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Cauchy problem with general discontinuous initial data along a smooth curve for 2-d Euler system

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Received 13 January 2014; revised 7 May 2014

Abstract

We study the Cauchy problems for the isentropic 2-d Euler system with discontinuous initial data along a smooth curve. All three singularities are present in the solution: shock wave, rarefaction wave and contact discontinuity. We show that the usual restrictive high order compatibility conditions for the initial data are automatically satisfied. The local existence of piecewise smooth solution containing all three waves is established.

MSC: 35L60; 35L65; 35L67; 76L05

Keywords: Cauchy problem; Euler systems; Discontinuous initial data; Shock; Rarefaction wave; Contact discontinuity

1. Introduction

The study of the quasilinear hyperbolic systems of conservation laws originates from many physical problems. A fundamental problem for the systems is the Cauchy problem: to determine the solution satisfying given initial data. An important phenomenon for nonlinear hyperbolic systems of conservation laws is that a general solution may develop singularities, no matter how smooth the initial data are. Therefore, one must study the weak solutions of the systems, i.e., the

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http://dx.doi.org/10.1016/j.jde.2014.05.027
0022-0396/

Please cite this article in press as: S. Chen, D. Li, Cauchy problem with general discontinuous initial data along a smooth curve for 2-d Euler system, J. Differential Equations (2014), http://dx.doi.org/10.1016/j.jde.2014.05.027

solutions with singularities. Hence it is natural to study the Cauchy problem with discontinuous initial data.

In one-space-dimensional case the typical Cauchy problem with discontinuous data is the Riemann problem, in which the initial data are two constant states with discontinuity at the origin. For the Riemann problem of the genuinely nonlinear or linearly degenerate hyperbolic systems the theory on existence, uniqueness and stability has been established. In accordance, the Riemann problem for the gas dynamic system is completely solved (see [13,25]). Furthermore, when the initial data are not piecewise constant, but only piecewise smooth, the local existence of the solution to quasilinear hyperbolic system is also established (see [16]). Particularly, if the initial data have only one discontinuity at the origin, and are smooth up to this point, then the solution often has the following structure: several waves, including shocks, rarefaction waves and/or contact discontinuity, issuing from the origin. These waves shape like a fan, and such a structure is called *fan-shaped structure* (see [7]).

The one-space-dimensional model assumes that all quantities under consideration are uniform with respect to other space variables. Obviously, many physical problems do not have such a property. Therefore, it is necessary to study the multi-dimensional quasilinear hyperbolic systems including their Cauchy problems or various boundary value problems. Due to the complexity of characteristic varieties of multi-dimensional systems, the nonlinear wave structure for these systems is abundant.

In the two-space dimensional case considered in this paper, the initial data are assumed to be discontinuous along a smooth curve, and the data are smooth up to the curve. Then the solution would usually contain several nonlinear waves issuing from such an initial curve. The nonlinear waves are composed of shocks, simple waves and contact discontinuity. They form a twisted fan so that the wave structure is still called *fan-shaped structure*.

The study of the Cauchy problem of multi-dimensional hyperbolic system of conservation laws with discontinuous data attracted one's attention for a long time. In 1983 A. Majda started the study of weak solutions to multidimensional system of conservation laws. He applied the theory of microlocal analysis to prove the stability and existence of the solution to the Cauchy problem of nonlinear multidimensional hyperbolic systems involving a shock front, which issues from a curve carrying the discontinuity of the initial data [17]. Later, in 1989 S. Alinhac [1] employed Nash–Moser iterative scheme to overcome the "derivative loss" difficulty on the characteristic boundary and proved the existence of the solution with a rarefaction wave for the Cauchy problem of nonlinear multidimensional hyperbolic systems, where the rarefaction wave also issues from the curve carrying the discontinuity of the initial data. More recently, J. Coulombel and P. Secchi [8] proved the corresponding local existence of solution with a contact discontinuity to Cauchy problem of the Euler system, using a delicate analysis on the Kreiss–Lopatinskii condition.

In all these works the initial data are highly restricted to ensure that one and only one nonlinear wave will issue from the initial curve of discontinuity. Such demanding restrictions are called compatibility conditions which consist of many equalities involving the value of the initial data and their derivatives along the given curve of discontinuity. Obviously, such conditions are not only difficult to satisfy, but also difficult to check.

When the initial data do not satisfy the above restrictions, the weak solution may contain more than one nonlinear wave, like two shocks (see [4,20,23]), one shock and one rarefaction wave (see [15]). Other results on physical problems with fan-shaped wave configurations can also be found in [5] for the supersonic flow past a curved wedge, in [6] for shock reflection by a smooth surface, in [20] for propagation of sound waves etc.

In general, for the smooth data containing discontinuity on a given smooth curve, one would expect that the weak solution should develop all three kinds of nonlinear waves (shock, rarefaction wave and contact discontinuity), without satisfying the complex and demanding compatibility conditions. We notice that in 1-d case, for both Riemann problem and general Riemann problem, such restrictions are not necessary. That is, if the difference of the right and left limit of the initial data at the point carrying the discontinuity of the data is small, then the Cauchy problem is solvable, and a solution with fan-shaped wave structure can be constructed. It is then natural to try to remove or simplify such restrictions given in [1,8,17]. Correspondingly, we try to answer the following questions. Are such general multidimensional Cauchy problems still solvable? What is the wave structure near the curve carrying initial discontinuity?

In this paper, we will prove that the similar conclusion for the two-dimensional Euler systems is still valid. For convenience, we will consider only isentropic Euler system, while the discussion can be extended to the general quasilinear hyperbolic system later. Our main conclusion in this paper can be described as follows: if the frozen Riemann problem at the origin has a corresponding 1-d piecewise solution with stable complete nonlinear wave structure and the stability condition of the 2-d contact discontinuity is satisfied, then the two-dimensional Cauchy problem also has a local piecewise smooth solution with the same fan-shaped wave structure near the initial curve of discontinuity.

Denote ρ, p the density and pressure of the fluid, (u, v) the velocity in the (x, y) direction. In two space dimension, the Euler system of isentropic flow can be written as follows:

$$\begin{cases} \dfrac{\partial\rho}{\partial t}+\dfrac{\partial(\rho u)}{\partial x}+\dfrac{\partial(\rho v)}{\partial y}=0, \\ \dfrac{\partial(\rho u)}{\partial t}+\dfrac{\partial(p+\rho u^2)}{\partial x}+\dfrac{\partial(\rho uv)}{\partial y}=0, \\ \dfrac{\partial\rho v}{\partial t}+\dfrac{\partial(\rho uv)}{\partial x}+\dfrac{\partial(p+\rho v^2)}{\partial y}=0, \end{cases} \tag{1.1}$$

or simply written as

$$\partial_t H_0+\partial_x H_1+\partial_y H_2=0,$$

with

$$H_0=\begin{pmatrix}\rho\\ \rho u\\ \rho v\end{pmatrix},\qquad H_1=\begin{pmatrix}\rho u\\ p+\rho u^2\\ \rho uv\end{pmatrix},\qquad H_2=\begin{pmatrix}\rho v\\ \rho uv\\ p+\rho v^2\end{pmatrix}.$$

Denote $U=(\rho, u, v)$ the unknown functions. For smooth solutions, the system (1.1) is equivalent to the following

$$H_0'U_t+H_1'U_x+H_2'U_y=0,$$

or equivalently,

$$LU\equiv\partial_t U+A_1(U)\partial_x U+A_2(U)\partial_y U=0. \tag{1.2}$$

Here, $U = (\rho, u, v)$ and

$$H_0' = \begin{bmatrix} 1 & 0 & 0 \\ u & \rho & 0 \\ v & 0 & \rho \end{bmatrix}, \qquad H_1' = \begin{bmatrix} u & \rho & 0 \\ c^2 + u^2 & 2\rho u & 0 \\ uv & \rho v & \rho u \end{bmatrix}, \qquad H_2' = \begin{bmatrix} v & 0 & \rho \\ uv & \rho v & \rho u \\ c^2 + v^2 & 0 & 2\rho v \end{bmatrix},$$

$$A_1 = \begin{bmatrix} u & \rho & 0 \\ c^2/\rho & u & 0 \\ 0 & 0 & u \end{bmatrix}, \qquad A_2 = \begin{bmatrix} v & 0 & \rho \\ 0 & v & 0 \\ c^2/\rho & 0 & v \end{bmatrix},$$

with $c^2 = p'(\rho) > 0$.

Consider the Cauchy problem for (1.1). Let Γ: $x = \phi_0(y)$ be a smooth curve on the initial plane with $\phi_0(0) = 0$, $\phi_0'(0) = 0$. The initial data are given as

$$U = \begin{cases} U_-(x, y), & \text{if } x < \phi_0(y), \\ U_+(x, y), & \text{if } x > \phi_0(y). \end{cases} \tag{1.3}$$

We assume that $U_-(x, y)$, $U_+(x, y)$ are smooth up to the curve Γ. Here and afterwards, the word "smooth" means C^∞-smooth, unless specified otherwise.

The matrix $A_1(U) + A_2(U)\phi_y$ has three distinct real eigenvalues

$$\lambda_- = u - v\phi_y - a\sqrt{1 + \phi_y^2},$$
$$\lambda_0 = u - v\phi_y,$$
$$\lambda_+ = u - v\phi_y + a\sqrt{1 + \phi_y^2}$$

with $\lambda_- < \lambda_0 < \lambda_+$.

Remark 1.1. Since we only consider the local existence and structure of the solution near the origin, then we may assume the flow, including the initial data and the curve $x = \phi_0(y)$ being periodic with respect to y. Therefore, we only have to consider the problem (1.3), (1.5) in a period of the variable y, so that one can avoid the trouble of divergence of integration in the variable y (see [5]). Such a remark or a corresponding treatment is omitted in [1,9] and tacitly assumed.

In our discussion of (1.1), (1.3), we will refer to the following *accompanying 1-d Riemann problem* with constant initial data

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial(\rho u)}{\partial x} = 0, \\ \dfrac{\partial \rho u}{\partial t} + \dfrac{\partial(p + \rho u^2)}{\partial x} = 0, \end{cases} \tag{1.4}$$

$$(\rho, u)(0, x) = \begin{cases} (\rho_-(0,0), u_-(0,0)), & \text{if } x < 0, \\ (\rho_+(0,0), u_+(0,0)), & \text{if } x > 0. \end{cases} \tag{1.5}$$

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The Riemann problem (1.4), (1.5) is the 1-d version of (1.1), (1.3) with the constant initial data with the values of $(\rho_\pm, u_\pm)$ at the origin $(0, 0)$.

It is well known that for the Riemann problem (1.4), (1.5), shocks or rarefaction waves may be produced from the initial discontinuity. The four possible combinations are SS, SR, RS and RR, where “S” stands for a shock, and “R” stands for a rarefaction wave [25], while the first letter represents a left-propagating wave, and the second letter represents a right-propagating wave.

In some special cases, shock or rarefaction waves may degenerate into a characteristic carrying weak discontinuities (of the derivatives of the solution), called sound wave [22]. In this paper we will not consider these degenerate cases. Meanwhile, we assume $v_-^0 \neq v_+^0$, hence a contact discontinuity for the 2-d Cauchy problem will appear.

The main theorem of this paper can be stated as follows.

Theorem 1.2. *For the Cauchy problem (1.1), (1.3), assuming that*

- (C1) *The problem (1.4), (1.5) has a solution with complete nonlinear wave configuration, i.e., one of the four combinations: SS, SR, RS or RR;*
- (C2) *The shocks satisfy the Lax’s inequality, i.e., supersonic flowing into the shock front and subsonic flowing out of the shock front;*
- (C3) *At origin $(0, 0, 0)$: $|v_-^0 - v_+^0| > 2\sqrt{2}c$, with c being the sonic speed of the center state between two nonlinear waves produced by (1.4), (1.5),*

the 2-d Cauchy problem of the isentropic Euler system (1.1), (1.3) admits a unique piecewise smooth solution with fan-shaped wave structure in a neighborhood of the origin.

Q7 **Remark 1.3.** In the study of 1-d Riemann problem (1.4), (1.5) (see e.g. [25]), the tool of wave curve is often used. Given a state (p_-, u_-), it can be connected from right with (p, u) either by a left-propagating shock, or by a left-propagating rarefaction wave. The possible state (p, u) thus connected to (p_-, u_-) forms a curve issuing from (p_-, u_-) on the (p, u) plane, called the wave curve $\Sigma(p_-, u_-)$. There is also a similar wave curve $\Sigma(p_+, u_+)$ for right-propagating waves.

Q9 In terms of wave curves, the condition (C1) in Theorem 1.2 is equivalent to the requirement that the two wave curves $\Sigma(p_-, u_-)$ and $\Sigma(p_+, u_+)$ intersect transversally at a point other than (p_-, u_-) or (p_+, u_+).

Remark 1.4. If the shocks in (C1) are understood to be stable in the sense of Lax, then the condition (C2) is automatically satisfied.

In Theorem 1.2, the neighborhood of the origin is divided by nonlinear waves to several sectors, while the solution of the Euler system is C^∞ smooth in each sector. If the initial data, including the curve $x = \phi_0(y)$, are only finitely smooth up to the curve Γ, then we can also obtain a piecewise finitely-smooth solution. Moreover, the smoothness of the solution will be much lower than the smoothness of the initial data on both sides of Γ, as in the discussions of the rarefaction wave or the contact discontinuity (see [1,9]). However, to focus our attention to the existence and wave structure of the solutions we will not discuss the case for the piecewise finitely smooth data.

The Cauchy problems with discontinuous data for Euler system involving only one nonlinear wave were discussed in [1,9,17] separately. The linear estimates in these works are the basis of

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the linear estimates used in proving Theorem 1.2. By localization, we are able to use the results for linear estimates in [1,8,9,17].

