Compatibility of Jump Cauchy Data for Non-isentropic Euler Equations *†

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Abstract

The paper studies the compatibility of the Cauchy data which have a jump discontinuity for the non-isentropic 3-d Euler system. For a complete range of combinations of waves including shocks, rarefaction waves, and contact discontinuity, it is shown that the data is compatible of infinite order if the corresponding 1-d Riemann problem admits such a solution.

1 Introduction

In the study of the initial-boundary value problems for evolutionary equations, such as the Euler system in gas-dynamics, one of the necessary conditions for the existence of smooth solution is the compatibility [15, 17]. For the more complicated free-boundary problems where solutions may contain various discontinuities, such as shock wave, rarefaction wave, or vortex sheet, the compatibility requirements are also always imposed, see [1, 11, 12, 5], also [2, 9].

Look at the typical initial-boundary value problem for an $n\times n$ hyperbolic system

$$\partial_t u + A_1\partial_x u + A_2\partial_y u + A_3\partial_z u + Cu = f. \tag{1.1}$$

$$u(0,x,y,z) = u_0(x,y,z), \quad x \geq 0, \tag{1.2}$$

$$Bu(t,0,y,z) = g(t,y,z),\ t\geq 0, \quad (y,z)\in\mathbb{R}^2. \tag{1.3}$$

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In order to have a solution which is continuous at $t = 0, x = 0$, we must have by (1.2)(1.3), $Bu_0(0, y, z) = g(0, y, z)$ for all $(y, z) \in \mathbb{R}^2$. In order that the solution has continuous first order derivative, we must have the value of u_t determined by (1.1)(1.2) satisfying (1.3), i.e., $Bu_t(0, 0, y, z) = g_t(0, y, z)$. Similarly are derived the higher order compatibilities which contain many equations for the values of the initial and boundary data and their derivatives along the intersection of the initial plane and the boundary surface.

For Euler equations, a piecewise smooth solution may contain such discontinuities such as shock, rarefaction wave, contact discontinuity and/or vortex sheet [8, 16]. The mathematical formulation for such solution is a free-boundary value problem which requires similar compatibility of the initial data. Such compatibility conditions consist of a large simultaneous system of algebraic equations for the values of initial data and their derivatives along the initial discontinuity surface. For high order of compatibility, the conditions are not only difficult to satisfy, but also very tedious to verify for a given set of initial data.

The compatibility requirement is not a big concern in the one space dimension [10], where the solution is sought in the space of C^1 is required to be only continuous up to the boundary. In other word, the required compatibility is of order 0.

However, in high space dimension where the solution is usually found in the Sobolev space H^k, high value of k is needed in order to perform linear iteration and to obtain classical solution by the imbedding theorem. The compatibility becomes especially an issue if the linearized problems have only weak estimates (tame estimate [7]) and Nash-Moser iteration is needed in establishing the existence of solution, see [1, 5]. In such cases, it is usually required that one works in the space H^k with k sufficiently large which can only be determined in the process of the iteration. In addition, when rarefaction wave is involved or two waves intersect, [2, 9], the high order weighted norms also demand a high order approximate solution [1, 2, 9]. The magnitude of such high order k is usually very complicated to determine explicitly.

In studying the existence of a specific wave structure with central vortex sheet for 2-d isentropic Euler equations, it was shown in [3] that if all three waves are non-degenerate, the high order compatibility requirement can be satisfied from the 0-order compatibility with usual Lax' shock inequality. This is a big simplification on the usually cumbersome but necessary compatibility requirement.

This paper intends to expands the result for the special case in [3] to the more general cases. We will consider not only the non-isentropic 3-

d Euler system, but also the more general wave combinations, including the contact discontinuity which could be a density jump, a vortex sheet, or their combination or even their degeneration – a weak discontinuity. In addition, the result will also apply (with minor adjustment) to the sound waves, treated as a degenerate rarefaction wave, see also [13].

It is important to emphasize that we are here concerned only with the compatibility aspect of all these wave combinations, with no regard to their linear stability or existence. Indeed, it is well-known that 3-d vortex sheet are highly unstable, see e.g. [4]. However, the compatibility result obtained here can be a useful tool in the study of the existence of various waves where a weak linear stability does hold, like the 2-d non-isentropic vortex sheet with tangential speed difference larger than a critical value [5, 14], or the genuine contact discontinuity without tangential velocity jump [4], or sound wave cases [13]. These topics are not addressed here and will be studied in future works.

In the following, the Cauchy problem for the Euler system will be formulated and the main theorem presented in section 2. Sections 3, 4, 5 will discuss the compatibility of three different combinations of the waves.

2 Cauchy problem for 3-D non-isentropic Euler system with piecewise smooth initial data

Let (ρ, p, e) be the density, pressure, and the internal energy of the fluid, (u, v, w) be the velocity in the (x, y, z) direction. The Euler system of compressible flow in 3-D space can be written as follows:

$$\begin{cases} \partial_t\rho + \partial_x(\rho u) + \partial_y(\rho v) + \partial_z(\rho w) = 0, \\ \partial_t(\rho u) + \partial_x(p + \rho u^2) + \partial_y(\rho uv) + \partial_z(\rho uw) = 0, \\ \partial_t(\rho v) + \partial_x(\rho uv) + \partial_y(p + \rho v^2) + \partial_z(\rho vw) = 0, \\ \partial_t(\rho w) + \partial_x(\rho uw) + \partial_y(\rho vw) + \partial_z(p + \rho w^2) = 0, \\ \partial_t(\rho E) + \partial_x(\rho Eu + pu) + \partial_y(\rho Ev + pv) + \partial_z(\rho Ew + pw) = 0, \end{cases} \tag{2.1}$$

with $E = e + \frac{1}{2}(u^2 + v^2 + w^2)$.

Denote

$$H_0 = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho E \end{pmatrix}, \ H_1 = \begin{pmatrix} \rho u \\ p + \rho u^2 \\ \rho uv \\ \rho uw \\ (\rho E + p)u \end{pmatrix}, \ H_2 = \begin{pmatrix} \rho v \\ \rho uv \\ p + \rho v^2 \\ \rho vw \\ (\rho E + p)v \end{pmatrix}, \ H_3 = \begin{pmatrix} \rho w \\ \rho uw \\ \rho vw \\ p + \rho w^2 \\ (\rho E + p)w \end{pmatrix}.$$

Then system (2.1) can be written briefly as

$$\partial_t H_0 + \partial_x H_1 + \partial_y H_2 + \partial_z H_3 = 0,$$

and the corresponding Rankine-Hugoniot condition on the shock front $x = \phi(t, y, z)$ is

$$\phi_t [H_0]_0^1 - [H_1]_0^1 + \phi_y [H_2]_0^1 + \phi_z [H_3]_0^1 = 0. \tag{2.2}$$

Here as usual, $[f]_0^1$ denotes the difference between the status ahead of and behind the shock front $x = \phi(t, y, z)$.

Introducing the unknown vector of functions $U = (p, u, v, w, S)$, it is well-known (see e.g., [6]) that for smooth solutions, the system (2.1) is equivalent to the following system

$$\begin{cases} \partial_t p + (u, v, w) \cdot \nabla p + \rho c^2 \nabla \cdot (u, v, w) = 0, \\ \rho \partial_t u + \rho (u, v, w) \cdot \nabla u + \partial_x p = 0, \\ \rho \partial_t v + \rho (u, v, w) \cdot \nabla v + \partial_y p = 0, \\ \rho \partial_t w + \rho (u, v, w) \cdot \nabla w + \partial_z p = 0, \\ \partial_t S + (u, v, w) \cdot \nabla S = 0, \end{cases} \tag{2.3}$$

with $c^2 = p'_\rho(\rho, S) > 0$.

System (2.3) can be written as the following symmetric form

$$LU \equiv A_0 \partial_t U + A_1(U) \partial_x U + A_2(U) \partial_y U + A_3(U) \partial_z U = 0,$$

where

$$A_0 = \begin{bmatrix} \frac{1}{\rho c^2} & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & \rho & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} \frac{u}{\rho c^2} & 1 & 0 & 0 & 0 \\ 1 & \rho u & 0 & 0 & 0 \\ 0 & 0 & \rho u & 0 & 0 \\ 0 & 0 & 0 & \rho u & 0 \\ 0 & 0 & 0 & 0 & u \end{bmatrix},$$

$$A_2 = \begin{bmatrix} \frac{v}{\rho c^2} & 0 & 1 & 0 & 0 \\ 0 & \rho v & 0 & 0 & 0 \\ 1 & 0 & \rho v & 0 & 0 \\ 0 & 0 & 0 & \rho v & 0 \\ 0 & 0 & 0 & 0 & v \end{bmatrix}, \quad A_3 = \begin{bmatrix} \frac{w}{\rho c^2} & 0 & 0 & 1 & 0 \\ 0 & \rho w & 0 & 0 & 0 \\ 0 & 0 & \rho w & 0 & 0 \\ 1 & 0 & 0 & \rho w & 0 \\ 0 & 0 & 0 & 0 & w \end{bmatrix}.$$

The matrix $A_0^{-1}[A_1(U)+A_2(U)\phi_y+A_3(U)\phi_z]$ has two simple eigenvalues $\lambda_\pm$ and one triple eigenvalue λ_0:

$$\begin{aligned} \lambda_- &= u - v\phi_y - w\phi_z - a\sqrt{1+\phi_y^2+\phi_z^2}, \\ \lambda_0 &= u - v\phi_y - w\phi_z, \\ \lambda_+ &= u - v\phi_y - w\phi_z + a\sqrt{1+\phi_y^2+\phi_z^2}, \end{aligned} \tag{2.4}$$

with $\lambda_- < \lambda_0 < \lambda_+$.

