Then (5.5) *has a unique piece-wise smooth vortex sheet solution* (U_-, U_+, ϕ) *near the origin, provided the corresponding compatibility condition on* $x = \phi_0(y)$.

6 GRP for Euler equations with all three waves

In all the above work, three nonlinear waves are studied separately [1, 12, 13, 41, 42] or at most the combination of two waves. Special care has to be taken in the above studies to ensure that only the desired waves are present, usually in the form of complicated compatibility conditions. As is well known from the 1-D GRP, given an arbitrary piece-wise smooth initial data, the solution should contain all three different waves in general. To obtain such a solution, we need to combine the techniques used in all the previous studies for different waves.

A solution containing all three possible nonlinear waves (shock, rarefaction wave and vortex sheet) for isentropic 2-d Euler system has been obtained by Chen and Li [10]. The main result of [10] is the following.

Theorem 6.1. *Assume that* P *is a point on the initial discontinuity curve of the initial data* (U_{0-}, U_{0+}), *and*

(1) *the* 1*-D simplified model at* P (*which is a* 1*-D Riemann problem*) *has a solution containing a complete nonlinear wave patterns* (*non-degenerate shock or rarefaction wave*) *with shock satisfying Lax's shock inequality*;

(2) *at the point* P, *the supersonic tangential velocity condition* (5.3) *is satisfied.*

Then the 2*-D generalized Riemann problem of the isentropic Euler system admits a unique piece-wise smooth solution with fan-shaped wave structure containing shock, rarefaction wave and vortex sheet in a neighborhood of* P.

Main steps of the proof are as follows:

(1) First of all, the compatibility issue is discussed and an approximate solution of infinite order is constructed.

(2) Reformulate the problem by fixing and flattening the free boundaries.

(3) Linearize the problem and obtain the energy estimates first locally for each wave structure and then combine them together to establish a unified estimate in the whole domain. Here, because of the different regularity in the energy estimates in separate domains, special attention needs to be paid when combining these estimates.

(4) Finally, Nash-Moser iteration is used to obtain the existence of the desired solution by choosing appropriate parameters to ensure the convergence of a sequence of approximate solutions.

One common feature for all the three waves discussed above, whether they are shock, rarefaction waves, or 2-D contact discontinuity, is that the iteration to establish the existence of solution need to be carried out near an approximate solution, and the existence of such an approximate solution is a necessary condition to have a piece-wise smooth solution. In other words, the existence of any such waves requires the initial data to be compatible. The compatibility condition is the set of algebraic equations on the value of the initial data $U_{0\pm}$ at the initial jump curve $x = \phi_0(y, z)$, together with all their derivatives up to the order k. They consist of many algebraic equations, and the higher the order k is, the more complicated the system of algebraic equations become, and the harder it is to check and satisfy all these conditions.

However, since it is a necessary condition, one cannot avoid it but has to assume it as a prerequisite condition in all the above-listed work, see [1, 4, 13, 35, 41]. In the 1-D case, this is not an issue since the classical solution only requires the first order compatibility. In the high-dimensional case, the issue is no more negligible because of the use of Sobolev space H^k where $k > \frac{n+1}{2}$ is larger than one. It becomes very important in the case of rarefaction wave or contact discontinuity when Nash-Moser iteration is used and a very high order of compatibility is required.

It is remarkable to notice that such complicated high order compatibility requirements become automatically satisfied, if the 0-order compatibility condition is satisfied. In other words, the compatibility issue becomes trivial, by treating all three waves together, rather than each separately. Indeed, the following theorem is proved in [10] (and later on expanded to the general case in [36]).

Theorem 6.2. *For the generalized Riemann problem* (1.8)*, if the Riemann problem with constant data frozen at the origin has a solution with complete nonlinear wave configuration, then the initial data in* (1.8) *for the M-D Euler system is automatically compatible of infinite order for a corresponding M-D solution.*

7 Further considerations

In the previous sections we reviewed the progress in the study of the GRP for multi-dimensional Euler equations. As we can see, the picture is still incomplete compared with the 1-D case.

(1) First of all, even for the 2-D Euler equations, the supersonic condition (5.3) is required to ensure the existence of the solution to GRP. Comparing with the known result in 1-D case, it is natural to ask if such a condition can be eliminated or weakened. Because of the instability of the linearized vortex sheet, it seems impossible to try to establish the existence by usual linear iteration. However, since the complete set of nonlinear waves are considered, it is still unknown if the appearance of shock waves could offer some stability effect to get rid of the condition (5.3).

(2) The similar question can be asked to the existence of solution containing all three waves for genuine 3-D Euler system. The difficulty involved is fundamental, because of the nature of instability of linearized vortex sheet.

(3) Using the same idea as in [10] the initial-boundary value problems for multi-dimensional Euler system are studied with general data [37]. It has been attempted to combine the solutions of two initial-boundary value problems as in [37] to construct the solution containing vortex sheet, but so far without much progress in this direction.

(4) One promising direction in the study of GRP is the degenerate case. In [10], all possible waves (whether shock wave, rarefaction wave, or contact discontinuity) are non-degenerate. However, as mentioned in [36], from the viewpoint of compatibility, these waves could be degenerate, as long as some kind of uniformness is maintained. The existence problem containing such degenerate waves is still open. We believe that similar existence results could be obtained along similar line, or with some minor modification of methods.

(5) Finally, the GRP considered above for multi-dimensional Euler system is indeed quasi-one dimensional. However, even for the 2-D case, one may consider another generalization of Riemann problem, when the initial data are assumed to be different constants in different angular domains (see [5, 6, 50]). Even with such simple data the problems are much more difficult than in 1-D case. With their importance in application, they become a big topic in partial differential equations. So far there are few rigorous theoretical results on this topic. A rigorous proof on the existence of the solution is established for the Chaplygin gas in [11]. As for the polytropic gas, an existence result is obtained in [38], where the gas-expansion problem is treated and is regarded as a special case of the corresponding multi-dimensional Riemann problem. Other discussions on multi-dimensional Riemann problems can also be found in [7–9, 39, 50] and the references cited therein.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant Nos. 11031001, 11101101 and 11421061).

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