When only one nonlinear wave was discussed for the Cauchy problems with discontinuous data, a very strong compatibility condition is always required to obtain the existence of the solution, see [1,9,17]. The advantage of treating all three waves at the same time lets us to reduce the compatibility requirement to the minimum (C1), (C2), and to obtain an approximate solution which is C^∞ compatible. Such approximate solution will serve as starting term in the process of iteration to establish the precise solution.

The rest of the paper is arranged as follows. The compatibility conditions will be carefully discussed in Section 2. We will show how to determine the derivatives at the origin of the piecewise smooth solution in all sectors under the assumptions (C1), (C2). Then we use Borel technique to construct a C^∞ smooth approximate solution. In Section 3 we reformulate the Cauchy problem for two-dimensional Euler system to a set of boundary value problems. In Section 4 we describe the Nash–Moser iteration scheme employed for all reduced boundary value problems. In Section 5 by employing the estimates for linearized problems in each case, we summarize a unified estimate to the whole problem. Finally, in Section 6 we prove the convergence of the revised Nash–Maser iteration and hence prove the main Theorem 1.2.

2. Compatibility

In the study of initial–boundary value problems or free boundary problems, the compatibility is a standard requirement for the existence of smooth or piecewise smooth solutions. Such requirement is necessary so that the initial and boundary conditions do not conflict with the partial differential equations at the intersection curve of the initial manifold and the boundary. The compatibility conditions usually consist of a system of algebraic equations for the initial data and their derivatives. The higher the order of compatibility, the higher the order of the derivatives are involved, and the more equations are contained in the system.

Even though such conditions are necessary for the existence of the expected solution, practically it is very tedious to verify and difficult to satisfy them for a given set of initial data. For the problem considered in this paper, due to the fact that the solution includes the complete set of wave patterns, the situation actually becomes much better, i.e.,

Theorem 2.1. *For the Cauchy problem (1.1), (1.3) under the assumptions* (C1) *and* (C2)*, the compatibility conditions are automatically satisfied up to any order k for any smooth initial data $U_\pm$.*

In this section, we are going to prove Theorem 2.1 for all three possible fan-shaped wave structures: SCS, SCR, and RCR (here, "S" stands for shock, "R" stands for rarefaction wave and "C" stands for contact discontinuity) respectively.

2.1. Compatibility condition for the case SCS

The wave configuration of the case SCS is illustrated in Fig. 2.1. The left-propagating shock and right-propagating shock are denoted by S_l: $x = \phi_l(t, y)$ and S_r: $x = \phi_r(t, y)$, the contact discontinuity is denoted by C: $x = \theta(t, y)$.

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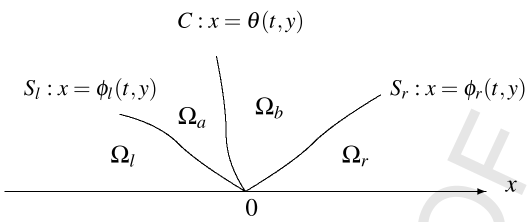

Fig. 2.1. SCS wave configuration.

Let $U_a(t, x, y)$ ($U_b(t, x, y)$, resp.) be the solution in the angular domain Ω_a (Ω_b, resp.) between S_l (S_r, resp.) and C. And $U_l(t, x, y)$ ($U_r(t, x, y)$, resp.) is the solution in the domain Ω_l (Ω_r, resp.) left (right, resp.) to S_l (S_r, resp.).

Then U_a, U_b, U_l and U_r satisfy Eqs. (1.1) in their individual domain and the initial condition:

$$\begin{aligned} &U_l(0, x, y) = U_-(x, y) \quad \text{in } x < \phi_0(y); \\ &U_r(0, x, y) = U_+(x, y) \quad \text{in } x > \phi_0(y); \\ &\phi_l(0, y) = \phi_r(0, y) = \theta(0, y) = \phi_0(y). \end{aligned} \tag{2.1}$$

Besides, they also satisfy the following Rankine–Hugoniot conditions:

$$\phi_{lt}[H_0]_-^a + \phi_{ly}[H_2]_-^a - [H_1]_-^a = 0 \quad \text{on } x = \phi_l(t, y); \tag{2.2}$$

$$\phi_{rt}[H_0]_+^b + \phi_{ry}[H_2]_+^b - [H_1]_+^b = 0 \quad \text{on } x = \phi_r(t, y); \tag{2.3}$$

$$\begin{cases} \theta_t + v_b\theta_y - u_b = 0, \\ \rho_b - \rho_a = 0, \\ \theta_y(v_b - v_a) - (u_b - u_a) = 0 \end{cases} \quad \text{on } x = \theta(t, y); \tag{2.4}$$

with the notation $[f]_-^a = f_a - f_-$ as usual.

The compatibility requires that one can uniquely determine the values of the functions $(U_a, U_b, \phi_l, \phi_r, \theta)$ and their derivatives at Γ from Eqs. (1.1) and the boundary conditions (2.2)–(2.4). It is equivalent to the existence of an approximate solution which satisfies (1.1) and (2.2)–(2.4) near Γ up to the order $O(t^{k+1})$ (for the k-th order compatibility).

The 0-order compatibility does not involve the derivatives of (U_a, U_b) and we have 9 variables

$$U_a\big(0, \phi_0(y), y\big),\ U_b\big(0, \phi_0(y), y\big),\ \partial_t\phi_l(0, y),\ \partial_t\phi_r(0, y),\ \partial_t\theta(0, y)$$

to satisfy 9 equations in the boundary conditions (2.2)–(2.4).

Due to the continuity in the variable y, by the implicit function theorem we need to show that at the origin $(0, 0, 0)$, the system (2.2)–(2.4) has one solution

$$U_a(0, 0, 0),\ U_b(0, 0, 0),\ \partial_t\phi_l(0, 0),\ \partial_t\phi_r(0, 0),\ \partial_t\theta(0, 0),$$

and the corresponding Jacobian non-degenerate.

The existence of one solution at $(0, 0, 0)$ is guaranteed by the non-degenerate condition (C1) in Theorem 1.2. Indeed, at $(0, 0, 0)$ the equations in (2.2)–(2.4) become

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$$\phi_{lt}(0)\begin{pmatrix}\rho_a-\rho_-\\ \rho_a u_a-\rho_-u_-\\ \rho_a v_a-\rho_-v_-\end{pmatrix}-\begin{pmatrix}\rho_a u_a-\rho_-u_-\\ p_a+\rho_a u_a^2-p_--\rho_-u_-^2\\ \rho_a u_a v_a-\rho_-u_-v_-\end{pmatrix}=0, \tag{2.5}$$

$$\phi_{rt}(0)\begin{pmatrix}\rho_b-\rho_+\\ \rho_b u_b-\rho_+u_+\\ \rho_b v_b-\rho_+v_+\end{pmatrix}-\begin{pmatrix}\rho_b u_b-\rho_+u_+\\ p_b+\rho_b u_b^2-p_+-\rho_+u_+^2\\ \rho_b u_b v_b-\rho_+-u_+v_+\end{pmatrix}=0, \tag{2.6}$$

$$\rho_a=\rho_b, \qquad u_a=u_b=\theta_t(0). \tag{2.7}$$

v_a and v_b, each appears only in one equation of (2.5), (2.6) and each has non-zero coefficient $\rho_a(\phi_{lt}-u_a)$ or $\rho_b(\phi_{rt}-u_b)$ by (C2). Eliminating v_a, v_b, ρ_b, u_b and $\theta_t(0)$ from (2.5)–(2.7), we end up with a 4×4 system for $(\rho_a, u_a, \phi_{lt}, \phi_{rt})$:

$$\phi_{lt}(0)\begin{pmatrix}\rho_a-\rho_-\\ \rho_a u_a-\rho_-u_-\end{pmatrix}-\begin{pmatrix}\rho_a u_a-\rho_-u_-\\ p_a+\rho_a u_a^2-p_--\rho_-u_-^2\end{pmatrix}=0, \tag{2.8}$$

$$\phi_{rt}(0)\begin{pmatrix}\rho_b-\rho_+\\ \rho_b u_b-\rho_+u_+\end{pmatrix}-\begin{pmatrix}\rho_b u_b-\rho_+u_+\\ p_b+\rho_b u_b^2-p_+-\rho_+u_+^2\end{pmatrix}=0. \tag{2.9}$$

These equations are nothing but the Rankine–Hugoniot conditions for the Riemann problem (1.4), (1.5), for which the existence of solution is provided by the condition (C1).

Denote the left hand sides of (2.2)–(2.4) by F_1, F_2, F_3, then we need to prove the Jacobian to be non-zero at the origin $(0,0,0)$

$$\det J|_0=\det\frac{\partial(F_1,F_2,F_3)}{\partial(\phi_{lt},\phi_{rt},\theta_t,\rho_a,u_a,v_a,\rho_b,u_b,v_b)}(0,0,0)\neq 0. \tag{2.10}$$

The Jacobian J is the coefficient matrix of the linearized system (2.2)–(2.4). Similarly as above, we can eliminate (θ_t, v_a, v_b) from this linear system. By (2.7), we can also eliminate ρ_b, u_b and (2.10) can be reduced to

$$\det\begin{pmatrix}\rho_a-\rho_- & 0 & \phi_{lt}-u_a & -\rho_a\\ \rho_a u_a-\rho_-u_- & 0 & \phi_{lt}u_a-c_a^2-u_a^2 & \phi_{lt}\rho_a-2\rho_a u_a\\ 0 & \rho_a-\rho_+ & \phi_{rt}-u_a & -\rho_a\\ 0 & \rho_a u_a-\rho_+u_+ & \phi_{rt}u_a-c_a^2-u_a^2 & \phi_{rt}\rho_a-2\rho_a u_a\end{pmatrix}\neq 0.$$

Noticing that $\phi_{lt}(\rho_a-\rho_-)=\rho_a u_a-\rho_-u_-$ and $\phi_{rt}(\rho_a-\rho_+)=\rho_a u_a-\rho_+u_+$, (2.10) is equivalent to

$$\det J|_0=2\rho_a(\rho_a-\rho_-)(\rho_a-\rho_+)\det\begin{pmatrix}c_a^2+(\phi_{lt}-u_a)^2 & \phi_{lt}-u_a\\ c_a^2+(\phi_{rt}-u_a)^2 & \phi_{rt}-u_a\end{pmatrix}\neq 0. \tag{2.11}$$

That (2.11) is true follows from the Lax' shock inequality (C2):

$$u_a-\phi_{lt}>0>u_a-\phi_{rt}.$$

This finishes the proof of 0-order compatibility.

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For the *first order compatibility*, we notice that once the values of $(U_a, U_b, \phi_l, \phi_r, \theta)$ are determined at the initial discontinuity Γ, then all their derivatives tangential to Γ are uniquely determined. Therefore, the first order compatibility consists of 15 linear equations for the 15 variables

$$
\begin{aligned}
&U_{at}\big(0,\phi_0(y),y\big),\ U_{an}\big(0,\phi_0(y),y\big),\ U_{bt}\big(0,\phi_0(y),y\big),\ U_{bn}\big(0,\phi_0(y),y\big),\\
&\phi_{ltt}(0,y),\ \phi_{rtt}(0,y),\ \theta_{tt}(0,y).
\end{aligned}
$$

Here (U_{an}, U_{bn}) denote the normal derivative to Γ.

Again by the continuity in y and the implicit function theorem, we need only to show that the Jacobian of these 15 equations is non-degenerate at $(0,0,0)$. At the origin, $(U_{an}, U_{bn}) = (U_{ax}, U_{bx})$, $\phi_{ly} = \phi_{ry} = \theta_y = 0$ and $\theta_t = u_a = u_b = u$.

Let (D_l, D_c, D_r) denote the differential operators:

$$
D_l = \partial_t + \phi_{lt}\partial_x, \qquad D_c = \partial_t + u\partial_x, \qquad D_r = \partial_t + \phi_{rt}\partial_x.
$$

Taking tangential derivatives of Eqs. (2.2)–(2.4) in the t–x plane and evaluating them at $(0,0,0)$, we obtain

$$
\phi_{ltt}[H_0]_-^a + \big(\phi_{lt}H_0' - H_1'\big)D_lU_a = *, \tag{2.12}
$$

$$
\phi_{rtt}[H_0]_+^b + \big(\phi_{rt}H_0' - H_1'\big)D_rU_b = *, \tag{2.13}
$$

with

$$
\phi_{l(r)t}H_0' - H_1' = \begin{bmatrix} \phi_{l(r)t} - u & -\rho & 0 \\ u(\phi_{l(r)t} - u) - c^2 & \rho(\phi_{l(r)t} - 2u) & 0 \\ v(\phi_{l(r)t} - u) & -\rho v & \rho(\phi_{l(r)t} - u) \end{bmatrix}
$$

and

$$
\begin{aligned}
&\theta_{tt} - D_cu_a = *,\\
&D_c\rho_a - D_c\rho_b = *,\\
&D_cu_a - D_cu_b = *,
\end{aligned} \tag{2.14}
$$

where $*$ stands for terms already determined by lower order compatibility.

At the origin $(0,0,0)$, the interior equation (1.2) becomes

$$
\begin{cases}
D_c\rho_{a(b)} + \rho\partial_x u_{a(b)} = *,\\
D_cu_{a(b)} + c^2/\rho\,\partial_x\rho_{a(b)} = *, \quad \text{in } \Omega_{a(b)}.\\
D_cv_{a(b)} = *,
\end{cases} \tag{2.15}
$$

The linear system (2.12)–(2.15) consists of 15 equations for the 15 variables

$$
(\phi_{ltt}, \phi_{rtt}, \theta_{tt}, U_{at}, U_{ax}, U_{bt}, U_{bx}),
$$

we are going to confirm that the system has a unique solution for these variables.

Please cite this article in press as: S. Chen, D. Li, Cauchy problem with general discontinuous initial data along a smooth curve for 2-d Euler system, J. Differential Equations (2014), http://dx.doi.org/10.1016/j.jde.2014.05.027

Now the system (2.12)–(2.15) can be simplified.