Let $\Gamma: \ x = \phi_0(y,z)$ be a smooth surface in the initial space $\mathbb{R}^3$ with $\phi_0(0,0) = 0$ and $\nabla\phi_0(0,0) = 0$. Consider the Cauchy problem for (2.1) with the initial data given as

$$U = \begin{cases} U_-(x,y,z), & \text{if } \ x < \phi_0(y,z), \\ U_+(x,y,z), & \text{if } \ x > \phi_0(y,z). \end{cases} \tag{2.5}$$

Here $U_-(x,y,z), U_+(x,y,z)$ are smooth in their respective domains up to Γ.

Corresponding to the Cauchy problem (2.1), (2.5), we will refer to the following *accompanying 1-d Riemann problem* with constant initial data

$$\begin{cases} \partial_t\rho + \partial_x(\rho u) = 0, \\ \partial_t\rho u + \partial_x(p+\rho u^2) = 0, \\ \partial_t\rho E + \partial_x(\rho E u + pu) = 0, \end{cases} \tag{2.6}$$

$$(\rho,u,e)(0,x) = \begin{cases} (\rho_-,u_-,e_-)(0,0,0), & \text{if } \ x<0, \\ (\rho_+,u_+,e_+)(0,0,0), & \text{if } \ x>0. \end{cases} \tag{2.7}$$

The Riemann problem (2.6), (2.7) is the 1-d version of (2.1), (2.5) with the constant initial data taking the values of $(\rho_\pm, u_\pm)$ at the origin (0,0,0).

The solutions for the Riemann problem (2.6), (2.7) have been well-studied and may contain in general shocks, rarefaction waves or contact discontinuity emanating from the initial discontinuity. The possible combinations are SCS, SCR, RCS and RCR, where “S” stands for a shock, “R” stands for a rarefaction wave, and “C” stands for contact discontinuity, while the first letter represents a wave propagating to left, and the third letter represents a right-propagating wave.

Depending upon the initial data (2.7), shock or rarefaction waves may be degenerate into a characteristic carrying weak discontinuities (of the derivatives of the solution), also called sound wave [13]. The contact discontinuity may also be similarly degenerate.

In the study of high-dimensional free boundary problems involving shocks [11, 12], rarefaction waves [1], contact discontinuity [5], or some combination thereof [2, 9], complicated compatibility conditions are always required. The higher the order of compatibility, the higher the order of the derivatives are involved, and the more the equations are contained in the system.

Even though such conditions are necessary for the solution's existence, for a given set of the initial data, it is practically very difficult to verify their compatibility, and it is also very tedious to explicitly construct a compatible set of initial data, see [1, 12, 5], except for the trivial constant cases.

The purpose of this paper is to show that as long as we consider the complete set of wave patterns, even including some degenerate cases, the high-order compatibility can actually be derived from basic stability of the wave patterns and hence the infinite order compatibility is automatically satisfied for piecewise smooth initial data. The main theorem is stated as follows.

Theorem 2.1 *For the Cauchy problem (2.1), (2.5), assume that*

(C) the problem (2.6), (2.7) has a solution with complete nonlinear wave configuration, i.e., one of the four combinations: SCS, SCR, RCS or RCR; with the shocks satisfying the usual Lax' condition,

then the initial data in (2.5) for the non-isentropic 3-d Euler system (2.1) is automatically compatible of infinite order for a corresponding 3-d solution.

Remark 2.1 The corresponding 3-d solution in Theorem 2.1 means a solution which has the same left or right propagating waves "S" or "R", while in the center, there is a contact discontinuity. This contact discontinuity "C" could be either a vortex sheet, or a discontinuity in the density ρ only, or a combination of the two. It could even be degenerate, i.e., it is missing in the solution for (2.6), (2.7), but appears in the corresponding 3-d solution for (2.1), (2.5) as a weak discontinuity with solution being continuous, but only possible discontinuous normal derivatives.

Remark 2.2 Theorem 2.1 can be expanded to cover the case when the shock or rarefaction wave also degenerates into a sound wave. In this case, the initial data in (2.5) should be required to be uniformly degenerate at every point along Γ, not only at the origin. The proof can be easily obtained by adapting slightly the proof for Theorem 2.1 (see Remark 4.2) and will not be explicitly given here.

For convenience, we will assume in the following that the non-isentropic flow is the ideal polytropic gas with

$$p = A(S)\rho^{\gamma}, \quad \gamma > 1.$$

Then we have the relation (see, e.g., [6]) with $\tau = \rho^{-1}$

$$p'_{\rho} = a^2 = \gamma p \tau; \quad e = \frac{1}{\gamma - 1} p \tau.$$

For our purpose of studying compatibility, it is convenient to choose the unknown vector function as $U = (\rho, u, v, s, p)$ (denoted again as U with the abuse of notation), and consider the following equivalent system to (2.1):

$$\begin{cases} \partial_t \rho + (u, v, w) \cdot \nabla \rho + \rho \nabla \cdot (u, v, w) = 0, \\ \partial_t u + (u, v, w) \cdot \nabla u + \rho^{-1} \partial_x p = 0, \\ \partial_t v + (u, v, w) \cdot \nabla v + \rho^{-1} \partial_y p = 0, \\ \partial_t w + (u, v, w) \cdot \nabla w + \rho^{-1} \partial_z p = 0, \\ \partial_t p + (u, v, w) \cdot \nabla p + \rho c^2 \nabla \cdot (u, v, w) = 0. \end{cases} \tag{2.8}$$

The last equation of energy conservation in (2.2) can be replaced by the following thermodynamic relation [6]

$$e_1 - e_0 = (\tau_1 - \tau_0) \frac{p_1 + p_0}{2}. \tag{2.9}$$

Here as usual, subscripts "$_0$" and "$_1$" denote the status before and after the shock front. In particular for the ideal polytropic gas, (2.9) becomes

$$(\rho_0 - \mu^2 \rho_1) p_1 = (\rho_1 - \mu^2 \rho_0) p_0, \tag{2.10}$$

here $\mu^2 = (\gamma - 1)/(\gamma + 1)$.

3 Compatibility for the SCS combination of waves

The SCS combination of waves consist of a left-propagating shock, a right-propagating shock, and a contact discontinuity (or vortex sheet) at the center. Let the left-propagating shock be denoted by $S_l: \ x = \phi_l(t, y, z)$, the right-propagating shock by $S_r: \ x = \phi_r(t, y, z)$, and the contact discontinuity be denoted by $C: \ x = \theta(t, y, z)$. See Fig. 3.1.

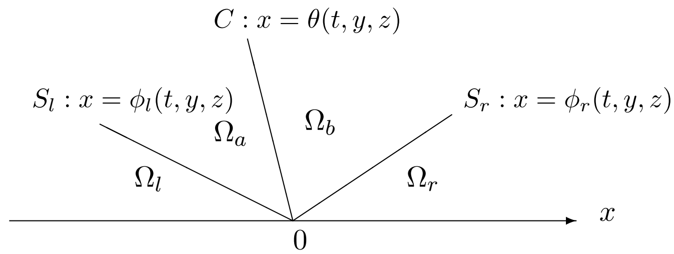

Figure 3.1: SCS wave configuration

Let $U_a(t, x, y, z)$ ($U_b(t, x, y, z)$, resp.) be the solution of (2.8) (which is equivalent to (2.1)) in the angular domain Ω_a (Ω_b, resp.) between S_l (S_r, resp.) and C. And $U_l(t, x, y, z)$ ($U_r(t, x, y, z)$, resp.) is the solution of (2.8) in the domain Ω_l (Ω_r, resp.) left (right, resp.) to S_l (S_r, resp.).

In addition to the equations of (2.8), (U_l, U_r) and (ϕ_l, ϕ_r, θ) also satisfy the initial condition:

$$\begin{cases} U_l(0, x, y, z) = U_-(x, y, z) \ \text{ in } \ x < \phi_0(y, z); \\ U_r(0, x, y, z) = U_+(x, y, z) \ \text{ in } \ x > \phi_0(y, z); \\ \phi_l(0, y, z) = \phi_r(0, y, z) = \theta(0, y, z) = \phi_0(y, z). \end{cases} \tag{3.1}$$

Besides, they also satisfy the following Rankine-Hugoniot conditions on two shock fronts $x = \phi_l$, $x = \phi_r$:

$$\phi_{lt}[H_0]^a_- + \phi_{ly}[H_2]^a_- + \phi_{lz}[H_3]^a_- - [H_1]^a_- = 0 \ \text{ at } \ x = \phi_l(t, y, z); \tag{3.2}$$

$$\phi_{rt}[H_0]^b_+ + \phi_{ry}[H_2]^b_+ + \phi_{rz}[H_3]^b_+ - [H_1]^b_+ = 0 \ \text{ at } \ x = \phi_r(t, y, z); \tag{3.3}$$

and on the contact discontinuity $x = \theta$:

$$\begin{cases} \theta_t + v_b\theta_y + w_b\theta_z - u_b = 0, \\ p_b - p_a = 0, \\ \theta_y(v_b - v_a) + \theta_z(w_b - w_a) - (u_b - u_a) = 0 \end{cases} \quad \text{at } \ x = \theta(t, y, z). \tag{3.4}$$

Remark 3.1 The condition (3.4) does not impose any restriction on the density ρ or the tangential velocity. It has the same form for all different type of contact discontinuity. In particular, it also applies to the degenerate case where only weak discontinuity exits. This shows that the proof in the following is valid for all these cases, hence follows Remark 2.1.

The k-th order compatibility is the requirement that one should be able to uniquely determine the values of the functions $(U_a, U_b, \phi_{lt}, \phi_{rt}, \theta_t)$ and their derivatives up to the order k at the initial manifold Γ from Eqs (2.8) and the boundary conditions (3.2)-(3.4). It is equivalent to the existence of an approximate solution which satisfies (2.8) and (3.2)-(3.4) near Γ up to the order $O(t^k)$. In the following, we show that it is true for any k under the condition in Theorem 2.1.

3.1 The 0-order compatibility

The 0-order compatibility does not involve any derivatives of (U_a, U_b) and we have 13 variables

$$U_a(0, \phi_0(y,z), y, z),\ U_b(0, \phi_0(y,z), y, z),\ \partial_t\phi_l(0,y,z),\ \partial_t\phi_r(0,y,z),\ \partial_t\theta(0,y,z).$$

to satisfy 13 equations in the boundary conditions (3.2)-(3.4).

Due to the continuity in the variables y and z, by the implicit function theorem we need only to show that at the origin (0,0,0,0), the system (3.2)-(3.4) has one solution

$$U_a(0,0,0,0),\ U_b(0,0,0,0),\ \partial_t\phi_l(0,0,0),\ \partial_t\phi_r(0,0,0),\ \partial_t\theta(0,0,0).$$

and the corresponding Jacobian matrix is non-degenerate.

The existence of a solution at (0,0,0,0) is provided by the condition (C) in Theorem 2.1. Indeed by (2.10), Eqs (3.2)-(3.4) at (0,0,0,0) become

$$\begin{aligned}
&\phi_{lt}(0)\begin{pmatrix}\rho_a-\rho_-\\ \rho_a u_a-\rho_- u_-\\ \rho_a v_a-\rho_- v_-\\ \rho_a w_a-\rho_- w_-\end{pmatrix}=\begin{pmatrix}\rho_a u_a-\rho_- u_-\\ p_a+\rho_a u_a^2-p_--\rho_- u_-^2\\ \rho_a u_a v_a-\rho_- u_- v_-\\ \rho_a u_a w_a-\rho_- u_- w_-\end{pmatrix};\\
&(\rho_--\mu^2\rho_a)p-(\rho_a-\mu^2\rho_-)p_-=0
\end{aligned}\tag{3.5}$$

$$\begin{aligned}
&\phi_{rt}(0)\begin{pmatrix}\rho_b-\rho_+\\ \rho_b u_b-\rho_+ u_+\\ \rho_b v_b-\rho_+ v_+\\ \rho_b w_b-\rho_+ w_+\end{pmatrix}=\begin{pmatrix}\rho_b u_b-\rho_+ u_+\\ p_b+\rho_b u_b^2-p_+-\rho_+ u_+^2\\ \rho_b u_b v_b-\rho_+ u_+ v_+\\ \rho_b u_b w_b-\rho_+ u_+ w_+\end{pmatrix};\\
&(\rho_+-\mu^2\rho_b)p-(\rho_b-\mu^2\rho_+)p_+=0;
\end{aligned}\tag{3.6}$$

$$p_a=p_b=p,\ u_a=u_b=u=\theta_t(0).\tag{3.7}$$

The four variables (v_a, v_b, w_a, w_b), each appears only in one equation of (3.5) and (3.6), and they all have non-zero coefficients ($\rho_a(\phi_{lt}-u_a)$ and

$\rho_b(\phi_{rt} - u_b))$ by condition (C) in Theorem 2.1. Hence they can be solved independently after all other variables are determined.

Eliminating these four variables together with $(p_b, u_b, \theta_t(0))$ from (3.5)-(3.7), we end up with a 6×6 system for $(\phi_{lt}, \phi_{rt}, \rho_a, u, p, \rho_b)$:

$$\begin{cases} \phi_{lt}(0) \begin{pmatrix} \rho_a - \rho_- \\ \rho_a u - \rho_- u_- \end{pmatrix} - \begin{pmatrix} \rho_a u - \rho_- u_- \\ p + \rho_a u^2 - p_- - \rho_- u_-^2 \end{pmatrix} = 0; \\ (\rho_- - \mu^2 \rho_a) p - (\rho_a - \mu^2 \rho_-) p_- = 0; \end{cases} \tag{3.8}$$

$$\begin{cases} \phi_{rt}(0) \begin{pmatrix} \rho_b - \rho_+ \\ \rho_b u - \rho_+ u_+ \end{pmatrix} - \begin{pmatrix} \rho_b u - \rho_+ u_+ \\ p + \rho_b u^2 - p_+ - \rho_+ u_+^2 \end{pmatrix} = 0; \\ (\rho_+ - \mu^2 \rho_b) p - (\rho_b - \mu^2 \rho_+) p_+ = 0; \end{cases} \tag{3.9}$$

These equations are exactly the Rankine-Hugoniot conditions for the Riemann problem (2.6)(2.7), for which the existence of solution is assumed by the condition (C).

Denote the left hand sides of (3.2)-(3.4) by F_1, F_2, F_3, and let $\det J$ be the determinant of the following Jacobian matrix

$$J = \frac{\partial(F_1, F_2, F_3)}{\partial(\phi_{lt}, \phi_{rt}, \theta_t, \rho_a, u_a, v_a, w_a, p_a, \rho_b, u_b, v_b, w_b, p_b)}. \tag{3.10}$$

We need to show that $\det J \neq 0$ at the point $(0,0,0,0)$.

The Jacobian J is the coefficient matrix of the linearized system (3.2)-(3.4). The linearization of (3.4) at the point $(0,0,0,0)$ becomes simply

$$\theta_t = u_a, \quad u_a = u_b, \quad p_a = p_b.$$

So we can eliminate the variables (θ_t, u_b, p_b) in the linearized system. For the same reason listed above, the variables (v_a, w_a, v_b, w_b) in the linearized system all appear only in one equation with non-zero coefficient, hence we can also eliminate them from the linear system.

The condition $\det J \neq 0$ is equivalent to the non-degeneracy of the coefficient matrix of the linearization of (3.8), (3.9) at $(0,0,0,0)$, which is

$$\dot{\phi}_{lt} \begin{bmatrix} \rho \\ \rho u \\ 0 \end{bmatrix}_-^a + \begin{bmatrix} \phi_{lt} - u & -\rho_a & 0 \\ u(\phi_{lt} - u) & \rho_a(\phi_{lt} - 2u) & -1 \\ p_- + \mu^2 p_a & 0 & \mu^2 \rho_a - \rho_- \end{bmatrix} \begin{bmatrix} \dot{\rho}_a \\ \dot{u} \\ \dot{p} \end{bmatrix} = 0. \tag{3.11}$$

$$\dot{\phi}_{rt} \begin{bmatrix} \rho \\ \rho u \\ 0 \end{bmatrix}_+^b + \begin{bmatrix} \phi_{rt} - u & -\rho_b & 0 \\ u(\phi_{rt} - u) & \rho_b(\phi_{rt} - 2u) & -1 \\ p_+ + \mu^2 p_b & 0 & \mu^2 \rho_b - \rho_+ \end{bmatrix} \begin{bmatrix} \dot{\rho}_b \\ \dot{u} \\ \dot{p} \end{bmatrix} = 0. \tag{3.12}$$

Here $(\dot\phi_{lt}, \dot\phi_{rt}, \dot\rho_a, \dot\rho_b, \dot u, \dot p)$ denote the small increments of the corresponding variables.

Using the relations $\phi_{lt}(\rho_a - \rho_-) = \rho_a u - \rho_- u_-$ and $\phi_{rt}(\rho_b - \rho_+) = \rho_b u - \rho_+ u_+$ to eliminate the terms $\dot\phi_{lt}$ and $\dot\phi_{rt}$ in (3.11) and (3.12), we obtain

$$\begin{aligned}
&\begin{bmatrix} (u-\phi_{lt})^2 & 2\rho_a(u-\phi_{lt}) & 1 \\ p_- + \mu^2 p_a & 0 & \mu^2\rho_a - \rho_- \end{bmatrix} \begin{bmatrix} \dot\rho_a \\ \dot u \\ \dot p \end{bmatrix}. \\
&\begin{bmatrix} (u-\phi_{rt})^2 & 2\rho_b(u-\phi_{rt}) & 1 \\ p_+ + \mu^2 p_b & 0 & \mu^2\rho_b - \rho_+ \end{bmatrix} \begin{bmatrix} \dot\rho_b \\ \dot u \\ \dot p \end{bmatrix}.
\end{aligned} \tag{3.13}$$

Further eliminating $\dot\rho_a$ and $\dot\rho_b$ from (3.11) yields a 2×2 system for $(\dot u, \dot p)$

$$M\begin{bmatrix} \dot u \\ \dot p \end{bmatrix} \equiv \begin{bmatrix} m_{l1} & m_{l2} \\ m_{r1} & m_{r2} \end{bmatrix} \begin{bmatrix} \dot u \\ \dot p \end{bmatrix} = 0. \tag{3.14}$$

with

$$\begin{cases} m_{l1} = 2\rho_a(u-\phi_{lt})(p_- + \mu^2 p_a), \\ m_{l2} = (p_- + \mu^2 p_a) - (\mu^2\rho_a - \rho_-)(u-\phi_{lt})^2 \\ m_{r1} = 2\rho_b(u-\phi_{rt})(p_+ + \mu^2 p_b), \\ m_{r2} = (p_+ + \mu^2 p_b) - (\mu^2\rho_b - \rho_+)(u-\phi_{rt})^2 \end{cases} \tag{3.15}$$

To see that $\det M \neq 0$, it suffices to show that

$$m_{l1} > 0, \quad m_{l2} > 0, \quad m_{r1} < 0, \quad m_{r2} > 0. \tag{3.16}$$

First of all, one obtains $m_{l1} > 0$ and $m_{r1} < 0$ readily from $u - \phi_{lt} > 0$ and $u - \phi_{rt} < 0$.