The variable θ_{tt} appears only in one equation (2.14) and can be eliminated. Since there is no restriction in (2.14) on (v_t, v_x), so (v_{at}, v_{ax}) and (v_{bt}, v_{bx}) are uncoupled with each other. In addition, (v_{at}, v_{ax}) appear only in the third equation of (2.15) in the form $D_c v_a$ and they appear only in the third equation of (2.12) in the form $D_l v$, both with non-zero coefficients. Since D_c, D_l are not parallel, (v_{at}, v_{ax}) can be uniquely determined by $(D_c v_a, D_r v_a)$, which in turn can be uniquely determined by other variables from (2.15), (2.12). Same argument also applies to (v_{bt}, v_{bx}).

Therefore, we can eliminate the variables $(\theta_{tt}, v_{at}, v_{ax}, v_{bt}, v_{bx})$ from (2.12)–(2.15) and obtain 10 equations for the 10 variables $(\phi_{ltt}, \rho_{at}, \rho_{ax}, u_{at}, u_{ax}, \phi_{rtt}, \rho_{bt}, \rho_{bx}, u_{bt}, u_{bx})$.

From (2.14) and (2.15), we have

$$D_c\rho_a - D_c\rho_b = *, \qquad D_c u_a - D_c u_b = *;$$
$$\partial_x\rho_a - \partial_x\rho_b = *, \qquad \partial_x u_a - \partial_x u_b = *.$$

Hence, we can further eliminate $D_c\rho_b$, $D_c u_b$, $\partial_x\rho_b$ and $\partial_x u_b$ and use the same equation (2.15) for both (ρ_a, u_a) and (ρ_b, u_b). For convenience, we will drop the subscript "a" and denote in the following $\rho_a = \rho$ and $u_a = u$.

Now there remain 6 equations for the 6 variables $(\phi_{ltt}, D_c\rho, D_c u, \rho_x, u_x, \phi_{rtt})$:

$$\phi_{ltt}\begin{bmatrix}\rho-\rho_-\\ \phi_{lt}(\rho-\rho_-)\end{bmatrix}+\begin{bmatrix}\phi_{lt}-u & -\rho\\ u(\phi_{lt}-u)-c^2 & \rho(\phi_{lt}-2u)\end{bmatrix}D_l\begin{bmatrix}\rho\\ u\end{bmatrix}=*; \tag{2.16}$$

$$\phi_{rtt}\begin{bmatrix}\rho-\rho_-\\ \phi_{rt}(\rho-\rho_-)\end{bmatrix}+\begin{bmatrix}\phi_{rt}-u & -\rho\\ u(\phi_{rt}-u)-c^2 & \rho(\phi_{rt}-2u)\end{bmatrix}D_r\begin{bmatrix}\rho\\ u\end{bmatrix}=*; \tag{2.17}$$

$$\begin{cases}D_c\rho+\rho\partial_x u=*,\\ D_c u+c^2/\rho\partial_x\rho=*.\end{cases} \tag{2.18}$$

The system (2.18) can be written as

$$D_c\begin{bmatrix}\rho\\ u\end{bmatrix}+c\mathscr{E}\partial_x\begin{bmatrix}\rho\\ u\end{bmatrix}=*,$$

with

$$\mathscr{E}\equiv\begin{bmatrix}0 & \rho/c\\ c/\rho & 0\end{bmatrix}=\mathscr{E}^{-1}. \tag{2.19}$$

Hence

$$\begin{cases}D_l=D_c+(\phi_{lt}-u)\partial_x=(I-\beta_l\mathscr{E})D_c,\\ D_r=D_c+(\phi_{rt}-u)\partial_x=(I-\beta_r\mathscr{E})D_c\end{cases} \tag{2.20}$$

with

$$\beta_l\equiv\frac{\phi_{lt}-u}{c}, \qquad \beta_r\equiv\frac{\phi_{rt}-u}{c}. \tag{2.21}$$

By the Lax' shock inequality (C2), we have $|\beta_l|<1$ and $|\beta_r|<1$ in (2.21).

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Using (2.20) to replace (ρ_x, u_x) in (2.16), (2.17), we obtain a system of 4 equations for the 4 variables $(\phi_{ltt}, \phi_{rtt}, D_c\rho, D_cu)$.

Eliminating ϕ_{ltt} and ϕ_{rtt} from (2.16), (2.17), we obtain a 2×2 system for $(D_c\rho, D_cu)$

$$\begin{cases} (-\phi_{lt} \quad 1)\begin{bmatrix} \phi_{lt}-u & -\rho \\ u(\phi_{lt}-u)-c^2 & \rho(\phi_{lt}-2u) \end{bmatrix}(I-\beta_l\mathscr{E})D_c\begin{bmatrix}\rho\\ u\end{bmatrix}=*; \\ (-\phi_{rt} \quad 1)\begin{bmatrix} \phi_{rt}-u & -\rho \\ u(\phi_{rt}-u)-c^2 & \rho(\phi_{rt}-2u) \end{bmatrix}(I-\beta_r\mathscr{E})D_c\begin{bmatrix}\rho\\ u\end{bmatrix}=*, \end{cases}$$

which can be simplified into

$$\begin{cases} [-(\phi_{lt}-u)^2-c^2 \quad 2\rho(\phi_{lt}-u)]\begin{bmatrix} 1 & -\beta_l\frac{\rho}{c} \\ -\beta_l\frac{c}{\rho} & 1 \end{bmatrix}D_c\begin{bmatrix}\rho\\ u\end{bmatrix}=*; \\ [-(\phi_{rt}-u)^2-c^2 \quad 2\rho(\phi_{rt}-2u)]\begin{bmatrix} 1 & -\beta_l\frac{\rho}{c} \\ -\beta_r\frac{c}{\rho} & 1 \end{bmatrix}D_c\begin{bmatrix}\rho\\ u\end{bmatrix}=*. \end{cases} \tag{2.22}$$

The system (2.22) has a unique solution if and only if the following determinant is non-zero

$$\begin{bmatrix} [(\phi_{lt}-u)^2+c^2]+2\beta_lc(\phi_{lt}-u) & \beta_l[(\phi_{lt}-u)^2+c^2]+2c(\phi_{lt}-u) \\ [(\phi_{rt}-u)^2+c^2]+2\beta_rc(\phi_{rt}-u) & \beta_r[(\phi_{rt}-u)^2+c^2]+2c(\phi_{rt}-u) \end{bmatrix}.$$

This is true because from $\beta_l<0<\beta_r$ and $|\beta_l|<1$, $|\beta_r|<1$, we have

$$\begin{aligned} &\left[(\phi_{lt}-u)^2+c^2\right]+2\beta_lc(\phi_{lt}-u)>0, &&\beta_l\left[(\phi_{lt}-u)^2+c^2\right]+2c(\phi_{lt}-u)<0, \\ &\left[(\phi_{rt}-u)^2+c^2\right]+2\beta_rc(\phi_{lt}-u)>0, &&\beta_r\left[(\phi_{rt}-u)^2+c^2\right]+2c(\phi_{rt}-u)>0. \end{aligned}$$

Consider now the general *k-th order compatibility*. Taking the tangential derivatives of Eqs. (2.2)–(2.4) and then evaluating them at the origin $(0,0,0)$, we obtain

$$\begin{cases} \partial_t^{k+1}\phi_l[H_0]_-^a+\left(\phi_{lt}H_0'-H_1'\right)D_l^kU_a=*, \\ \partial_t^{k+1}\phi_r[H_0]_+^b+\left(\phi_{rt}H_0'-H_1'\right)D_r^kU_b=*, \end{cases} \tag{2.23}$$

$$\begin{cases} \partial_t^{k+1}\theta-D_c^ku_a=*, \\ D_c^k\rho_a-D_c^k\rho_b=*, \\ D_c^ku_a-D_c^ku_b=*. \end{cases} \tag{2.24}$$

From the interior equation (1.2), we have

$$\begin{cases} D_c^k\rho+\rho\partial_xD_c^{k-1}u=*, \\ D_c^ku+c^2/\rho\partial_xD_c^{k-1}\rho=*, \quad \text{in } \Omega_{a,b}. \\ D_c^kv=*, \end{cases} \tag{2.25}$$

The linear system (2.23)–(2.25) consists of 15 equations for the 15 variables

$$\partial_t^{k+1}\phi_l,\ \partial_t^{k+1}\phi_r,\ \partial_t^{k+1}\theta,\ \partial_t^k U_a,\ \partial_t^{k-1}U_{ax},\ \partial_t^k U_b,\ \partial_t^{k-1}U_{bx}.$$

As in the first order case, the variables $(\partial_t^{k+1}\theta, \partial_t^k v_a, \partial_t^{k-1}v_{ax}, \partial_t^k v_b, \partial_t^{k-1}v_{bx})$, as well as the variables $(D_c^k\rho_b, D_c^k u_b, D_c^{k-1}\rho_{bx}, D_c^{k-1}u_{bx})$ can be eliminated.

We end up with 4 equations for the 4 variables $(\partial_t^{k+1}\phi_l, \partial_t^{k+1}\phi_r, D_c^k\rho, D_c^k u)$:

$$\partial_t^{k+1}\phi_l\begin{bmatrix}\rho-\rho_-\\ \phi_{lt}(\rho-\rho_-)\end{bmatrix}+\begin{bmatrix}\phi_{lt}-u & -\rho\\ u(\phi_{lt}-u)-c^2 & \rho(\phi_{lt}-2u)\end{bmatrix}D_l^k\begin{bmatrix}\rho\\ u\end{bmatrix}=*; \tag{2.26}$$

$$\partial_t^{k+1}\phi_r\begin{bmatrix}\rho-\rho_+\\ \phi_{rt}(\rho-\rho_+)\end{bmatrix}+\begin{bmatrix}\phi_{rt}-u & -\rho\\ u(\phi_{rt}-u)-c^2 & \rho(\phi_{rt}-2u)\end{bmatrix}D_r^k\begin{bmatrix}\rho\\ u\end{bmatrix}=*. \tag{2.27}$$

Eqs. (2.26), (2.27) are similar to Eqs. (2.16), (2.17), except for the terms (D_l^k, D_r^k).

Using the interior equation (2.25) and (2.20) to replace $(D_l^k\rho, D_l^k u)$ and $(D_r^k\rho, D_r^k u)$ by $(D_c^k\rho, D_c^k u)$, we need the following lemma.

Lemma 2.1. *The operators (D_l^k, D_r^k) always have the following form*

$$D_l^k=\delta_l(\alpha_{lk}-\beta_l\mathscr{E})D_c^k,\qquad D_r^k=\delta_r(\alpha_{rk}-\beta_r\mathscr{E})D_c^k, \tag{2.28}$$

where $0<|\beta_l|<\alpha_{lk}\le 1$ and $0<|\beta_r|<\alpha_{rk}\le 1$. δ_l and δ_r are two positive constants which may depend on k and the explicit form of which is of no consequence in our discussion.

Proof. For $k=1$, (2.28) is (2.20), which is obvious by Lax' shock inequality (C2).

By induction, assume (2.28) for $k-1$, then

$$D_l^k=\delta_l(\alpha_{l(k-1)}-\beta_l\mathscr{E})(1-\beta_l\mathscr{E})D_c^k=(1+\alpha_{l(k-1)})(\alpha_{lk}-\beta_l\mathscr{E})D_c^k.$$

We need only to show that

$$\alpha_{lk}=\frac{\alpha_{l(k-1)}+\beta_l^2}{1+\alpha_{l(k-1)}}>|\beta_l|. \tag{2.29}$$

Eq. (2.29) follows from

$$\alpha_{l(k-1)}-|\beta_l|-\alpha_{l(k-1)}|\beta_l|+\beta_l^2=\big(\alpha_{l(k-1)}-|\beta_l|\big)\big(1-|\beta_l|\big)>0$$

by induction assumption. The same argument also applies to D_r^k. This concludes the proof of Lemma 2.1. □

Eliminating $(\partial_t^{k+1}\phi_l, \partial_t^{k+1}\phi_r)$ from (2.26), (2.27) and applying Lemma 2.1, we obtain two equations for two variables $(D_c^k\rho, D_c^k u)$:

$$\begin{aligned}
&\begin{pmatrix} -\phi_{lt} & 1 \end{pmatrix}\begin{bmatrix} \phi_{lt}-u & -\rho \\ u(\phi_{lt}-u)-c^2 & \rho(\phi_{lt}-2u) \end{bmatrix}(\alpha_{lk}-\beta_l\mathscr{E})D_c^k\begin{bmatrix} \rho \\ u \end{bmatrix}=*; \\
&\begin{pmatrix} -\phi_{rt} & 1 \end{pmatrix}\begin{bmatrix} \phi_{rt}-u & -\rho \\ u(\phi_{rt}-u)-c^2 & \rho(\phi_{rt}-2u) \end{bmatrix}(\alpha_{rk}-\beta_r\mathscr{E})D_c^k\begin{bmatrix} \rho \\ u \end{bmatrix}=*.
\end{aligned} \tag{2.30}$$

Q4 The system (2.30) can be written as

$$\begin{aligned}
&\begin{bmatrix} -(\phi_{lt}-u)^2-c^2 & 2\rho(\phi_{lt}-u) \end{bmatrix}\begin{bmatrix} \alpha_{lk} & -\beta_l\frac{\rho}{c} \\ -\beta_l\frac{c}{\rho} & \alpha_{lk} \end{bmatrix}D_c^k\begin{bmatrix} \rho \\ u \end{bmatrix}=*; \\
&\begin{bmatrix} -(\phi_{rt}-u)^2-c^2 & 2\rho(\phi_{rt}-2u) \end{bmatrix}\begin{bmatrix} \alpha_{rk} & -\beta_l\frac{\rho}{c} \\ -\beta_r\frac{c}{\rho} & \alpha_{rk} \end{bmatrix}D_c^k\begin{bmatrix} \rho \\ u \end{bmatrix}=*;
\end{aligned}$$

which has a unique solution $(D_c^k\rho, D_c^k u)$ if and only if the following matrix is non-degenerate

$$\begin{bmatrix} \alpha_{lk}[(\phi_{lt}-u)^2+c^2]+2\beta_l c(\phi_{lt}-u), & \beta_l[(\phi_{lt}-u)^2+c^2]+2\alpha_{lk}c(\phi_{lt}-u) \\ \alpha_{rk}[(\phi_{rt}-u)^2+c^2]+2\beta_r c(\phi_{rt}-u), & \beta_r[(\phi_{rt}-u)^2+c^2]+2\alpha_{rk}c(\phi_{rt}-u) \end{bmatrix}. \tag{2.31}$$

Because $\alpha_{lk} > |\beta_l|$ and $\alpha_{rk} > |\beta_r|$ by Lemma 2.1, we have for the elements in the first column of (2.31)

$$\begin{aligned}
&\alpha_{lk}\big[(\phi_{lt}-u)^2+c^2\big]+2\beta_l c(\phi_{lt}-u)>0, \\
&\alpha_{rk}\big[(\phi_{rt}-u)^2+c^2\big]+2\beta_r c(\phi_{rt}-u)>0.
\end{aligned}$$

Since $\beta_l < 0 < \beta_r$ and $\phi_{lt}-u < 0 < \phi_{rt}-u$, we have for the elements in the second column of (2.31)

$$\begin{aligned}
&\beta_l\big[(\phi_{lt}-u)^2+c^2\big]+2\alpha_{lk}c(\phi_{lt}-u)=\frac{\phi_{lt}-u}{c}\big[2\alpha_{lk}c^2+(\phi_{lt}-u)^2+c^2\big]<0, \\
&\beta_r\big[(\phi_{rt}-u)^2+c^2\big]+2\alpha_{rk}c(\phi_{rt}-u)=\frac{\phi_{rt}-u}{c}\big[2\alpha_{rk}c^2+(\phi_{rt}-u)^2+c^2\big]>0.
\end{aligned}$$

Therefore, (2.31) is non-degenerate.