For polytropic gas $p(\rho, S) = A(S)\rho^\gamma$, hence $p = \frac{c^2\rho}{\gamma}$. Since the supersonic or subsonic character of the flow can be determined by the flow speed with the critical speed c_* [6], so by Lax' shock inequality,

$$(\phi_{lt} - u)^2 < c_-^2, \quad (\phi_{lt} - u)^2 < c_a^2, \quad (\phi_{rt} - u)^2 < c_b^2, \quad (\phi_{rt} - u)^2 < c_+^2.$$

Hence $m_{l2} > 0$ and $m_{r2} > 0$ if

$$(\rho_- + \mu^2\rho_a) > \gamma(\mu^2\rho_a - \rho_-), \quad (\rho_+ + \mu^2\rho_b) > \gamma(\mu^2\rho_b - \rho_+).$$

or equivalently

$$\frac{\rho_-}{\rho_a} > \mu^4, \quad \frac{\rho_+}{\rho_b} > \mu^4. \tag{3.17}$$

(3.17) follows readily from the restriction on compression ratio ρ_1/ρ_0 (see [6], p. 148)

$$\mu^2 < \frac{\rho_1}{\rho_0} < \frac{1}{\mu^2}.$$

Therefore (3.16) is true. This finishes the proof of 0-order compatibility.

3.2 The first order compatibility

First, we notice that once the values of $(U_a, U_b, \phi_{lt}, \phi_{rt}, \theta_t)$ are determined at the initial discontinuity Γ, then all their derivatives tangential to Γ are uniquely determined. Therefore, the first order compatibility consists of 23 linear equations for the 23 variables

$$\begin{aligned}
&U_{at}(0, \phi_0(y,z), y, z),\ U_{an}(0, \phi_0(y,z), y, z),\\
&U_{bt}(0, \phi_0(y,z), y, z),\ U_{bn}(0, \phi_0(y,z), y, z),\\
&\phi_{ltt}(0, y, z),\ \phi_{rtt}(0, y, z),\ \theta_{tt}(0, y, z).
\end{aligned}$$

Here (U_{an}, U_{bn}) denote the normal derivative to Γ.

Again by the continuity in (y, z) and the implicit function theorem, we need only to show the Jacobian of these 23 equations is non-degenerate at the origin (0,0,0,0). In particular at the origin, $(U_{an}, U_{bn}) = (U_{ax}, U_{bx})$, $\phi_{ly} = \phi_{lz} = \phi_{ry} = \phi_{rz} = \theta_y = \theta_z = 0$ and $\theta_t = u_a = u_b = u$.

Let (D_l, D_c, D_r) be the tangential differential operators in the $(x = \phi_l, \theta, \phi_r)$ directions in the t-x plane:

$$D_l = \partial_t + \phi_{lt}\partial_x,\ \ D_c = \partial_t + u\partial_x,\ \ D_r = \partial_t + \phi_{rt}\partial_x.$$

Replace the last equations in (3.2)-(3.3) by (2.10), and then denote by (H_{04}, H_{14}) the first four components of (H_0, H_1). Taking tangential derivatives (D_l, D_r) of thus modified equations (3.2) and (3.3) in the t-x plane and evaluating them at (0,0,0,0), we obtain (here and in the following in this paper, $*$ stands for terms already determined by lower order compatibility):

$$\begin{cases} \phi_{ltt}[H_{04}]^a_- + (\phi_{lt}H'_{04} - H'_{14})D_l U_a = *, \\ (p_- + \mu^2 p)D_l\rho_a + (\mu^2\rho_a - \rho_-)D_l p_a = *. \end{cases} \tag{3.18}$$

$$\begin{cases} \phi_{rtt}[H_{04}]^b_+ + (\phi_{rt}H'_{04} - H'_{14})D_l U_b = *, \\ (p_+ + \mu^2 p)D_r\rho_b + (\mu^2\rho_b - \rho_+)D_r p_b = *. \end{cases} \tag{3.19}$$

where the 4×5 matrix $(\phi_{l(r)t}H'_{04}-H'_{14})$ is the following

$$\begin{bmatrix} \phi_{l(r)t}-u & -\rho_{a(b)} & 0 & 0 & 0 \\ u(\phi_{l(r)t}-u) & \rho_{a(b)}(\phi_{l(r)t}-2u) & 0 & 0 & -1 \\ 0 & 0 & \rho_{a(b)}(\phi_{l(r)t}-u) & 0 & 0 \\ 0 & 0 & 0 & \rho_{a(b)}(\phi_{l(r)t}-u) & 0 \end{bmatrix}.$$

Taking tangential derivatives D_c of the equations in (3.4) and evaluating them at (0,0,0) yields

$$\begin{aligned} &\theta_{tt}-D_c u_a = *, \\ &D_c p_a - D_c p_b = *, \\ &D_c u_a - D_c u_b = *. \end{aligned} \tag{3.20}$$

At the origin (0,0,0,0), the interior equation (2.8) becomes

$$\begin{cases} D_c\rho_{a(b)}+\rho_{a(b)}\partial_x u_{a(b)} = *, \\ D_c u_{a(b)}+\frac{1}{\rho_{a(b)}}\partial_x p_{a(b)} = *, \\ D_c v_{a(b)} = *, \\ D_c w_{a(b)} = *, \\ D_c p_{a(b)}+\rho_{a(b)}c^2_{a(b)}\partial_x u_{a(b)} = *. \end{cases} \quad \text{in } \Omega_{a(b)} \tag{3.21}$$

The linear system (3.18)-(3.21) consists of 23 equations for the 23 variables

$$(\phi_{ltt},\phi_{rtt},\theta_{tt},U_{at},U_{ax},U_{bt},U_{bx}),$$

where $U=(\rho,u,v,w,p)$. They can be simplified as follows.

- θ_{tt} appears only in one equation in (3.20) and can be eliminated.
- There is no restriction in (3.14) on (v_t,v_x,w_t,w_x), and so $(v_{at},v_{ax},w_{at},w_{ax})$ and $(v_{bt},v_{bx},w_{bt},w_{bx})$ are decoupled with each other.

  Because D_c, D_l are not parallel, (v_{at},v_{ax}) are uniquely determined by $(D_c v_a, D_l v_a,)$. Since Dv_a appears only in one equation in (3.21) in the form $D_c v_a$, and appears only in one equation in (3.18) in the form $D_l v$, both with non-zero coefficients, hence $(D_c v_a, D_l v_a)$ can be uniquely determined. Therefore (v_{at},v_{ax}) can be eliminated.
- Same argument also applies to $(w_{at},w_{ax},v_{bt},v_{bx},w_{bt},w_{bx})$.

Therefore we can eliminate the 9 variables

$$(\theta_{tt},v_{at},v_{ax},w_{at},w_{ax},v_{bt},v_{bx},w_{bt},w_{bx})$$

from (3.18)-(3.21) and obtain 14 equations for the remaining 14 variables

$$(\phi_{ltt}, \rho_{at}, \rho_{ax}, u_{at}, u_{ax}, p_{at}, p_{ax}, \phi_{rtt}, \rho_{bt}, \rho_{bx}, u_{bt}, u_{bx}, p_{bt}, p_{bx}).$$

Eliminating ϕ_{ltt} from (3.18) yields

$$\begin{bmatrix} (u-\phi_{lt})^2 & 2\rho_a(u-\phi_{lt}) & 1 \\ p_- + \mu^2 p_a & 0 & \mu^2\rho_a - \rho_- \end{bmatrix} D_l \begin{bmatrix} \rho_a \\ u_a \\ p_a \end{bmatrix} = *. \tag{3.22}$$

Eliminating $D_l\rho_a$ from (3.22) yields

$$(m_{l1}, m_{l2}) D_l \begin{bmatrix} u_a \\ p_a \end{bmatrix} = *, \tag{3.23}$$

where (m_{l1}, m_{l2}) are defined as in (3.15).

Similarly from (3.19), we obtain

$$(m_{r1}, m_{r2}) D_r \begin{bmatrix} u_b \\ p_b \end{bmatrix} = *. \tag{3.24}$$

with (m_{r1}, m_{r2}) defined as in (3.15).

Removing the two equations involving $D_c v_{a(b)}$ and $D_c w_{a(b)}$ in (3.21), the remaining last two equations are independent of $D\rho$:

$$\begin{cases} D_c u_{a(b)} + \frac{1}{\rho_{a(b)}} \partial_x p_{a(b)} = *, \\ D_c p_{a(b)} + \rho_{a(b)} c^2_{a(b)} \partial_x u_{a(b)} = *. \end{cases} \tag{3.25}$$

Or equivalently

$$D_c \begin{bmatrix} u_{a(b)} \\ p_{a(b)} \end{bmatrix} + c_{a(b)} \mathscr{E}_{a(b)} \partial_x \begin{bmatrix} u_{a(b)} \\ p_{a(b)} \end{bmatrix} = *, \tag{3.26}$$

with

$$\mathscr{E}_{a(b)} \equiv \begin{bmatrix} 0 & (c_{a(b)}\rho_{a(b)})^{-1} \\ c_{a(b)}\rho_{a(b)} & 0 \end{bmatrix} = \mathscr{E}^{-1}_{a(b)}. \tag{3.27}$$

Since $D_l = D_c + (\phi_{lt} - u)\partial_x$ and $D_r = D_c + (\phi_{rt} - u)\partial_x$, then they can be written by (3.26) as

$$\begin{cases} D_l \begin{bmatrix} u_a \\ p_a \end{bmatrix} = (I - \beta_l \mathscr{E}_a)\, D_c \begin{bmatrix} u_a \\ p_a \end{bmatrix} + *, \\ D_r \begin{bmatrix} u_b \\ p_b \end{bmatrix} = (I - \beta_r \mathscr{E}_b)\, D_c \begin{bmatrix} u_b \\ p_b \end{bmatrix} + *, \end{cases} \tag{3.28}$$

with

$$\beta_l \equiv \frac{\phi_{lt}-u}{c_a} < 0, \quad \beta_r \equiv \frac{\phi_{rt}-u}{c_b} > 0. \tag{3.29}$$

By the Lax' shock inequality in condition (C), we have $|\beta_l| < 1$ and $|\beta_r| < 1$.

Replacing $(D_l u_a, D_l p_a)$ and $(D_r u_b, D_r p_b)$ in (3.23) and (3.24) using (3.28), we obtain two equations for two variables $(D_c u, D_c p)$:

$$\begin{bmatrix} m_{l1} - m_{l2}\beta_l c_a \rho_a & -m_{l1}\beta_l (c_a\rho_a)^{-1} + m_{l2} \\ m_{r1} - m_{r2}\beta_r c_b \rho_b & -m_{r1}\beta_r (c_b\rho_b)^{-1} + m_{r2} \end{bmatrix} D_c \begin{bmatrix} u \\ p \end{bmatrix} = *. \tag{3.30}$$

By (3.16) and (3.29), we have

$$\begin{aligned} m_{l1} - m_{l2}\beta_l c_a \rho_a > 0, \quad -m_{l1}\beta_l (c_a\rho_a)^{-1} + m_{l2} > 0, \\ m_{r1} - m_{r2}\beta_r c_b \rho_b < 0, \quad -m_{r1}\beta_r (c_b\rho_b)^{-1} + m_{r2} > 0. \end{aligned}$$

Consequently (3.30) is non-degenerate and $(D_c u, D_c p)$ are uniquely determined.

This finishes the proof of the 1st order compatibility.

3.3 k-th order compatibility

As in the case of the first order compatibility, let's take the k-th order tangential derivatives of the modified equations (3.2)-(3.4), and then evaluate them at the origin (0,0,0,0). Apply D_c^{k-1} to the interior equations (2.8), and then evaluate them at (0,0,0,0). Similar to the case of the 1st order compatibility, the 9 variables $(\partial_t^{k+1}\theta, D_{l(r)}^k v_{a(b)}, D_{l(r)}^{k-1}\partial_x v_{a(b)}, D_{l(r)}^k w_{a(b)}, D_{l(r)}^{k-1}\partial_x w_{a(b)})$ can be determined independently and thus be eliminated.

From (3.20) we also have

$$D_c^k u_a = D_c^k u_b + *, \quad D_c^k p_a = D_c^k p_b + *.$$

For the remaining 12 variables

$$\partial_t^{k+1}\phi_{l(r)}, D_c^k \rho_{a(b)}, D_c^{k-1}\partial_x \rho_{a(b)}, D_c^k u_a, D_c^{k-1}\partial_x u_{a(b)}, D_c^k p_a, D_c^{k-1}\partial_x p_{a(b)},$$

we have 12 equations

$$\begin{cases} \begin{bmatrix} \rho \\ \rho u \end{bmatrix}_-^a \partial_t^{k+1}\phi_l + \begin{bmatrix} \phi_{lt}-u & -\rho_a & 0 \\ u(\phi_{lt}-u) & \rho_a(\phi_{lt}-2u) & -1 \end{bmatrix} D_l^k \begin{bmatrix} \rho_a \\ u_a \\ p_a \end{bmatrix} = *; \\ (p_- + \mu^2 p) D_l^k \rho_a + (\mu^2\rho_a - \rho_-) D_l^k p_a = *. \end{cases} \tag{3.31}$$

$$\begin{cases} \begin{bmatrix} \rho \\ \rho u \end{bmatrix}_+^b \partial_t^{k+1}\phi_r + \begin{bmatrix} \phi_{rt}-u & -\rho_b & 0 \\ u(\phi_{rt}-u) & \rho_b(\phi_{rt}-2u) & -1 \end{bmatrix} D_r^k \begin{bmatrix} \rho_b \\ u_b \\ p_b \end{bmatrix} = *; \\ (p_+ + \mu^2 p) D_r^k \rho_b + (\mu^2\rho_b - \rho_+) D_r^k p_b = *. \end{cases} \tag{3.32}$$

$$\begin{cases} D_c^k \rho_{a(b)} + \rho_{a(b)} D_c^{k-1}\partial_x u_{a(b)} = *, \\ D_c^k u + \frac{1}{\rho_{a(b)}} D_c^{k-1}\partial_x p_{a(b)} = *, & \text{in } \Omega_{a(b)} \\ D_c^k p + \rho_{a(b)} c_{a(b)}^2 D_c^{k-1}\partial_x u_{a(b)} = *. \end{cases} \tag{3.33}$$

The system (3.31)-(3.33) consists of 12 equations for the 12 variables (with $D_c^k u \equiv D_c^k u_a$ and $D_c^k p \equiv D_c^k p_a$)

$$\partial_t^{k+1}\phi_{l(r)},\ D_c^k\rho_{a(b)},\ D_c^{k-1}\partial_x\rho_{a(b)},\ D_c^k u,\ D_c^{k-1}\partial_x u_{a(b)},\ D_c^k p,\ D_c^{k-1}\partial_x p_{a(b)}.$$

As in the first order case, eliminating $\partial_t^{k+1}\phi_{l(r)}$ and $D_{l(r)}^k\rho_{a(b)}$ from (3.31) and (3.32), we obtain

$$\begin{cases} (m_{l1}, m_{l2}) D_l^k \begin{bmatrix} u_a \\ p_a \end{bmatrix} = *, \\ (m_{r1}, m_{r2}) D_r^k \begin{bmatrix} u_b \\ p_b \end{bmatrix} = *. \end{cases} \tag{3.34}$$

We can use (3.33) to replace $(D_{l(r)}^k u_{a(b)}, D_{l(r)}^k p_{a(b)})$ with $(D_c^k u, D_c^k p)$. For this, we have the following lemma (a little different one was used in [3])

Lemma 3.1 From (3.33), $(D_{l(r)}^k u_{a(b)}, D_{l(r)}^k p_{a(b)})$ can be expressed by $(D_c^k u, D_c^k p)$ as follows

$$\begin{aligned} D_l^k \begin{bmatrix} u_a \\ p_a \end{bmatrix} &= \delta_l(\alpha_{lk} - \beta_l\mathscr{E}) D_c^k \begin{bmatrix} u_a \\ p_a \end{bmatrix} = *, \\ D_r^k \begin{bmatrix} u_b \\ p_b \end{bmatrix} &= \delta_r(\alpha_{rk} - \beta_r\mathscr{E}) D_c^k \begin{bmatrix} u_b \\ p_b \end{bmatrix} = *, \end{aligned} \tag{3.35}$$

where $0 < |\beta_l| < \alpha_{lk} \leq 1$ and $0 < |\beta_r| < \alpha_{rk} \leq 1$. δ_l and δ_r are two positive constants which may depend on k and the explicit form of which is of no consequence in our discussion.

Proof: For $k = 1$, (3.35) is simply (3.28).

By induction, assume (3.35) for $k-1$, then

$$\begin{aligned} D_l^k \begin{bmatrix} u_a \\ p_a \end{bmatrix} &= \delta_l(\alpha_{l(k-1)} - \beta_l\mathscr{E})(1 - \beta_l\mathscr{E}) D_c^k \begin{bmatrix} u_a \\ p_a \end{bmatrix} + * \\ &= \delta_l(1 + \alpha_{l(k-1)})(\alpha_{lk} - \beta_l\mathscr{E}) D_c^k + *. \end{aligned}$$

We need only to show that

$$\alpha_{lk} = \frac{\alpha_{l(k-1)} + \beta_l^2}{1 + \alpha_{l(k-1)}} > |\beta_l|, \tag{3.36}$$

which follows by induction assumption from

$$\alpha_{l(k-1)} + \beta_l^2 - |\beta_l| - \alpha_{l(k-1)}|\beta_l| = (\alpha_{l(k-1)} - |\beta_l|)(1 - |\beta_l|) > 0$$

The same argument also applies to D_r^k. This concludes the proof of Lemma 3.1.

Applying Lemma 3.1, we can rewrite (3.34) into the following 2×2 system for $(D_c^k u, D_c^k p)$:

$$\begin{cases} (m_{l1}, m_{l2})\delta_l(\alpha_{lk} - \beta_l \mathscr{E}) D_c^k \begin{bmatrix} u \\ p \end{bmatrix} = *, \\ (m_{r1}, m_{r2})\delta_r(\alpha_{rk} - \beta_r \mathscr{E}) D_c^k \begin{bmatrix} u \\ p \end{bmatrix} = *. \end{cases} \tag{3.37}$$

i.e.,

$$\begin{bmatrix} m_{l1} & m_{l2} \end{bmatrix} \begin{bmatrix} \alpha_{lk} & -\beta_l (c_a \rho_a)^{-1} \\ -\beta_l (c_a \rho_a) & \alpha_{lk} \end{bmatrix} D_c^k \begin{bmatrix} u \\ p \end{bmatrix} = *;$$

$$\begin{bmatrix} m_{r1} & m_{r2} \end{bmatrix} \begin{bmatrix} \alpha_{rk} & -\beta_r (c_b \rho_b)^{-1} \\ -\beta_r (c_b \rho_b) & \alpha_{rk} \end{bmatrix} D_c^k \begin{bmatrix} u \\ p \end{bmatrix} = *.$$

(3.37) is non-degenerate iff the following matrix is non-degenerate

$$\begin{bmatrix} m_{l1}\alpha_{lk} - m_{l2}\beta_l(c_a\rho_a), & -m_{l1}\beta_l(c_a\rho_a)^{-1} + m_{l2}\alpha_{lk} \\ m_{r1}\alpha_{rk} - m_{r2}\beta_r(c_b\rho_b), & -m_{r1}\beta_r(c_b\rho_b)^{-1} + m_{r2}\alpha_{rk} \end{bmatrix} \tag{3.38}$$

From (3.16) and (3.29), we have

$$\begin{cases} m_{l1}\alpha_{lk} - m_{l2}\beta_l(c_a\rho_a) > 0, \\ -m_{l1}\beta_l(c_a\rho_a)^{-1} + m_{l2}\alpha_{lk} > 0, \\ m_{r1}\alpha_{rk} - m_{r2}\beta_r(c_b\rho_b) < 0, \\ -m_{r1}\beta_r(c_b\rho_b)^{-1} + m_{r2}\alpha_{rk} > 0. \end{cases}$$

Therefore (3.38) is non-degenerate.