Once $(D_c^k\rho, D_c^k u)$ is determined, one can obtain $(D_c^{k-1}\rho_x, D_c^{k-1}u_x)$ from Eqs. (2.25). Repeating using (2.25) yields all the k-th derivatives of (ρ, u). This finishes the proof of Theorem 2.1 for the SCS wave pattern.

2.2. *Compatibility condition for the case SCR*

The wave pattern for the case SCR is as follows. See Fig. 2.2.

The initial data are given as (1.3), with the curve Γ carrying the discontinuity of the initial data being $t=0$, $x_2=0$. The left-propagating shock front is denoted by S_l: $x=\phi_l(t,y)$, the contact discontinuity is denoted by C: $x=\theta(t,y)$. The rarefaction wave is described by an angular domain between two characteristics L^-: $x=\chi^-(t,y)$ and L^+: $x=\chi^+(t,y)$. The solution in the angular domain Ω_a (Ω_b, resp.) between S_l (L^-, resp.) and C is denoted by $U_a(t,x,y)$ ($U_b(t,x,y)$, resp.), the solution in the domain Ω_R formed by the rarefaction wave between L_-

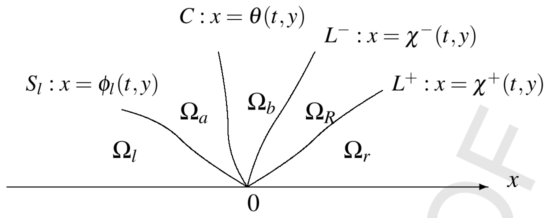

Fig. 2.2. SCR wave configuration.

and L^+ is denoted by U_c, the solution in the domain left to S_l is denoted by $U_l(t, x, y)$, and the solution in the domain right to S_r is denoted by $U_r(t, x, y)$.

By the finite speed propagation property for hyperbolic systems, the solution U_l, U_r, and the location of the characteristic L^+ are already uniquely determined by the initial data U_- and U_+.

The functions $U_a(t, x, y)$, $U_b(t, x, y)$, $U_c(t, x, y)$ satisfy the system (1.1)

$$\begin{aligned} LU_a &= 0 \quad \text{in } \Omega_a, \\ LU_b &= 0 \quad \text{in } \Omega_b, \\ LU_c &= 0 \quad \text{in } \Omega_R. \end{aligned} \tag{2.32}$$

Due to the multivaluedness of U_c at Γ, we introduce a parameter s to blow up the wedge area of Ω_R as in [1]. Let $x = \chi(t, s, y)$ $(1 \le s \le 2)$ be the family of characteristics issuing from Γ inside Ω_R. Then $\chi(t, s, y)$ satisfies

$$\det |A_1 - \chi_t - \chi_y A_2| = 0, \tag{2.33}$$

or more precisely

$$\chi_t = \lambda(U; -\chi_y), \tag{2.34}$$

where $\lambda(U; \eta)$ is the maximal eigenvalue of the matrix $A_1 + \eta A_2$.

Introduce the function $W(t, s, y)$:

$$W(t, s, y) = U_c\big(t, \chi(t, s, y), y\big), \tag{2.35}$$

which satisfies

$$\tilde{L}W \equiv \chi_s\left(\frac{\partial W}{\partial t} + A_2\frac{\partial W}{\partial y}\right) + (A_1 - \chi_t - \chi_y A_2)\frac{\partial W}{\partial s} = 0. \tag{2.36}$$

In addition to Eqs. (2.32) and (2.34), the functions $U_a(t, x, y)$, $U_b(t, x, y)$, $W(t, s, y)$, $\chi(t, s, y)$, $\phi_l(t, y)$, $\theta(t, y)$ should also satisfy the following boundary conditions:

$$\phi_{lt}[H_0] + \phi_{ly}[H_2] - [H_1] = 0 \quad \text{on } x = \phi_l(t, y), \tag{2.37}$$

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$$\begin{cases} \theta_t + v_b\theta_y - u_b = 0, \\ \rho_a - \rho_b = 0, \qquad\qquad\qquad \text{on } x = \theta(y), \\ \theta_y(v_b - v_a) - (u_b - u_a) = 0, \end{cases} \tag{2.38}$$

$$\begin{cases} \chi(t,1,y) = \chi^-(t,y), \\ \chi(t,2,y) = \chi^+(t,y), \\ W(t,1,y) = U_b\big(t,\chi(t,1,y),y\big), \\ W(t,2,y) = U_R\big(t,\chi(t,2,y),y\big). \end{cases} \tag{2.39}$$

Finally, we have the initial conditions at Γ:

$$\phi(0,y) = \theta(0,y) = 0 = \chi(0,s,y) = 0, \tag{2.40}$$

with the assumption as in [1]

$$\chi_s = \gamma(t,s,y)t \quad \text{with } \gamma(t,s,y) \geq \delta > 0. \tag{2.41}$$

The k-th order compatibility conditions require that one can find an approximate solution $\tilde{U}_a(t,x,y)$, $\tilde{U}_b(t,x,y)$, $\tilde{\phi}(t,y)$, $\tilde{\theta}(t,y)$ and $\tilde{W}(t,s,y)$, $\tilde{\chi}(t,s,y)$ such that Eqs. (2.32), (2.36)–(2.39) are satisfied up to the order $O(t^{k+1})$.

The 0-*order compatibility* $k = 0$.

As in the SCS case, we need only to consider the case that U_l $(= U_-)$, U_r $(= U_+)$, U_a, U_b are all constant. Also we can assume that $(\phi_{lt}, \theta_t, \chi_t^-, \chi_t^+)$ are all constant, and χ_t depends only on s. $W(t,s,y) = W(s)$ satisfies

$$\begin{cases} \big(A_1 - \chi_t(s)\big)W'(s) = 0, \\ W(0) = U_+. \end{cases} \tag{2.42}$$

From (2.38) we see that $\rho_a = \rho_b$, $u_a = u_b$, and v_a, v_b, θ_0 can be determined separately. Therefore, the problem becomes connecting the state (ρ_-, u_-) with the state (ρ_+, u_+) by a shock $x = \phi_l(t)$ and a rarefaction wave $x = \chi(t,s)$. The existence of $(\rho_a, u_a = u_b, \phi_{lt}, \chi_t(s))$ is guaranteed by the condition (C1) on the accompanying problem (1.4), (1.5).

Specifically, let $w(s) = (\rho(s), u(s))$ be the solution of (1.4), (1.5) such that

$$\begin{aligned} &\big(A_1' - \chi_t(s)I\big)w'(s) = 0, \\ &w(2) = (\rho_+, u_+), \qquad w(1) = (\rho_a, u_a), \qquad \chi_t(1) = \chi_t^-, \end{aligned} \tag{2.43}$$

with

$$A_1' = \begin{bmatrix} u & \rho \\ c^2/\rho & u \end{bmatrix}.$$

From (2.43) we see that $\chi_t(s) = \lambda(w(s))$ and $w'(s) = r(w)$, where $r(w)$ is the right eigenvector of A_1', satisfying

$$r \cdot \frac{\partial\lambda}{\partial w} = 1$$

and $w(s)$ is the solution of the system of the ordinary differential equation

$$\frac{dw}{ds} = r(w) \tag{2.44}$$

with $w(2) = (\rho_+, u_+)$.

Write the solution of (2.44) as $G(\rho, u) = 0$ and denote the Rankine–Hugoniot conditions on $x = \phi_l(t)$ $(= \phi_0 t)$ as

$$\begin{aligned} F_1 &= \phi_0(\rho_a - \rho_-) - (\rho_a u_a - \rho_- u_-) = 0, \\ F_2 &= \phi_0(\rho_a u_a - \rho_- u_-) - \left(p_a + \rho_a u_a^2 - p_- - \rho_- u_-^2\right) = 0. \end{aligned} \tag{2.45}$$

Then (ϕ_0, ρ_a, u_a) is uniquely determined if and only if

$$\Delta = \det\left(\frac{\partial(F_1, F_2, G)}{\partial(\phi_0, \rho_a, u_a)}\right) = \det\begin{pmatrix} F_{1\phi_0} & F_{1\rho_a} & F_{1u_a} \\ F_{2\phi_0} & F_{2\rho_a} & F_{2u_a} \\ 0 & G_{\rho_a} & G_{u_a} \end{pmatrix} \neq 0. \tag{2.46}$$

Noticing that the right-eigenvector $r(w)$ in (2.44) is parallel to the vector (ρ, c) and hence (G_{ρ_a}, G_{u_a}) is parallel to the vector $(c, -\rho)$, we find that (2.46) is equivalent to

$$\Delta_1 = \det\begin{pmatrix} \rho - \rho_- & \phi_0 - u & -\rho \\ \phi_0(\rho - \rho_-) & u(\phi_0 - u) - c^2 & \rho(\phi_0 - 2u) \\ 0 & c & -\rho \end{pmatrix} \neq 0. \tag{2.47}$$

Because the flow is subsonic behind the shock front by (C2), direct computation of the determinant in (2.47) yields

$$\Delta_1 \sim \det\begin{pmatrix} -(\phi_0 - u)^2 - c^2 & 2\rho(\phi_0 - u) \\ c & -\rho \end{pmatrix} = \rho(\phi_0 - u - c)^2 \neq 0. \tag{2.48}$$

This concludes the proof of 0-order compatibility.

Next we consider the *first order compatibility* $k = 1$. We need to determine the first order derivatives of $U_a(t, x, y)$, $U_b(t, x, y)$, $W(t, s, y)$ and $\phi_{lt}(t, y)$, $\theta_t(t, y)$, $\chi_t(t, s, y)$ at Γ. From the 0-order compatibility, these functions and their tangential derivatives with respect to Γ are already known.

The following computation in the domain Ω_R follows [1], and for readers' convenience we briefly repeat it here.

Let $H(v, \eta)$ be the matrix satisfying

$$H^{-1}(A_1 - \chi_y A_2)H = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^b \end{pmatrix} \quad (\triangleq d), \tag{2.49}$$

where the superscript b means the last two rows.

From (2.36) we have

$$H^{-1}(W_t + A_2 W_y) = -H^{-1}(A_1 - \chi_t I - A_2\chi_y)\frac{W_s}{\chi_s}$$
$$= \begin{pmatrix} \chi_t - \lambda & 0 \\ * & * \end{pmatrix} H^{-1}\frac{W_s}{\chi_s} = \begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix} H^{-1}\frac{W_s}{\chi_s}. \tag{2.50}$$

Then the first row becomes

$$\big(H^{-1}(W_t + A_2 W_y)\big)^1 = 0. \tag{2.51}$$

Multiplying (2.36) by H^{-1} we have

$$\chi_s H^{-1}(W_t + A_2 W_y) + (d - \chi_t)H^{-1}W_s u = 0.$$

Differentiating with respect to t gives

$$\chi_{ts} H^{-1}(W_t + A_2 W_y) + \chi_s\big(H^{-1}(W_t + A_2 W_y)\big)_t$$
$$+ (d - \chi_t)_t H^{-1}W_s + (d - \chi_t)\big(H^{-1}W_s\big)_t = 0. \tag{2.52}$$

Since $\chi_s = 0$ at $t = 0$, (2.52) gives

$$\chi_{ts}\big(H^{-1}(W_t + A_2 W_y)\big)^b + \big(\lambda^b - \chi_t\big)_t\big(H^{-1}W_s\big)^b + \big(\lambda^b - \chi_t\big)\big(H^{-1}W_s\big)_t^b = 0. \tag{2.53}$$

On the other hand, by differentiating (2.34) with respect to t we can obtain

$$\chi_{tt} = \lambda_W W_t - \lambda_\eta \chi_{yt}. \tag{2.54}$$

Substituting χ_{tt} into (2.53) we have

$$\big(\lambda^b - \chi_t\big)\big(\big(H^{-1}W_t\big)^b\big)_s + \big(H^{-1}W_t\big)^b \cdot * = *, \tag{2.55}$$

where $*$ stands for known terms. Therefore, the value of $(H^{-1}W_t)^b$ at any $s \in [1, 2]$ can be uniquely determined by its value at $s = 2$. On the other hand, the value $(H^{-1}W_t)^1$ is determined from (2.51). Hence the value of all the components of $H^{-1}W_t$ are uniquely determined, and so are all the components of W_t.