Once $(D_c^k u, D_c^k p)$ is known, $(D_c^{k-1}\partial_x u_{a(b)}, D_c^{k-1}\partial_x p_{a(b)})$ as well as $D_c^k \rho_{a(b)}$ can be obtained from (3.33). Then $D_{l(r)}^k \rho_{a(b)}$ and consequently $D_c^{k-1}\partial_x \rho_{a(b)}$

can be determined from (3.31), (3.32). Induction on the index j would give all k-th order derivatives for $D_c^{k-j}\partial_x^j\rho_{a(b)}, D_c^{k-j}\partial_x^j u_{a(b)}, D_c^{k-j}\partial_x^j p_{a(b)}$ $(j = 2, \cdots, k)$.

This concludes the proof of Theorem 2.1 for the SCS wave combination.

4 Compatibility for the SCR/RCS combination of waves

We need only to consider only the SCR combination of waves whch consist of a left-propagating shock, a right-propagating rarefaction wave, and a contact discontinuity at the center. As in Section 3, the contact discontinuity here is understood in the most general sense, including the degenerate situation. See Fig. 4.1.

Let the left-propagating shock be denoted by $S_l : \; x = \phi_l(t, y, z)$, the contact discontinuity be denoted by $C : \; x = \theta(t, y, z)$, and the right-propagating rarefaction wave is represented by an angular domain between two characteristics $L_- : \; x = \chi^-(t, y, z)$ and $L_+ : \; x = \chi^+(t, y, z)$.

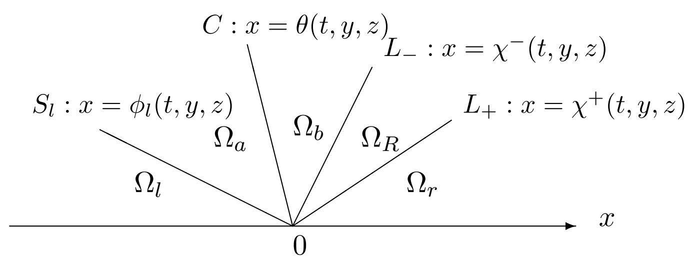

Figure 4.1: SCR wave configuration

The solution of (2.8) in the angular domain Ω_a (Ω_b, resp.) between S_l (L_-, resp.) and C is denoted by $U_a(t, x, y)$ ($U_b(t, x, y)$, resp.), the solution in the domain Ω_R formed by the rarefaction wave between L_- and L_+ is denoted by U_c, the solution in the domain left to S_l is denoted by $U_l(t, x, y)$, and the solution in the domain right to S_r is denoted by $U_r(t, x, y)$.

By the finite speed propagation property for hyperbolic systems, the solution U_l, U_r, and the location of the characteristic L_+ are already uniquely determined by the initial data U_- and U_+.

Due to the multi-valuedness of U_c at Γ, a parameter s is introduced to blow up the wedge area of Ω_R as [1]. Let $x = \chi(t, s, y, z)$ $(1 \leq s \leq 2)$ be the family of characteristics issuing from Γ inside Ω_R. Then $\chi(t,s,y,z)$ satisfies

$$\det|A_1 - \chi_t - \chi_y A_2 - \chi_z A_3| = 0, \tag{4.1}$$

or more precisely

$$\chi_t = \lambda_+(U_c; \nabla\chi), \tag{4.2}$$

where $\lambda_+(U;\phi)$ is the maximal eigenvalue in (2.4).

Introduce the function $W(t,s,y,z)$:

$$W(t,s,y,z) = U_c(t, \chi(t,s,y,z), y, z), \tag{4.3}$$

which satisfies

$$\begin{aligned} \tilde{L}W \equiv \chi_s\Big(\frac{\partial W}{\partial t} + A_2\frac{\partial W}{\partial y} + A_3\frac{\partial W}{\partial z}\Big) \\ +(A_1 - \chi_t - \chi_y A_2 - \chi_z A_3)\frac{\partial W}{\partial s} = 0. \end{aligned} \tag{4.4}$$

In summary, the SCR combination of waves is represented by the set of functions $U_a(t,x,y,z), U_b(t,x,y,z)$, $W(t,s,y,z)$, $\chi(t,s,y,z)$, $\phi_l(t,y,z)$, $\theta(t,y,z)$ satisfying, in addition to (4.2),

$$\begin{cases} LU_a = 0 \quad \text{in} \ \ \Omega_a, \\ LU_b = 0 \quad \text{in} \ \ \Omega_b, \\ \tilde{L}W = 0 \quad \text{in} \ \ 1 < s < 2. \end{cases} \tag{4.5}$$

$$\phi_{lt}[H_0] + \phi_{ly}[H_2] - [H_1] = 0 \quad \text{on} \ \ x = \phi_l(t,y). \tag{4.6}$$

$$\begin{cases} \theta_t - u_a - v_a\theta_y - w_a\theta_z = 0, \\ p_a - p_b = 0, \\ (u_b - u_a) - \theta_y(v_b - v_a) - \theta_z(w_b - w_a) = 0. \end{cases} \tag{4.7}$$

$$\begin{cases} \chi(t,1,y,z) = \chi^-(t,y,z), \\ \chi(t,2,y,z) = \chi^+(t,y,z), \\ W(t,1,y,z) = U_b(t,\chi(t,1,y,z),y,z), \\ W(t,2,y,z) = U_R(t,\chi(t,2,y,z),y,z). \end{cases} \tag{4.8}$$

Finally, we have the initial conditions at Γ:

$$\phi(0,y,z) = \theta(0,y,z) = \chi(0,s,y,z) = 0, \tag{4.9}$$

with the assumption as in [1]

$$\chi_s = \gamma(t,s,y,z)t \quad \text{with} \ \ \gamma(t,s,y,z) \geq \delta > 0. \tag{4.10}$$

Remark 4.1 As noted in Remark 2.2, for sound wave treated as a degenerate rarefaction wave, the last equations in (4.5) are void, as well as (4.8). And $\chi^-(t,y,z) = \chi^+(t,y,z)$ are determined by the initial value U_+.

4.1 The 0-order compatibility

As in the SCS case, we may assume $(U_l, U_r) = (U_-, U_+)$ and (U_a, U_b) as well as $(\phi_{lt}, \theta_t, \chi_t^-, \chi_t^+)$ being all constant. Also assume χ_t depending only on s, with $W(t,s,y,z) = W(s)$ satisfying

$$\begin{cases} (A_1 - \chi_t(s))W(s) = 0, \\ W(2) = U_+. \end{cases} \tag{4.11}$$

By (4.7), $p_a = p_b, u_a = u_b$, and only one equation in (4.7) contains θ_t. In addition, (v_a, w_a, v_b, w_b) are decoupled and can be determined independently from (4.6) and (4.8). Therefore, the 0-order compatibility becomes connecting the state (ρ_-, u_-, p_-) with the state (ρ_+, u_+, p_+) by a shock $x = \phi_l(t)$ and a rarefaction wave $x = \chi$. The existence of $(\rho_a, \rho_b, u_a = u_b, p, \phi_{lt}, \chi_t(s))$ is provided by the condition (C) on the accompanying problem (2.6)(2.7).

Let $\zeta(s) = (\rho(s), u(s), p(s))$ be the solution of (2.6)(2.7) such that

$$\begin{cases} (A_1' - \chi_t(s)I)\zeta'(s) = 0, \\ \zeta(2) = (\rho_+, u_+, p_+), \quad \zeta(1) = (\rho_b, u_a, p_a), \quad \chi_t(1) = \chi_t^-, \end{cases} \tag{4.12}$$

with

$$A_1' = \begin{bmatrix} u & \rho & 0 \\ 0 & u & \rho^{-1} \\ 0 & \rho c^2 & u \end{bmatrix}.$$

From (4.12) we see that $\chi_t(s) = \lambda_+(\zeta(s)) = u+c$ and $\zeta'(s) = r(\zeta)$, where the $r(\zeta)$ being parallel to (ρ, c, c^2), is the right eigenvector of A_1', satisfying

$$r \cdot \frac{\partial \lambda}{\partial \zeta} = 1$$

and $\zeta(s)$ is the solution of the system of the ordinary differential equation

$$\frac{d\zeta}{ds} = r(\zeta) \tag{4.13}$$

with $\zeta(2) = (\rho_+, u_+, p_+)$.

Replace the last equation in (4.6) by the thermodynamic relation (2.10), then the boundary conditions on $x = \phi_l$ can be written as

$$\begin{aligned} &F_1 = \phi_{lt}(\rho_a - \rho_-) - (\rho_a u_a - \rho_- u_-) = 0, \\ &F_2 = \phi_{lt}(\rho_a u_a - \rho_- u_-) - (p_a + \rho_a u_a^2 - p_- - \rho_- u_-^2) = 0, \\ &F_3 = (\rho_- - \mu^2 \rho_a)p - (\rho_a - \mu^2 \rho_-)p_- = 0. \end{aligned} \tag{4.14}$$

Denote the solution (ρ_b, u, p) of (4.13) as two equations

$$G_1(\rho_b, u, p) = 0, \quad G_2(\rho_b, u, p) = 0.$$

Then $(\phi_{lt}, \rho_a, \rho_b, u, p)$ can be uniquely determined if and only if

$$\Delta = \det\left(\frac{\partial(F_1, F_2, F_3, G_1, G_2)}{\partial(\phi_{lt}, \rho_a, u, p, \rho_b)}\right) \neq 0. \tag{4.15}$$

The Jacobian in (4.15) can be computed explicitly

$$\begin{pmatrix} \rho_a - \rho_- & \phi_{lt} - u & -\rho_a & 0 & 0 \\ \rho_a u_a - \rho_- u_- & (\phi_{lt} - u)u & \rho_a(\phi_{lt} - 2u) & -1 & 0 \\ 0 & F_{3\rho_a} & 0 & F_{3p} & 0 \\ 0 & 0 & G_{1u} & G_{1p} & G_{1\rho_b} \\ 0 & 0 & G_{2u} & G_{2p} & G_{2\rho_b} \end{pmatrix} \tag{4.16}$$

Obviously (4.16) is non-singular if and only if

$$\det\begin{pmatrix} (u - \phi_{lt})^2 & 2\rho_a(u - \phi_{lt}) & 1 & 0 \\ -(p_- + \mu^2 p) & 0 & \rho_- - \mu^2 \rho_a & 0 \\ 0 & G_{1u} & G_{1p} & G_{1\rho_b} \\ 0 & G_{2u} & G_{2p} & G_{2\rho_b} \end{pmatrix} \neq 0, \tag{4.17}$$

or equivalently

$$\det\begin{pmatrix} m_{l1} & m_{l2} & 0 \\ G_{1u} & G_{1p} & G_{1\rho_b} \\ G_{2u} & G_{2p} & G_{2\rho_b} \end{pmatrix} \neq 0, \tag{4.18}$$

where (m_{l1}, m_{l2}) are defined as in (3.15).