Since the tangential derivatives of U_b and U_c on $x = \chi(t, 1, y) \equiv \chi^-(t, y)$ are equal, therefore the value of the tangential derivative $D_r U_b \equiv (\partial_t + \chi_t^- \partial_x)U_b$ is known. Evaluating the tangential derivatives at the origin and noticing $\chi^-(0, 0) = u + c$,

$$D_r = \partial_t + (u + c)\partial_x = D_c + c\partial_x,$$

we have

$$\begin{aligned} D_r\rho_b &= *, \\ D_r u_b &= *, \\ D_r v_b &= *. \end{aligned} \tag{2.56}$$

Since $\chi_{1t} = \lambda_+$ is the eigenvalue for the system (1.2), we obtain only two independent relations for (ρ_b, u_b, v_b) from (2.56):

$$\begin{aligned} D_r\rho_b &= *, \\ D_r v_b &= *. \end{aligned} \tag{2.57}$$

Similarly as in the SCS case, we can derive the conditions on $(\phi_{ltt}, \theta_{tt}, U_{at}, U_{ax})$. Differentiating (2.37) in the tangential direction and evaluating at the origin yields

$$\phi_{ltt}[H_0] + \big(\phi_{lt} H_0' - H_1'\big) D_l U_a = *. \tag{2.58}$$

Differentiating (2.38) leads to the following same equations as in (2.14):

$$\begin{aligned} \theta_{tt} - D_c u_a &= *, \\ D_c\rho_a - D_c\rho_b &= *, \\ D_c u_a - D_c u_b &= *. \end{aligned} \tag{2.14}$$

The linear system (2.54), (2.57), (2.58), (2.14), together with the system (2.15) in the Ω_a and Ω_b consists of 15 equations (one in (2.54), two in (2.57), three in (2.58), three in (2.14), six in (2.15)) for the 15 variables $(\phi_{ltt}, \theta_{tt}, \chi_{tt}^-, U_{at}, U_{ax}, U_{bt}, U_{bx})$.

As in the SCS case, $(\theta_{tt}, v_{at}, v_{ax}, v_{bt}, v_{bx})$ as well as $(\rho_{bt}, \rho_{bx}, u_{bt}, u_{bx})$ can be eliminated from the system. And χ_{tt}^- can be eliminated by (2.55).

After the simplification, we end up with five linear equations for the five variables $(\phi_{ltt}, \rho_{at}, \rho_{ax}, u_{at}, u_{ax})$: (We will drop the subscript "a" in the following.)

$$\phi_{ltt}\begin{bmatrix} \rho - \rho_- \\ \phi_{lt}(\rho - \rho_-) \end{bmatrix} + \begin{bmatrix} \phi_{lt} - u & -\rho \\ u(\phi_{lt} - u) - c^2 & \rho(\phi_{lt} - 2u) \end{bmatrix} D_l \begin{bmatrix} \rho \\ u \end{bmatrix} = *; \tag{2.59}$$

$$(D_c + c\mathscr{E}\partial_x)\begin{pmatrix} \rho \\ u \end{pmatrix} = *, \tag{2.60}$$

$$(D_c + c\partial_x)\rho = *. \tag{2.61}$$

As in the case of SCS, we can use (2.60) to replace (ρ_x, u_x) by $(D_c\rho, D_c u)$:

$$\partial_x\begin{pmatrix} \rho \\ u \end{pmatrix} = -\frac{1}{c}\mathscr{E} D_c\begin{pmatrix} \rho \\ u \end{pmatrix} + * \quad \text{and} \quad D_l = (I - \beta_l\mathscr{E})D_c.$$

Eliminating ϕ_{ltt} from (2.59) yields

$$\big[-(\phi_{lt} - u)^2 - c^2 \quad 2\rho(\phi_{lt} - u)\big](I - \beta_l\mathscr{E})D_c\begin{bmatrix} \rho \\ u \end{bmatrix} = *. \tag{2.62}$$

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By (2.60), (2.61) becomes

$$(1 \quad 0)\,(I-\mathscr{E})D_c\begin{bmatrix}\rho\\ u\end{bmatrix}=*. \tag{2.63}$$

The coefficient matrix for (2.62) and (2.63) is

$$\begin{bmatrix}[-(\phi_{lt}-u)^2-c^2]-2\beta_l c(\phi_{lt}-u) & \beta_l\frac{\rho}{c}[(\phi_{lt}-u)^2+c^2]+2\rho(\phi_{lt}-u)\\ 1 & -\rho/c\end{bmatrix}$$

which is non-degenerate because

$$\left[-(\phi_{lt}-u)^2-c^2\right]-2\beta_l c(\phi_{lt}-u)<0, \qquad \beta_l\frac{\rho}{c}\left[(\phi_{lt}-u)^2+c^2\right]+2\rho(\phi_{lt}-u)<0.$$

This completes the proof of the first order compatibility.

For the general *k-th order compatibility*, we apply the tangential derivatives D_r^{k-1} to (2.57), D_l^{k-1} to (2.58), D_c^{k-1} to (2.14), as well as ∂_t^{k-1} to the interior equations in (2.15). By the same argument as in the first order compatibility, we can eliminate the variables $(\partial_t^{k+1}\phi_l$, $\partial_t^{k+1}\theta$, $\partial_t^{k+1}\chi^-$, D^kU_b, $D^kv_a)$ and obtain the four linear equations for $(D_c^k\rho_a$, $D_c^k u_a$, $D_c^{k-1}\rho_{ax}$, $D_c^{k-1}u_{ax})$ (the subscript "a" is dropped again in the following):

$$[\,-(\phi_{lt}-u)^2-c^2 \quad 2\rho(\phi_{lt}-u)\,]\,(I-\beta_l\mathscr{E})^k D_c^k\begin{bmatrix}\rho\\ u\end{bmatrix}=*; \tag{2.64}$$

$$(D_c+c\mathscr{E}\partial_x)D_c^{k-1}\begin{pmatrix}\rho\\ u\end{pmatrix}=*, \tag{2.65}$$

$$(1 \quad 0)\,(D_c+c\partial_x)^k\begin{bmatrix}\rho\\ u\end{bmatrix}=*. \tag{2.66}$$

Using (2.60) to replace ∂_x by D_c, we have

$$\partial_x\begin{bmatrix}\rho\\ u\end{bmatrix}=-\frac{1}{c}\mathscr{E}D_c\begin{bmatrix}\rho\\ u\end{bmatrix}. \tag{2.67}$$

Then (2.66) becomes

$$(1 \quad 0)\,(I-\mathscr{E})^k D_c^k\begin{bmatrix}\rho\\ u\end{bmatrix}=*. \tag{2.68}$$

It is readily checked that $(I\pm\mathscr{E})^k=2^{k-1}(I\pm\mathscr{E})$. Hence (2.68) is reduced to

$$(1 \quad -\tfrac{\rho}{c})\,D_c^k\begin{bmatrix}\rho\\ u\end{bmatrix}=*. \tag{2.69}$$

On the other hand, by Lemma 2.1 we have from (2.64)

$$\delta_l\,[\,-(\phi_{lt}-u)^2-c^2 \quad 2\rho(\phi_{lt}-u)\,]\,(\alpha_{lk}-\beta_l\mathscr{E})D_c^k\begin{bmatrix}\rho\\ u\end{bmatrix}=*. \tag{2.70}$$

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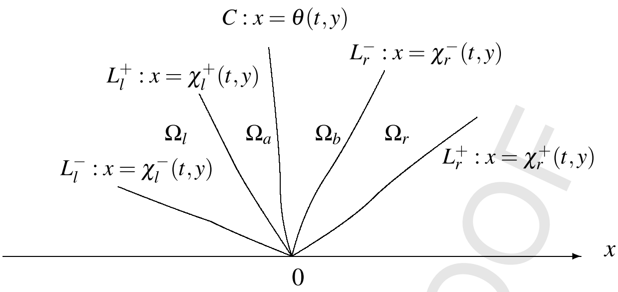

Fig. 2.3. RCR wave configuration.

Similarly as in the first order case, (2.69), (2.70) have a unique solution if and only if the following matrix is non-degenerate:

$$\begin{bmatrix} -\alpha_{lk}((\phi_{lt} - u)^2 + c^2) - 2\beta_l c(\phi_{lt} - u) & \beta_l \frac{\rho}{c}((\phi_{lt} - u)^2 + c^2) + 2\alpha_{lk}\rho(\phi_{lt} - u) \\ 1 & -\frac{\rho}{c} \end{bmatrix}$$

which has non-zero determinant because

$$-\alpha_{lk}\big((\phi_{lt} - u)^2 + c^2\big) - 2\beta_l c(\phi_{lt} - u) < 0, \qquad \beta_l \frac{\rho}{c}\big((\phi_{lt} - u)^2 + c^2\big) + 2\alpha_{lk}\rho(\phi_{lt} - u) < 0$$

by Lax' inequality (C2) and $|\beta_l| < \alpha_{lk}$ in Lemma 2.1.

2.3. Compatibility condition for the case RCR

The discussion of compatibility condition for the case RCR follows almost exactly the case SCR. See Fig. 2.3. Omitting the tedious details, we just mention the following main issues:

(1) The 0-order compatibility follows from the condition (C1);
(2) The eigenvalues corresponding to the two rarefaction waves are $\lambda = u \pm c$, the k-th order compatibility can be reduced, just as in the SCR case, to

$$\begin{aligned} (1 \quad 0)(D_c - c\partial_x)^k \begin{bmatrix} \rho \\ u \end{bmatrix} &= *, \\ (1 \quad 0)(D_c + c\partial_x)^k \begin{bmatrix} \rho \\ u \end{bmatrix} &= *. \end{aligned} \tag{2.71}$$

Replacing ∂_x by D_c in (2.71) according to (2.67), we obtain

$$\begin{aligned} (1 \quad 0)(I + \mathscr{E})^k D_c^k \begin{bmatrix} \rho \\ u \end{bmatrix} &= (2^{k-1} \quad 0)(I + \mathscr{E}) D_c^k \begin{bmatrix} \rho \\ u \end{bmatrix} = *, \\ (1 \quad 0)(I - \mathscr{E})^k D_c^k \begin{bmatrix} \rho \\ u \end{bmatrix} &= (2^{k-1} \quad 0)(I - \mathscr{E}) D_c^k \begin{bmatrix} \rho \\ u \end{bmatrix} = *. \end{aligned} \tag{2.72}$$

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It is equivalent to

$$\begin{cases} D_c^k \rho + \dfrac{\rho}{c} D_c^k u = *, \\ D_c^k \rho - \dfrac{\rho}{c} D_c^k u = *. \end{cases} \tag{2.73}$$

So that the derivatives of k-th order for ρ, u can be uniquely determined.

2.4. Approximate solution of infinite compatibility

Usually, the k-th order approximate solution follows immediately from the k-th order compatibility. One needs only to construct the approximate solution by using the Taylor series. However, for the construction of infinite order approximate solution, one should use Borel technique to construct a C^∞ smooth approximate solution.

Denote briefly the nonlinear waves by ϕ and the solution by U. More precisely, in SCS case $\phi = (\phi_l, \theta, \phi_r)$, $U = (U_a, U_b)$, in SCR case $\phi = (\phi_l, \theta, \chi^-)$, $U = (U_a, U_b, U_R)$ and in RCR case $\phi = (\chi_l^+, \theta, \chi_r^-)$, $U = (U_{R_l}, U_a, U_b, U_{R_r})$. Then from Theorem 2.1, we obtain the following existence of approximate solutions.

Theorem 2.2. *Under the assumptions* (C1) *and* (C2)*, for the Cauchy problem* (1.1), (1.3) *with smooth initial data* $U_\pm$*, there exists an approximate solution* (U^a, ϕ^a) *which:*

- *Is C^∞ in their respective domains;*
- *Satisfies the initial conditions* (2.1)*;*
- *For any $k \in \mathbb{N}$, (U^a, ϕ^a) satisfies the interior equations* (1.1) *and boundary conditions* (2.2)–(2.4) *up to the order of t^k near $t = 0$.*

Proof. Obviously the explicit form of the proof of Theorem 2.2 depends upon the specific SCS, SCR, or RCR wave configurations. But the general idea behind all the proofs is the same: given the values of all the (t, x) derivatives of a function w along the initial curve Γ: $x = \phi_0(y)$, construct a C^∞ function w in the neighborhood of Γ and $t = 0$, assuming the given values of all the derivatives on Γ and $t = 0$. Next we give a generic construction of the function w. To simplify notations we omit the variable y.

Let $\alpha = (\alpha_0, \alpha_1)$ be the multi-index corresponding to the variables (t, x) with the convention $|\alpha| = \alpha_0 + \alpha_1$, $\alpha! = \alpha_0!\alpha_1!$ and

$$\partial^\alpha w = \partial_t^{\alpha_0} \partial_x^{\alpha_1} w.$$

Let $\varphi(t, x) \in C_0^\infty(\mathbb{R}^2)$ satisfy

$$\operatorname{supp} \varphi \subset (-1, 1) \times (-1, 1) \quad \text{and} \quad \varphi \equiv 1 \quad \text{in } (-1/2, 1/2) \times (-1/2, 1/2).$$

Let $\{s_n\}$ be an increasing sequence defined by

$$s_n = \sum_{|\alpha|=n, |\beta| \le n-1} \max \left|\phi^{(\beta)}\right| \left(1 + |a_\alpha|\right) n!, \tag{2.74}$$

where $\phi^{(\beta)}$ denotes the derivatives of ϕ.