Since the eigenvector $r(\zeta)$ is parallel to (ρ_b, c_b, c_b^2), hence (G_1, G_2) can be chosen such that $(G_{1\rho_a}, G_{1u}, G_{1p})$ is parallel to the vector $(-\rho_b, c_b, 0)$ and $(G_{2\rho_a}, G_{2u}, G_{2p})$ is parallel to the vector $(0, 1, -c_b)$.

Then (4.18) becomes

$$\det\begin{pmatrix} m_{l1} & m_{l2} & 0 \\ c_b & 0 & -\rho_b \\ 1 & -c_b & 0 \end{pmatrix} = \rho_b c_b(m_{l1} + m_{l2}) \neq 0, \tag{4.19}$$

which is obvious by (3.16). This concludes the proof of 0-order compatibility.

4.2 The first order compatibility $k = 1$

We need to show that the first order derivatives of $U_{a(b)}(t,x,y,z), W(t,s,y,z)$ and $\phi_{lt}(t,y,z), \theta_t(t,y,z), \chi_t(t,s,y,z)$ can be uniquely determined at Γ. From the 0-order compatibility, these functions and their tangential derivatives with respect to Γ are already known.

As in [1], let $H(v,\eta)$ be the matrix satisfying

$$H^{-1}(A_1 - \chi_y A_2 - \chi_z A_3)H = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^b \end{pmatrix} \left(\triangleq d\right), \tag{4.20}$$

where the superscript b denotes the last four rows.

From (4.4) we have

$$\begin{aligned} & H^{-1}(W_t + A_2 W_y + A_3 W_z) \\ & \quad = -H^{-1}(A_1 - \chi_t I - A_2\chi_y - A_3\chi_z)W_s/\chi_s \\ & \quad = \begin{pmatrix} \chi_t - \lambda & 0 \\ * & * \end{pmatrix} W_s/\chi_s = \begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix} W_s/\chi_s. \end{aligned} \tag{4.21}$$

Then the first row becomes

$$(H^{-1}(W_t + A_2 W_y + A_3 W_z))^1 = 0. \tag{4.22}$$

Multiplying (4.4) by H^{-1} we have

$$\chi_s H^{-1}(W_t + A_2 W_y + A_3 W_z) + (d - \chi_t)H^{-1}W_s u = 0.$$

Differentiating with respect to t gives

$$\begin{aligned} & \chi_{ts} H^{-1}(W_t + A_2 W_y + A_3 W_z) + \chi_s(H^{-1}(W_t + A_2 W_y + A_3 W_z))_t \\ & \quad + (d - \chi_t)_t H^{-1}W_s + (d - \chi_t)(H^{-1}W_s)_t = 0. \end{aligned} \tag{4.23}$$

Since $\chi_s = 0$ at $t = 0$, (4.23) yields

$$\begin{aligned} & \chi_{ts}\left(H^{-1}(W_t + A_2 W_y + A_3 W_z)\right)^b \\ & \quad + (\lambda^b - \chi_t)_t \left(H^{-1}W_s\right)^b + (\lambda^b - \chi_t)\left(H^{-1}W_s\right)_t^b = 0. \end{aligned} \tag{4.24}$$

From (4.24) we have

$$\begin{aligned} & \chi_{ts}\left(H^{-1}(W_t + A_2 W_y + A_3 W_z)\right)^b \\ & \quad + (\lambda^b - \chi_t)_t((H^{-1}W_t)^b)_s + (H^{-1}W_t)^b \cdot * = *, \end{aligned} \tag{4.25}$$

here $*$ stands again for the known terms. Therefore, the value of $(H^{-1}W_t)^b$ at any $s \in [1,2]$ can be uniquely determined by its value at $s = 2$. On the other hand, the value $(H^{-1}W_t)^1$ is determined from (4.22). Hence the value of all the components of $H^{-1}W_t$ are uniquely determined, and so are all the components of W_t.

Once W_t is known, we can obtain χ_{tt} by differentiating (4.2) with respect to t:

$$\chi_{tt} = \lambda_W W_t - \lambda_\eta \chi_{yt} - \lambda_\zeta \chi_{zt}. \tag{4.26}$$

Since the tangential derivatives of U_b and U_c on $x = \chi(t,1,y,z) \equiv \chi_1(t,y,z)$ are equal, therefore the value of the tangential derivative $D_r U_b \equiv (\partial_t + \chi_{1t}\partial_x)U_b$ is known. Therefore

$$D_r\rho_b = *, \quad D_r u_b = *, \quad D_r p = *, \quad D_r v_b = *, \quad D_r w_b = *, \tag{4.27}$$

with

$$D_r = \partial_t + (u + c_b)\partial_x = D_c + c_b\partial_x,$$

since at the origin $\chi_1(0,0,0) = u + c_b$.

Because $\chi_{1t} = \lambda_+$ is an eigenvalue for the system (2.8), only two of the first three relations in (4.27) are independent. Hence (4.27) consists of only four independent relations for $(\rho_b, u_b, v_b, w_b, p_b)$ which we denote as

$$D_r\rho_b = *, \quad D_r p_b = *, \quad D_r v_b = *, \quad D_r w_b = *. \tag{4.28}$$

Similar as in the SCS case in section 3, we can also obtain (3.18) and (3.20) from (4.6) and (4.7).

Remark 4.2 For the sound wave which is treated as a degenerate rarefaction wave, $\chi_{1t} = \chi_{2t} = \lambda_+$ and we obtain the same (4.27) and (4.28). Since the degeneracy at one point on Γ does not ensure the degeneracy in its neighborhood, the additional requirement of uniform degeneracy is needed as in Remark 2.2. Then all the following arguments also apply to the degenerate case.

The linear system (3.18), (3.20), (3.21), (4.26), (4.28) consists of 23 equations for the 23 variables

$$(\phi_{ltt}, \theta_{tt}, \chi_{1tt}, U_{at}, U_{ax}, U_{bt}, U_{bx}).$$

By the same argument as in the SCS case, $(v_{at}, v_{ax}, w_{at}, w_{ax})$ can be eliminated from (3.18) and (3.21), $(v_{bt}, v_{bx}, w_{bt}, w_{bx})$ can be eliminated from (3.21) and (4.28). $(\theta_{tt}, \chi_{1tt})$ can be eliminated from (3.20) and (4.26).

Again eliminating ϕ_{ltt} from (3.18) yield (3.22). Then (ρ_{at}, ρ_{ax}) can be eliminated since $(D_l\rho_a, D_c\rho_a)$ can be eliminated from (3.21) and (3.22), (ρ_{bt}, ρ_{bx}) can be eliminated from (3.21) and (4.28). Finally (D_cp_b, D_cu_b) can also be eliminated from (3.20).

After these simplifications, we are left with 6 equations for 6 variables $(D_lu_a, D_lp_a, D_cu_a, D_cp_a, D_ru_b, D_rp_b)$. For convenience, we rewrite these 6 equations as follows:

$$\begin{cases} (m_{l1}, m_{l2})D_l\begin{bmatrix} u_a \\ p_a \end{bmatrix} = (m_{l1}, m_{l2})[D_c + (\phi_{lt} - u_a)\partial_x]\begin{bmatrix} u_a \\ p_a \end{bmatrix} = *; \\ D_cu_a + \frac{1}{\rho_{a(b)}}\partial_x p_{a(b)} = *; \\ D_cp_a + \rho_{a(b)}c^2_{a(b)}\partial_x u_{a(b)} = *; \\ D_rp_b = (0,1)[D_c + c_b\partial_x]\begin{bmatrix} u_b \\ p_b \end{bmatrix} = *. \end{cases} \tag{4.29}$$

Using four interior equations in (4.29) to eliminate $(\partial_x u_a, \partial_x p_a, \partial_x u_b, \partial_x p_b)$, we obtain two equations for (D_cu_a, D_cp_a)

$$\begin{cases} (m_{l1}, m_{l2})(I - \beta_l\mathscr{E}_a)D_c\begin{bmatrix} u_a \\ p_a \end{bmatrix} = *; \\ (0,1)(I - c_b^{-1}\mathscr{E}_a)D_c\begin{bmatrix} u_a \\ p_a \end{bmatrix} = *. \end{cases} \tag{4.30}$$

(4.30) is non-degenerate if

$$\det\begin{bmatrix} m_{l1} - m_{l2}\beta_l(c_a\rho_a) & -m_{l1}\beta_l(c_a\rho_a)^{-1} + m_{l2} \\ -c_b^{-1} & 1 \end{bmatrix} \neq 0, \tag{4.31}$$

which follows readily from the fact that $m_{l1} > 0$, $m_{l2} > 0$ by (3.16), and $\beta_l < 0$ by (3.29).