Now we define

$$w(t,x)=\sum_{|\alpha|\geq 0}\phi(s_{|\alpha|}t,s_{|\alpha|}x)\frac{a_\alpha}{\alpha!}t^{\alpha_0}x^{\alpha_1}. \tag{2.75}$$

Every term in (2.75) is C^∞ for all (t,x) and satisfies

$$D^\gamma\left(\phi(s_{|\alpha|}t,s_{|\alpha|}x)\frac{a_\alpha}{\alpha!}t^{\alpha_0}x^{\alpha_1}\right)\bigg|_{t=x=0}=\begin{cases}a_\gamma, & \gamma=\alpha;\\ 0, & \gamma\neq\alpha.\end{cases} \tag{2.76}$$

In order to prove that (2.75) is the C^∞ function, we need only to show that all its term-wise D^γ-derivative converges uniformly for all (t,x). Obviously we need only to consider the terms with $|\alpha|\geq|\gamma|+2$:

$$\begin{aligned}&D^\gamma\left(\phi(s_{|\alpha|}t,s_{|\alpha|}x)\right)\frac{a_\alpha}{\alpha!}t^{\alpha_0}x^{\alpha_1}\\ &\quad=\sum_{\beta\leq\gamma}\frac{\gamma!}{\beta!(\gamma-\beta)!(\alpha-\gamma+\beta)!}\phi^{(\beta)}s_{|\alpha|}^{|\beta|}a_\alpha t^{\alpha_0-\gamma_0+\beta_0}x^{\alpha_1-\gamma_1+\beta_1}.\end{aligned} \tag{2.77}$$

In view of the finite support of function ϕ and the choice of s_n in (2.74), we have in (2.77)

$$\begin{aligned}&s_{|\alpha|}\cdot|t|\leq 1, \qquad s_{|\alpha|}\cdot|x|\leq 1;\\ &\left|\phi^{(\beta)}a_\alpha\right|\cdot|t|\leq\frac{1}{|\alpha|!}, \qquad \left|\phi^{(\beta)}a_\alpha\right|\cdot|x|\leq\frac{1}{|\alpha|!}.\end{aligned} \tag{2.78}$$

Noticing that $\sum_{|\alpha|\geq 0}\frac{1}{\alpha!}=e^2$, we obtain the estimate of (2.77) for all (t,x)

$$\left|D^\gamma\left(\phi(s_{|\alpha|}t,s_{|\alpha|}x)\right)\frac{a_\alpha}{\alpha!}t^{\alpha_0}x^{\alpha_1}\right|\leq\sum_{\beta\leq\gamma}\frac{\gamma!}{|\alpha|!(\alpha-\gamma+\beta)!}\leq\frac{e^2}{|\alpha|(|\alpha|-1)}. \tag{2.79}$$

This implies the uniform convergence of $w^{(\gamma)}$ for (2.75). □

3. Transformation and reformulation

The proof of Theorem 1.2 should be carried out for all three cases – SCS, SCR, RCR – separately. Among the three cases, the SCR is the most general and typical case of the three possible wave combinations. As far as the approach and methods are concerned, the discussion on the other two cases can be found in SCR case. To avoid repetition and tediousness, we will here only present the proof for the SCR case.

Consider the initial value problem (1.1), (1.3), with a left-propagating shock front S_l: $x=\phi_l(t,y)$, a contact discontinuity C: $x=\theta(t,y)$, and a right-propagating rarefaction wave in an angular domain between two characteristics L^-: $x=\chi^-(t,y)$ and L^+: $x=\chi^+(t,y)$. These surfaces divides the upper half-space to the domain Ω_l, Ω_a, Ω_b, Ω_R and Ω_r, as shown in Fig. 3.1. In accordance, the solution in each domain is denoted by U_l, U_a, U_b, U_c and U_r respectively.

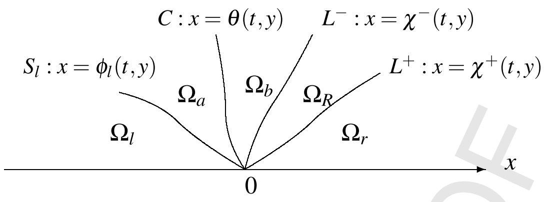

Fig. 3.1. SCR wave configuration.

By the finite speed propagation property for hyperbolic systems, the solution U_l, U_r, and the location of the characteristic L^+: $x = \chi^+(t, y)$ are already uniquely determined by the initial data U_- and U_+.

We are looking for the SCR solution of (1.1), (1.3) consisting of the functions $U_a(t, x, y)$, $U_b(t, x, y)$, $U_c(t, x, y)$ and the functions $\phi_l(t, y)$, $\theta(t, y)$, $\chi^-(t, y)$ such that

- (U_a, U_b, U_c) satisfy the system (1.1) in their respective domains Ω_a, Ω_b, Ω_R;
- $\phi_l(t, y), \theta(t, y), \chi^-(t, y)$ are the boundaries dividing the four domains Ω_l, Ω_a, Ω_b, Ω_R; and satisfying

$$\begin{aligned} &\phi_l(t, y) < \theta(t, y) < \chi_-(t, y) < \chi_+(t, y) \quad \text{when } t > 0, \\ &\phi_l(0, y) = \theta(0, y) = \chi^-(0, y) = \chi^+(0, y) = \phi_0(y), \\ &\phi_{lt}(0, y) < \theta_t(0, y) < \chi_t^-(0, y) < \chi_t^+(0, y). \end{aligned} \tag{3.1}$$

- On $x = \phi_l(t, y)$, the Rankine–Hugoniot condition (2.37) is satisfied;
  On $x = \theta(t, y)$, the contact-discontinuity condition (2.38) is satisfied;
  On $x = \chi^-(t, y)$, $U_b = U_c$; and $x = \chi^+(t, y)$, $U_c = U_r$.
- There exist two smooth functions $\Theta^\pm(t, x, y)$ defined on $\pm x \geq 0$ respectively, satisfying the eikonal equations

$$\begin{aligned} &\Theta_t^- + v_a \Theta_y^- - u_a = 0, \quad \text{in } x < 0, \\ &\Theta_t^+ + v_b \Theta_y^+ - u_b = 0, \quad \text{in } x > 0 \end{aligned} \tag{3.2}$$

  with

$$\Theta^\pm(t, 0, y) = \theta(t, y) \quad \text{and} \quad \partial_x \Theta^\pm(t, x, y) \geq \kappa > 0. \tag{3.3}$$

  As in [8], the satisfaction of eikonal equations (3.2) is required near $x = 0$ instead of only on the boundary $x = 0$ as in (2.38), because the weak Lopatinskii condition applies only for the uniformly characteristic boundary as discussed in [19]. For the linearly degenerate eigenvalue λ_0 for the Euler system (1.1), it can always be achieved by choosing an appropriate coordinate of variables (x, y).
- In Ω_R, U_c takes the form of (2.35), i.e., there exists a function $\chi(t, s, y)$, such that $W(t, s, y) = U_c(t, \chi(t, s, y), y)$ satisfies:
  (1) $x = \chi(t, s, y)$ $(0 \leq s \leq 1)$ is a family of characteristics (2.34): $\chi_t = \lambda_+(U; -\chi_y)$;
  (2) $\chi(0, s, y) = \phi_0(y)$;

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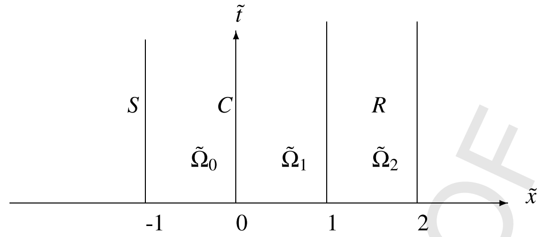

Fig. 3.2. SCR wave configuration on $(\tilde t, \tilde x)$ plane.

(3) $\chi(t,1,y) = \chi^-(t,y)$, $\chi(t,2,y) = \chi^+(t,y)$;
(4) $\chi_s = \gamma(t,s,y)t$ with $\gamma(t,s,y) \ge \delta > 0$ as in (2.41);
(5) the function $W(t,s,y)$ satisfies (2.36):

$$\tilde L W \equiv \chi_s\left(\frac{\partial W}{\partial t} + A_2\frac{\partial W}{\partial y}\right) + (A_1 - \chi_t - \chi_y A_2)\frac{\partial W}{\partial s} = 0. \tag{3.4}$$

Now we are going to make singular coordinates transforms in the three angular domains Ω_a, Ω_b, Ω_R to change them into standard cylindrical domains with fixed boundary. The surfaces $x = \phi_l(t,y), \theta(t,y), \chi^-(t,y), \chi^+(t,y)$ defined in (3.1) will be the boundary values of a family of surfaces which are also used in the construction of the rarefaction wave in the domain Ω_R.

Denote

$$\tilde\Omega_j = \left\{(\tilde t, \tilde x, \tilde y) : \tilde t > 0,\ j-1 < \tilde x < j\right\} \quad (j = 0,1,2). \tag{3.5}$$

Let $\phi^{(j)}(\tilde t, \tilde x, \tilde y)$ be defined on $\tilde\Omega_j$ as

$$\begin{aligned}
&\phi^{(0)}(\tilde t, \tilde x, \tilde y) = (1+\tilde x)\theta(t,y) - \tilde x\phi_l(\tilde t, \tilde y),\\
&\phi^{(1)}(\tilde t, \tilde x, \tilde y) = (1-\tilde x)\theta(t,y) + \tilde x\chi^-(\tilde t, \tilde y),\\
&\phi^{(2)}(\tilde t, \tilde x, \tilde y) = \chi(\tilde t, 2-\tilde x, \tilde y).
\end{aligned} \tag{3.6}$$

Then we have

$$\begin{aligned}
&\phi^{(0)}(t,-1,y) = \phi_l(t,y),\\
&\phi^{(0)}(t,0,y) = \phi^{(1)}(t,0,y) = \theta(t,y),\\
&\phi^{(1)}(t,1,y) = \phi^{(2)}(t,1,y) = \chi^-(t,y),\\
&\phi^{(2)}(t,2,y) = \chi^+(t,y).
\end{aligned} \tag{3.7}$$

For $\tilde t > 0$, the transformations

$$x = \phi^{(j)}(\tilde t, \tilde x, \tilde y), \qquad y = \tilde y, \qquad t = \tilde t \quad (j = 0,1,2) \tag{3.8}$$

Q6 are bijections from $\tilde\Omega_0$ to Ω_a, from $\tilde\Omega_1$ to Ω_b, and from $\tilde\Omega_2$ to Ω_R respectively. See Fig. 3.2.

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Under these transformations, the system of equations (1.2) becomes a singular system defined in $\tilde{\Omega}_j$. Denoting the new unknown function in $\tilde{\Omega}_j$ as $U^{(j)}$, we obtain the transformed system of (1.2) in $\tilde{\Omega}_j$ $(j=0,1,2)$

$$\partial_{\tilde{t}}U^{(j)}+A_2\big(U^{(j)}\big)\partial_{\tilde{y}}U^{(j)}+\frac{1}{\partial_{\tilde{x}}\phi^{(j)}}\big(A_1\big(U^{(j)}\big)-\phi_{\tilde{t}}^{(j)}-\phi_{\tilde{y}}^{(j)}A_2\big(U^{(j)}\big)\big)\partial_{\tilde{x}}U^{(j)}=0. \tag{3.9}$$

The system (3.9) is singular because $\partial_{\tilde{x}}\phi^{(j)}=O(\tilde{t})$.

To formally remove this singularity, as well as to derive the required estimates, we introduce another coordinate transform (see also [4]),

$$\tilde{t}=e^{\tau}, \quad \text{with } \partial_{\tilde{t}}=e^{-\tau}\partial_{\tau}. \tag{3.10}$$

With the transform (3.10), the domain $\tilde{\Omega}_j$ becomes ω_j:

$$\omega_j=\big\{(\tau,\tilde{x},\tilde{y}):\tau\in\mathbb{R},\ j-1<\tilde{x}<j\big\} \quad (j=0,1,2). \tag{3.11}$$

And the system (3.9) becomes

$$\begin{aligned}\mathscr{L}^{(j)}\big(U^{(j)},\phi^{(j)}\big)&\equiv\partial_{\tau}U^{(j)}+e^{\tau}A_2\big(U^{(j)}\big)\partial_{\tilde{y}}U^{(j)}\\&\quad+\frac{e^{\tau}}{\partial_{\tilde{x}}\phi^{(j)}}\big(A_1\big(U^{(j)}\big)-e^{-\tau}\partial_{\tau}\phi^{(j)}-\phi_{\tilde{y}}^{(j)}A_2\big(U^{(j)}\big)\big)\partial_{\tilde{x}}U^{(j)}\\&=0 \quad \text{in } \omega_j\ (j=0,1,2).\end{aligned} \tag{3.12}$$

In particular, the $\tilde{t}^{\eta}$-weighted integration in the domain $\tilde{\Omega}_j$ becomes the hyperbolic η-weighted integration in ω_j:

$$\int_{\tilde{\Omega}_j}\tilde{t}^{\eta}\big|U^{(j)}(\tilde{t},\tilde{x},\tilde{y})\big|^2d\tilde{t}d\tilde{x}d\tilde{y}=\int_{\omega_j}e^{(\eta+1)\tau}\big|U^{(j)}(\tau,\tilde{x},\tilde{y})\big|^2d\tau d\tilde{x}d\tilde{y}. \tag{3.13}$$

Besides, we denote

$$\mathscr{B}^{(-1)}\big(U^{(0)},\phi^{(0)}\big)\equiv\partial_{\tau}\phi^{(0)}[H_0]-e^{\tau}[H_1]+e^{\tau}\partial_{\tilde{y}}\phi^{(0)}[H_2], \quad \text{on } \tilde{x}=-1, \tag{3.14}$$

$$\mathscr{B}^{(0)}\big(U^{(0)},U^{(1)},\phi^{(0)}\big)\equiv\begin{cases}e^{-\tau}\phi_{\tau}^{(0)}-v_b^{(1)}\phi_{\tilde{y}}^{(0)}-u_b^{(1)},\\ \rho_a^{(0)}-\rho_b^{(1)},\\ \theta_{\tilde{y}}(v_b^{(1)}-v_a^{(0)})-(u_b^{(1)}-u_a^{(0)}),\end{cases} \quad \text{on } \tilde{x}=0. \tag{3.15}$$

To simplify the notation, we will drop the tilde in the new coordinates in the following and replace τ by t. In summary, the existence of SCR wave structure can be formulated equivalently as the following boundary value problem in the domain $\{(t,x,y):-1<x<2,\ t>-\infty\}$:

To find unknown functions $(U^{(j)}(t,x,y),\phi^{(j)}(t,x,y))$ $(j=0,1,2)$ *in the domain* $\omega_j=\{(t,x,y):j-1<x<j\}$ $(j=0,1,2)$ *satisfying*

- Interior equations:

$$\mathscr{L}^{(j)}\big(U^{(j)},\phi^{(j)}\big)=0 \quad \text{in } \omega_j \ (j=0,1,2); \tag{3.16}$$

- Boundary conditions for shock and contact discontinuity:

$$\mathscr{B}^{(-1)}\big(U^{(0)},\phi^{(0)}\big)=0 \quad \text{on } x=-1, \tag{3.17}$$

$$\mathscr{B}^{(0)}\big(U^{(0)},U^{(1)},\phi^{(0)}\big)=0 \quad \text{on } x=0; \tag{3.18}$$

- Continuous boundary conditions for rarefaction waves:

$$U^{(1)}(t,1,y)=U^{(2)}(t,1,y), \qquad U^{(2)}(t,2,y)=U^{(r)}(t,2,y); \tag{3.19}$$

- Rarefaction wave structure:

$$\phi_t^{(2)}(t,x,y)=\lambda_+\big(U^{(2)};-\phi_y^{(2)}\big),$$
$$\partial_x\phi^{(2)}(t,x,y)=\gamma(t,x,y)e^t \quad \text{with } \gamma\ge\delta>0; \tag{3.20}$$

- Boundary surfaces conditions:

$$\phi^{(j)}(-\infty,x,y)=\phi_0(y) \quad (j=0,1,2), \tag{3.21}$$

$$\phi^{(j)}(t,j,y)=\phi^{(j+1)}(t,j,y) \quad (j=0,1); \tag{3.22}$$

- Constraint: there exists a function $\Theta^{\pm}(t,x,y)$ satisfying (3.3) and

$$e^{-t}\Theta_t^-+v_a\Theta_y^--u_a=0 \quad \text{in } x<0,$$
$$e^{-t}\Theta_t^++v_b\Theta_y^+-u_b=0 \quad \text{in } x>0. \tag{3.23}$$

4. Iteration scheme

Our main task is to prove Theorem 1.2, the theorem on existence of the local solution, by using the Nash–Moser iteration technique near the C^∞ approximate solution constructed in Theorem 2.2. Due to the reformulation in Section 3, we need only to prove the existence of the solution $(U^{(j)},\phi^{(j)})$ ($j=0,1,2$) satisfying (3.16)–(3.23) in $-\infty<t<T$ for some $T\in\mathbb{R}$.

For the shock wave alone, the existence of the solution was established in [17], using Newton iteration. For the rarefaction wave and contact discontinuity, the existence was proven in [1] and [8] individually. In the proof for the latter cases a modified version of Nash–Moser iteration scheme is employed, in which an additional error term coming from the uniformly characteristic requirement.

In this paper we are going to combine these three cases and give a unified treatment to the iteration scheme. Indeed, with the transformation performed in Section 3, the problem in (3.16)–(3.23) has formally similar form near each wave. By localization in the x direction, we can use the result already obtained in [17,1,8], in particular the estimates for the corresponding linearized problem and the basic technique of iteration. Certainly, we should also treat the factor $e^{\eta t}$ appearing in the coefficients.

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Denote $U = (U^{(0)}, U^{(1)}, U^{(2)})$ and $\phi = (\phi^{(0)}, \phi^{(1)}, \phi^{(2)})$. Our aim is to construct a sequence of smooth approximate solutions $(U^a + U_k, \phi^a + \phi_k)$ $(k = 0, 1, 2, \ldots)$ near (U^a, ϕ^a), which converges in appropriate space to the solution of the problem (3.16)–(3.23).

Let $\{\theta_n\}$ be the sequence defined by

$$\theta_0 \geq 1, \qquad \theta_n = \sqrt{\theta_0^2 + n}, \qquad \Delta_n = \theta_{n+1} - \theta_n. \tag{4.1}$$

And the sequence $\{\Delta_n\}$ is decreasing with

$$\frac{1}{3\theta_n} \leq \Delta_n = \sqrt{\theta_n^2 + 1} - \theta_n \leq \frac{1}{2\theta_n}. \tag{4.2}$$

The parameter θ_0 will be chosen sufficiently large later.

Let (U^a, ϕ^a) be the C^∞ approximate solution we obtained in Theorem 2.2 which satisfies (3.16)–(3.23) near $t = -\infty$ up to any order of e^t, i.e., for any $\eta > 0$, the error decays faster than the order of $e^{\eta t}$ near $t = -\infty$.

We approximate the solution by a sequence of approximate solutions in the form of $(U^a + U_n, \phi^a + \phi_n)$, constructed as follows

$$\begin{aligned} &(U_0, \phi_0) = (0, 0), \\ &U_{n+1} = U_n + \Delta_n \dot{U}_n, \qquad \phi_{n+1} = \phi_n + \Delta_n \dot{\phi}_n \quad (n = 0, 1, 2, \ldots), \end{aligned} \tag{4.3}$$

where $\dot{U}_n$ and $\dot{\phi}_n$ will be the solution of an appropriate boundary value problem for a linear hyperbolic system specified as follows.

4.1. Interior equation

First, let's consider the interior equation part of the Nash–Moser iteration scheme, which yields the interior system the functions $(\dot{U}_n, \dot{\phi}_n)$ should satisfy.

For simplicity of notation, denote

$$\mathscr{L}(U, \phi) = \big(\mathscr{L}^{(0)}\big(U^{(0)}, \phi^{(0)}\big), \mathscr{L}^{(1)}\big(U^{(1)}, \phi^{(1)}\big), \mathscr{L}^{(2)}\big(U^{(2)}, \phi^{(2)}\big)\big).$$

Consider the linearization of the nonlinear operator $\mathscr{L}(U, \phi)$ at a given state (U, ϕ). Introduce a new variable $\dot{V}$ (see [1] and [8]):

$$\dot{V} = \dot{U} - \frac{U_x}{\phi_x} \dot{\phi}. \tag{4.4}$$

Then the linearized operator $\ell(U, \phi)(\dot{U}, \dot{\phi})$ of the nonlinear operator $\mathscr{L}(U, \phi)$ at the state (U, ϕ) can be expressed

$$\ell(U, \phi)(\dot{U}, \dot{\phi}) = \mathscr{L}'(U, \phi)\dot{V} + B(U, \phi)\dot{V} + \frac{\dot{\phi}}{\phi_x} \partial_x \mathscr{L}(U, \phi), \tag{4.5}$$

where the operators $\mathscr{L}'(U, \phi)$ and $B(U, \phi)$ are defined as

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$$\mathscr{L}'(U,\phi) \equiv \partial_t + e^t A_2(U)\partial_y + \frac{e^t}{\phi_x}\big(A_1(U) - e^{-t}\phi_t - \phi_y A_2(U)\big)\partial_x, \tag{4.6}$$

$$\begin{aligned} B(U,\phi) &\equiv \frac{e^t}{\phi_x}B_1(U,\phi) + e^t B_2(U,\phi) \\ &\equiv \frac{e^t}{\phi_x}\big(A_1'(U) - \phi_y A_2'(U)\big)U_x + e^t A_2'(U)U_y. \end{aligned} \tag{4.7}$$

For simplicity of notations, we introduce the notation $\mathscr{L}_a(U_n,\phi_n)$:

$$\mathscr{L}_a(U_n,\phi_n) \equiv \mathscr{L}\big(U^a + U_n, \phi^a + \phi_n\big).$$

Similarly, we will use the notations

$$\ell_a(U_n,\phi_n) \equiv \ell\big(U^a + U_n, \phi^a + \phi_n\big), \qquad B_a(U_n,\phi_n), \mathscr{B}_a(U_n,\phi_n), \ldots.$$

Then we have

$$\mathscr{L}(U_{n+1},\phi_{n+1}) - \mathscr{L}(U_n,\phi_n) = \ell_a(U_n,\phi_n)(\dot U_n,\dot\phi_n)\Delta_n + \Delta_n e_{n1}'$$

where e_{n1}' is the standard quadratic error in the Newtonian iteration.

Because of the loss of regularity in the tame estimate for the linearized problem, the Nash–Moser iteration applies a smoothing (or regularizing) operator $\mathscr{S}_n$ to the value of (U_n,ϕ_n) before iteration, therefore,

$$\mathscr{L}_a(U_{n+1},\phi_{n+1}) - \mathscr{L}_a(U_n,\phi_n) = \ell_a(\mathscr{S}_n U_n, \mathscr{S}_n\phi_n)(\dot U_n,\dot\phi_n)\Delta_n + \Delta_n e_{n1} \tag{4.8}$$

where $e_{n1} \equiv e_{n1}' + e_{n1}''$ with e_{n1}'' being the smoothing error.

Using the new variable $\dot V_n$ in (4.4) and introducing the operator $\tilde\ell_a(U,\phi)\dot V = \mathscr{L}_a'(U,\phi)\dot V + B(U,\phi)\dot V$, we have

$$\mathscr{L}_a(U_{n+1},\phi_{n+1}) - \mathscr{L}_a(U_n,\phi_n) = \Delta_n\tilde\ell_a(\mathscr{S}_n U_n,\mathscr{S}_n\phi_n)\dot V_n + \Delta_n(e_{n1}+e_{n2}), \tag{4.9}$$

where $e_{n2} \equiv \frac{\dot\phi_n}{\phi_x^a + \bar\phi_{nx}}\partial_x[\mathscr{L}_a(\mathscr{S}_n U_n,\mathscr{S}_n\phi_n)]$.

In order to apply the estimates established for the linearized system in [1] and [8] where the boundaries $x = 0$ as well as $x = \alpha$ with $1 \le \alpha \le 2$ are required to be uniformly characteristic, the values $(\mathscr{S}_n U_n, \mathscr{S}_n\phi_n)$ need to be further adjusted. Hence the introduction of the error term e_{n3} is required. In the rarefaction wave case the adjusted operator can be obtained by changing some coefficients in $\tilde\ell_a$. In the contact discontinuity case the adjusted operator is obtained by replacing $(\mathscr{S}_n U_n, \mathscr{S}\phi_n)$ by suitable $(\bar U_n,\bar\phi_n)$, which is denoted by $(U_{n+1/2},\phi_{n+1/2})$ in [8, Section 7.4]. Formally we denote the linear operator obtained from $\tilde\ell_a(U_n,\phi_n)$ through this adjustment as $\mathbb{L}_a$:

$$\mathbb{L}_a(U_n,\phi_n) \equiv \tilde\ell_a(\bar U_n,\bar\phi_n). \tag{4.10}$$

Then we obtain the following relation

$$\mathscr{L}_a(U_{n+1},\phi_{n+1}) - \mathscr{L}_a(U_n,\phi_n) = \Delta_n\mathbb{L}_a(U_n,\phi_n)\dot V_n + \Delta_n e_n, \tag{4.11}$$

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where

$$e_n \equiv e_{n1} + e_{n2} + e_{n3}.$$

Since in the domain $0 < x < 1$ ($-1 < x < 0$, resp.) two nonlinear waves – contact discontinuity and rarefaction wave (shock, resp.) are involved, the adjustment should be described more carefully. First, in the pure rarefaction wave region $1 < x < 2$, the adjustment $\mathbb{L}_a^{(2)}$ is the same as did in [1, Section 3.3], where the coefficient matrix of the system is decomposed according to its eigenvectors and then its coefficients are changed to make the boundaries $x = 1, 2$ uniformly characteristic. The adjusted operator is denoted by $\bar{\bar{L}}(U_n, \phi_n)$ in [1].

In the domain $0 < x < 1$ the adjusted operator $\mathbb{L}^{(1)}$ is chosen as

$$\mathbb{L}_a^{(1)}(U_n, \phi_n) = \varphi(x)\bar{\bar{L}}(U_n, \phi_n) + \big(1 - \varphi(x)\big)\tilde{\ell}_a(\bar{U}_n, \bar{\phi}_n),$$

where $\varphi(x)$ is the C^∞ function defined by

$$\varphi(x) = \begin{cases} 0 & x < 1/3, \\ 1 & x > 2/3, \end{cases}$$

and $(\bar{U}_n, \bar{\phi}_n)$ is the $(U_{n+1/2}, \phi_{n+1/2})$ introduced in [8, Section 7.4].

In the domain $-1 < x < 0$, the adjusted operator $\mathbb{L}^{(0)}$ is chosen as

$$\varphi(x+1)\tilde{\ell}_a(\bar{U}_n, \bar{\phi}_n) + \big(1 - \varphi(x+1)\big)\tilde{\ell}_a(U_n, \phi_n),$$

because in the neighborhood of shock wave $x = -1$, the linear iteration could proceed without introducing the smoothing operator $\mathscr{S}_n$ and no adjustment is needed for the uniform characteristic requirement. And the new variable V_n is also not needed. Since no characteristic adjustment is made, the error term e_{n3} at the shock wave $x = -1$ is zero.

Let $\dot{F}_n$ be chosen such that

$$\sum_{k=0}^{n} \Delta_k \dot{F}_k = -\mathscr{S}_n \mathscr{F}_T \mathscr{L}\big(U^a, \phi^a\big) - \mathscr{S}_n \sum_{k=0}^{n-1} \Delta_k e_k. \tag{4.12}$$

Here the operator $\mathscr{F}_T$ is the extension operator from $(-\infty, T)$ to $(-\infty, \infty)$ as in [1]. Then we finally obtain the iteration scheme in the interior, i.e., the value of increment $(\dot{V}_n, \dot{\phi}_n)$ should satisfy the hyperbolic system

$$\mathbb{L}_a(U_n, \phi_n)\dot{V}_n = \dot{F}_n. \tag{4.13}$$

4.2. Boundary conditions

There are three different types of conditions on the boundaries $x = -1$, $x = 0$, and $x = 1$.