4.3 The k-th order compatibility

We apply the tangential derivatives D_l^{k-1} to (3.18), D_c^{k-1} to both (3.20) and (3.21), D_r^{k-1} to (4.28), and ∂_t^{k-1} to (4.26), This gives 23 linear equations

$$\begin{cases} \partial_t^{k+1}\phi_l[H_{04}]^a_- + (\phi_{lt}H'_{04} - H'_{14})D_l^kU_a = *, \\ (p_- + \mu^2 p)D_l^k\rho_a + (\mu^2\rho_a - \rho_-)D_l^kp_a = *. \end{cases} \tag{4.32}$$

$$\begin{cases} D_c^k \rho_{a(b)} + \rho_{a(b)} D_c^{k-1} \partial_x u_{a(b)} = *, \\ D_c^k u_{a(b)} + \frac{1}{\rho_{a(b)}} D_c^{k-1} \partial_x p_{a(b)} = *, \\ D_c^k v_{a(b)} = *, \\ D_c^k w_{a(b)} = *, \\ D_c^k p_{a(b)} + \rho_{a(b)} c_{a(b)}^2 D_c^{k-1} \partial_x u_{a(b)} = *. \end{cases} \quad \text{in } \Omega_{a(b)} \tag{4.33}$$

$$\partial_t^{k+1} \theta - D_c^k u_a = *, \quad D_c^k p_a - D_c^k p_b = *, \quad D_c^k u_a - D_c^k u_b = *. \tag{4.34}$$

$$D_r^k \rho_b = *, \quad D_r^k p_b = *, \quad D_r^k v_b = *, \quad D_r^k w_b = *. \tag{4.35}$$

$$\partial_t^{k+1} \chi = \partial_t^{k-1} (\lambda_W W_t - \lambda_\eta \chi_{yt}). \tag{4.36}$$

For (4.32)-(4.36), there are 23 independent variables

$$\partial_t^{k+1} \phi_l, \quad \partial_t^{k+1} \theta, \quad \partial_t^{k+1} \chi_1, \quad D_c^k U_{a(b)}, \quad D_c^{k-1} \partial_x U_{a(b)}.$$

By the same argument as in the first order compatibility, the 12 variables

$$\partial_t^{k+1} \chi_1, \partial_t^{k+1} \theta, D_c^k v_{a(b)}, D_c^{k-1} \partial_x v_{a(b)}, D_c^k w_{a(b)}, D_c^{k-1} \partial_x w_{a(b)}, D_c^k u_b, D_c^k p_b$$

can be eliminated immediately. Straightforward computations further eliminate the following 5 variables

$$\partial_t^{k+1} \phi_l, D_c^k \rho_{a(b)}, D_c^{k-1} \partial_x \rho_{a(b)}$$

from (4.32), (4.33) and (4.35).

There are six equations left:

$$\begin{cases} (m_{l1}, m_{l2}) D_l^k \begin{bmatrix} u_a \\ p_a \end{bmatrix} = *, \\ D_c^k u + \frac{1}{\rho_{a(b)}} D_c^{k-1} \partial_x p_{a(b)} = *, \\ D_c^k p + \rho_{a(b)} c_{a(b)}^2 D_c^{k-1} \partial_x u_{a(b)} = *; \\ D_r^k p_b = (1, 0) D_r^k \begin{bmatrix} u_b \\ p_b \end{bmatrix} = *. \end{cases} \tag{4.37}$$

for the six variables

$$D_c^k u, \quad D_c^k p, \quad D_c^{k-1} \partial_x u_{a(b)}, \quad D_c^{k-1} \partial_x p_{a(b)}.$$

As shown in Lemma 3.1 for the SCS case, we can use (3.35) to replace $(D_l^k u_a, D_l^k p_a)$ by $(D_c^k u_a, D_c^k p_a)$. On the other hand, similar to (3.28), we have here

$$D_r \begin{bmatrix} u_b \\ p_b \end{bmatrix} = (I - \mathscr{E}_b) D_c \begin{bmatrix} u \\ p \end{bmatrix} + *, \tag{4.38}$$

and also

$$D_r^k \begin{bmatrix} u_b \\ p_b \end{bmatrix} = (I - \mathscr{E}_b)^k D_c^k \begin{bmatrix} u \\ p \end{bmatrix} + * = 2^{k-1} (I - \mathscr{E}_b) D_c^k \begin{bmatrix} u \\ p \end{bmatrix} + *. \quad (4.39)$$

Therefore (4.37) can be reduced to the following 2×2 system for $(D_c^k u, D_c^k p)$:

$$\begin{cases} (m_{l1}, m_{l2})\delta_l(\alpha_{lk} - \beta_l \mathscr{E}_a) D_c^k \begin{bmatrix} u \\ p \end{bmatrix} = *, \\ (1,0) 2^{k-1} (I - \mathscr{E}_b) D_c^k \begin{bmatrix} u \\ p \end{bmatrix} = *. \end{cases} \quad (4.40)$$

(4.40) is non-degenerate if

$$\det \begin{bmatrix} m_{l1}\alpha_{lk} - m_{l2}\beta_l(c_a\rho_a) & -m_{l1}\beta_l(c_a\rho_a)^{-1} + m_{l2}\alpha_{lk} \\ 1 & -(c_b\rho_b)^{-1} \end{bmatrix} \neq 0, \quad (4.41)$$

which follows readily from $m_{l1} > 0$, $m_{l2} > 0$, $\beta_l < 0$ and $\alpha_{lk} > 0$.

Once $(D_c^k u, D_c^k p)$ is known, we can determine $(D_c^{k-1}\partial_x u_{a(b)}, D_c^{k-1}\partial_x p_{a(b)})$. Then using interior equation (3.21) and induction, we can determine all the derivatives $(D_c^{k-j}\partial_x^j u_{a(b)}, D_c^{k-j}\partial_x^j p_{a(b)})$ for $j = 2, 3, \ldots, k$.

This finishes the proof of the compatibility for SCR wave combination.

5 Compatibility for the RCR combination of waves

The compatibility for the RCR wave combination can be obtained similarly as for the SCR case.

Omitting the tedious details, the proof is outlined as follows.

1. The 0-order compatibility follows from the condition (C);

2. For the first order compatibility, let

$$D_c = \partial_t + u\partial_x, \quad D_l = D_c - c_a\partial_x, \quad D_r = D_c + c_b\partial_x. \quad (5.1)$$

   Noticing that the eigenvalues corresponding to the two rarefaction waves are $\lambda_\pm = u \pm c$, and then taking tangential derivatives (D_l, D_r) to the boundary conditions on $x = \chi_{l(r)}$, and taking derivative D_c to the boundary condition on $x = \theta$, we obtain a linear system similar to (4.32)-(4.36) with $k = 1$, except the condition (4.32) is replaced by corresponding equations (4.35) and (4.36). After simplification, we have a system similar to (4.29), again with first equation replaced by a corresponding equation as the last one:

$$\begin{cases} D_l p_b = (0,1) D_l \begin{bmatrix} u_a \\ p_a \end{bmatrix} = *, \\ D_c u + \frac{1}{\rho_{a(b)}} \partial_x p_{a(b)} = *, \\ D_c p + \rho_{a(b)} c_{a(b)}^2 \partial_x u_{a(b)} = *; \\ D_r p_b = (0,1) D_r \begin{bmatrix} u_b \\ p_b \end{bmatrix} = *. \end{cases} \tag{5.2}$$

From the interior equations in (5.2), we have by (5.1)

$$\begin{aligned} D_l \begin{bmatrix} u_a \\ p_a \end{bmatrix} &= (I + \mathscr{E}_a)\, D_c \begin{bmatrix} u \\ p \end{bmatrix} + *, \\ D_r \begin{bmatrix} u_b \\ p_b \end{bmatrix} &= (I - \mathscr{E}_b)\, D_c \begin{bmatrix} u \\ p \end{bmatrix} + *. \end{aligned} \tag{5.3}$$

Therefore, we obtain from (4.39) and (5.2)

$$\begin{cases} (0,1)\,(I + \mathscr{E}_a)\, D_c \begin{bmatrix} u \\ p \end{bmatrix} = *, \\ (0,1)\,(I - \mathscr{E}_b)\, D_c \begin{bmatrix} u \\ p \end{bmatrix} = *, \end{cases} \tag{5.4}$$

i.e.,

$$\begin{bmatrix} c_a \rho_a & 1 \\ -c_b \rho_b & 1 \end{bmatrix} D_c \begin{bmatrix} u \\ p \end{bmatrix} = *, \tag{5.5}$$

which is obviously non-degenerate.

3. For the k-th order compatibility, take tangential derivatives (D_l^k, D_r^k) to the boundary conditions on $x = \chi_{l(r)}$, take derivative D_c^k to the boundary condition on $x = \theta$, and D_c^{k-1} to the interior equations in $\Omega_{a(b)}$.

   This yields a linear system similar to (4.32)-(4.36), except the condition (4.32) is replaced by corresponding Eqs (4.35),(4.36). After simplification, we are left with a system similar to (4.37):

$$\begin{cases} D_l^k p_b = (0,1) D_l^k \begin{bmatrix} u_a \\ p_a \end{bmatrix} = *, \\ D_c^k u + \frac{1}{\rho_{a(b)}} D_c^{k-1} \partial_x p_{a(b)} = *, \\ D_c^k p + \rho_{a(b)} c_{a(b)}^2 D_c^{k-1} \partial_x u_{a(b)} = *; \\ D_r^k p_b = (0,1) D_r^k \begin{bmatrix} u_b \\ p_b \end{bmatrix} = *. \end{cases} \tag{5.6}$$

From (5.3) and (5.6), we obtain

$$\begin{cases} (0,1)\,(I + \mathscr{E}_a)^k\, D_c^k \begin{bmatrix} u \\ p \end{bmatrix} = *, \\ (0,1)\,(I - \mathscr{E}_b)^k\, D_c^k \begin{bmatrix} u \\ p \end{bmatrix} = *. \end{cases} \tag{5.7}$$

By (4.39), (5.7) is equivalent to

$$\begin{bmatrix} c_a\rho_a & 1 \\ -c_b\rho_b & 1 \end{bmatrix} D_c^k \begin{bmatrix} u \\ p \end{bmatrix} = *, \tag{5.8}$$

which is obviously non-degenerate.

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