At the shock wave boundary $x = -1$, the linearization of the nonlinear operator $\mathscr{B}^{(-1)}(U^{(0)}, \phi^{(0)})$ at $(U^{(0)}, \phi^{(0)})$ is $\tilde{\mathbb{B}}^{(-1)}(U^{(0)}, \phi^{(0)})(\dot{U}^{(0)}, \dot{\phi}^{(0)})$

$$\begin{aligned}
&\tilde{\mathbb{B}}^{(-1)}\big(U^{(0)},\phi^{(0)}\big)\big(\dot{U}^{(0)},\dot{\phi}^{(0)}\big)\\
&\quad\equiv\big[H_0\big(U^{(0)}\big)\big]\partial_t\dot{\phi}^{(0)}+e^t\big[H_2\big(U^{(0)}\big)\big]\partial_y\dot{\phi}^{(0)}\\
&\qquad+\big[\phi_t^{(0)}H_0'\big(U^{(0)}\big)-e^tH_1'\big(U^{(0)}\big)+e^t\phi_y^{(0)}H_2'\big(U^{(0)}\big)\big]\dot{U}^{(0)}\\
&\qquad+\big[\phi_t^{(0)}H_0'\big(U^{(0)}\big)-e^tH_1'\big(U^{(0)}\big)+e^t\phi_y^{(0)}H_2'\big(U^{(0)}\big)\big]\frac{U_x^{(0)}}{\phi_x^{(0)}}\dot{\phi}^{(0)}\\
&\quad\equiv b^{(-1)}\big(U^{(0)},\phi^{(0)}\big)\dot{\phi}^{(0)}+M^{(-1)}\big(U^{(0)},\phi^{(0)}\big)\dot{U}^{(0)}.
\end{aligned}\tag{4.14}$$

Since $U_x^{(0)}/\phi_x^{(0)}$ is bounded, $b^{(-1)}(U^{(0)},\phi^{(0)})$ is an operator with bounded coefficients.

Replacing the variable $\dot{U}^{(0)}$ in (4.17) by $\dot{V}^{(0)}=\dot{U}-\frac{U_x}{\phi_x}\dot{\phi}$, we have

$$\begin{aligned}
&\mathbb{B}^{(-1)}\big(U^{(0)},\phi^{(0)}\big)\big(\dot{V}^{(0)},\dot{\phi}^{(0)}\big)\\
&\quad=b^{(-1)}\big(U^{(0)},\phi^{(0)}\big)\dot{\phi}^{(0)}+M^{(-1)}\big(U^{(0)},\phi^{(0)}\big)\frac{U_x}{\phi_x}\dot{\phi}^{(0)}+M^{(-1)}\big(U^{(0)},\phi^{(0)}\big)\dot{V}^{(0)}.
\end{aligned}\tag{4.15}$$

Here the notation $\mathbb{B}^{(-1)}$ means an operator acting on $(\dot{V},\dot{\phi})$ instead of $(\dot{U},\dot{\phi})$.

Therefore, on $x=-1$ we have

$$\begin{aligned}
&\mathscr{B}_a^{(-1)}\big(U_{n+1}^{(0)},\phi_{n+1}^{(0)}\big)-\mathscr{B}_a^{(-1)}\big(U_n^{(0)},\phi_n^{(0)}\big)\\
&\quad=\Delta_n\mathbb{B}_a^{(-1)}\big(\mathscr{S}_nU_n^{(0)},\mathscr{S}_n\phi_n^{(0)}\big)+\Delta_nd_n^{(-1)}
\end{aligned}\tag{4.16}$$

with $d_n^{(-1)}=d_{n1}^{(-1)}+d_{n2}^{(-1)}$ being the standard Nash–Moser error term consisting of quadratic error and regularization error:

$$\begin{aligned}
&\Delta_nd_{n1}^{(-1)}=\mathscr{B}_a^{(-1)}\big(U_{n+1}^{(0)},\phi_{n+1}^{(0)}\big)-\mathscr{B}_a^{(-1)}\big(U_n^{(0)},\phi_n^{(0)}\big)-\mathbb{B}_a^{(-1)}\big(U_n^{(0)},\phi_n^{(0)}\big)\big(\dot{U}_n^{(0)},\dot{\phi}_n^{(0)}\big)\Delta_n,\\
&d_{n2}^{(-1)}=\mathbb{B}_a^{(-1)}\big(U_n^{(0)},\phi_n^{(0)}\big)\big(\dot{V}_n^{(0)},\dot{\phi}_n^{(0)}\big)-\mathbb{B}_a^{(-1)}\big(\mathscr{S}_nU_n^{(0)},\mathscr{S}_n\phi_n^{(0)}\big)\big(\dot{V}_n^{(0)},\dot{\phi}_n^{(0)}\big).
\end{aligned}$$

Meanwhile, $\dot{G}_n^{(-1)}$ is chosen to satisfy

$$\sum_{k=0}^{n}\dot{G}_k^{(-1)}\Delta_k=-\mathscr{S}_n\mathscr{F}_T\mathscr{B}^{(-1)}\big(U^a,\phi^a\big)-\mathscr{S}_n\sum_{k=0}^{n-1}d_k^{(-1)}\Delta_k.\tag{4.17}$$

In accordance, the increment $(\dot{U}_n^{(0)},\dot{\phi}_n^{(0)})$ should satisfy

$$\mathbb{B}_a^{(-1)}\big(U_n^{(0)},\phi_n^{(0)}\big)\big(\dot{V}_n^{(0)},\dot{\phi}_n^{(0)}\big)=\dot{G}_n^{(-1)}.\tag{4.18}$$

This is the linearized boundary condition at $x=-1$.

At the contact discontinuity boundary $x=0$, we use the same iteration scheme as in [8]. First,

$$\mathscr{B}_a^{(0)}\big(U_{n+1}^{(0,1)},\phi_{n+1}^{(0)}\big)-\mathscr{B}_a^{(0)}\big(U_n^{(0,1)},\phi_n^{(0)}\big)=\beta_a^{(0)}\big(U_n^{(0,1)},\phi_n^{(0)}\big)\big(\dot{V}_n,\dot{\phi}_n^{(0)}\big)\Delta_n+d_{n1}^{(0)}\Delta_n\tag{4.19}$$

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with $\beta_a^{(0)}$ in (4.19) being the linearization of $\mathscr{B}_a^{(0)}$ at $(U_n^{(0,1)}, \phi_n^{(0)})$ and $d_{n1}^{(0)}$ being the quadratic error. Applying smoothing operator $\mathscr{S}_n$ to $(U_n^{(0,1)}, \phi_n^{(0)})$ in the coefficients of the operator $\beta_a^{(0)}$ and making further adjustment to satisfy the uniform characteristic conditions in eikonal equations in (3.2), the relation (4.19) changes into

$$\begin{aligned}&\mathscr{B}_a^{(0)}\big(U_{n+1}^{(0,1)}, \phi_{n+1}^{(0)}\big) - \mathscr{B}_a^{(0)}\big(U_n^{(0,1)}, \phi_n^{(0)}\big)\\ &\quad = \mathbb{B}_a^{(0)}\big(U_n^{(0,1)}, \phi_n^{(0)}\big)\big(\dot{V}_n^{(0)}, \dot{\phi}_n^{(0)}\big)\Delta_n + d_n^{(0)}\Delta_n,\end{aligned} \tag{4.20}$$

where

$$d_n^{(0)} = d_{n1}^{(0)} + d_{n2}^{(0)} + d_{n3}^{(0)}$$

with $d_{n2}^{(0)}$ and $d_{n3}^{(0)}$ being the substitution errors.

Then the boundary iteration scheme on $x = 0$ should be

$$\mathbb{B}_a^{(0)}\big(U_n^{(0,1)}, \phi_n^{(0)}\big)\big(\dot{V}_n^{(0)}, \dot{\phi}_n^{(0)}\big) = \dot{G}_n^{(0)} \tag{4.21}$$

with $\dot{G}_n^{(0)}$ chosen according to the following

$$\sum_{k=0}^{n} \dot{G}_k^{(0)}\Delta_k = -\mathscr{S}_n\mathscr{F}_T\mathscr{B}^{(0)}\big(U^a, \phi^a\big) - \mathscr{S}_n\sum_{k=0}^{n-1} d_k^{(0)}\Delta_k. \tag{4.22}$$

At the rarefaction wave boundaries $x = 1$, the solution should be continuous, i.e. $U^{(1)} = U^{(2)}$ at $x = 1$. Here we notice that the boundary L^+ and the value of $U^{(2)}$ on it are already known from the initial value $U_+(x, y)$ in (1.3) and Eqs. (1.1). Correspondingly, $V^{(2)}, \phi^{(2)}$ are also known.

We will adopt the boundary iteration scheme as in [1]:

$$U_{n+1}^{(1)} - U_{n+1}^{(2)} = U_n^{(1)} - U_n^{(2)} + \Delta\dot{G}_n^{(1)} + \Delta d_n^{(1)}, \tag{4.23}$$

where $d_n^{(1)}$ is the error and $\dot{G}_n^{(1)}$ is chosen to secure the convergence of the iteration.

Since the boundary L^+ is characteristic, the matrix $A_1(U^{(1)}) - \partial_t\phi^{(1)} - A_2(U^{(1)})\partial_y\phi^{(1)}$ is degenerate. In the process of iteration one must adjust the approximate solution U_n, ϕ_n to $\bar{U}_n, \bar{\phi}_n$, so that the adjusted boundary matrix

$$A_1\big(\bar{U}_n^{(1)}\big) - \partial_t\bar{\phi}_n^{(1)} - A_2\big(\bar{U}_n^{(1)}\big)\partial_y\bar{\phi}_n^{(1)} \tag{4.24}$$

is degenerate with rank 2 (correspondingly, the operator becomes the above-mentioned adjusted operator). Its eigenvectors form a new orthogonal basis in the space $\mathbb{R}^3$. Denote by $\Pi_n^{(1)} \equiv \Pi(\bar{U}_n^{(1)}, \bar{\phi}_n^{(1)})$ the matrix formed by these three unit column eigenvectors, we may obtain an orthogonal transformation from the original basis to the new basis. Without loss of generality we may assume that the first column vector corresponds to the right-propagation rarefaction wave. This vector spans an one-dimensional subspace in which the matrix (4.24) is degenerate,

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and non-degenerate in its orthogonal complement. Denoting the operators projecting to the non-degenerate and degenerate subspaces by P_n and $I - P_n$ respectively, we derive the boundary conditions on $x = 1$ as

$$P_n^{(1)}\dot V_n^{(1)} - P_n^{(2)}\dot V_n^{(2)} = P_n^{(1)}\dot G_n^{(1)} \quad \text{on } x = 1, \tag{4.25}$$

$$\big(1 - P_n^{(1)}\big)\dot V_n^{(1)} - \big(1 - P_n^{(2)}\big)\dot V_n^{(2)} = Z_n^{(1)}\dot\phi_n^{(1)} + \big(1 - P_n^{(1)}\big)\dot G_n^{(1)} \quad \text{on } x = 1. \tag{4.26}$$

Here $\dot G_n^{(j)}$ is the modified error as shown in the following (4.31),

$$Z_n^{(1)} \equiv \big(1 - P_n^{(2)}\big)\frac{U_x^{a(2)} + \bar U_{nx}^{(2)}}{\phi_x^{a(2)} + \bar\phi_{nx}^{(2)}} - \big(1 - P_n^{(1)}\big)\frac{U_x^{a(1)} + \bar U_{nx}^{(1)}}{\phi_x^{a(1)} + \bar\phi_{nx}^{(1)}}. \tag{4.27}$$

We remark here that

$$e^t Z_n^{(1)} \neq 0. \tag{4.28}$$

Indeed, in the above equality $\phi_x^{a(1)} + \bar\phi_{nx}^{(1)}$ is the approximation of $\phi_x^{a(1)}$ which satisfies

$$\phi_x^{a(1)} \geq C_1 e^t \tag{4.29}$$

with $C_1 > 0$ because of (2.41). Then the denominator of the second term in the right hand side of (4.27) obeys the inequality (4.29). In the meantime, we also have $U_x^{a(1)} + \bar U_{nx}^{(1)} = O(e^t)$ due to the smoothness of U in ω_1. Therefore, the fraction in (4.27) is bounded as $t \to -\infty$. Hence the argument in Proposition 6.3 of [1] implies (4.28).

For simplicity of notation, we will denote the boundary iteration scheme (4.24) and (4.25) on $x = 1$ as

$$\mathbb{B}_a^{(1)}\big(U_n^{(1)}, \phi_n^{(1)}\big)\big(\dot V_n^{(1)}, \dot\phi_n^{(1)}\big) = \dot G_n^{(1)} \tag{4.30}$$

with $\dot G_n^{(1)}$ chosen according to the following

$$\sum_{k=0}^{n} \dot G_k^{(1)}\Delta_k = -\mathscr{S}_n\mathscr{F}_T\mathscr{B}^{(1)}\big(U^a, \phi^a\big) - \mathscr{S}_n\sum_{k=0}^{n-1} d_k^{(1)}\Delta_k. \tag{4.31}$$

5. Estimate of linearized problem

5.1. Remarks on the reformulation

The Nash–Moser iteration method depends on obtaining the “tame” estimate in appropriate spaces for the linearized problem (see [12]). The tame estimate for the linearized problems involving rarefaction wave or contact discontinuity was established in [1,8], while the estimate established in [17] can also be regarded as a special case of tame estimate. Next we need to combine all three estimates in one framework. Meanwhile, we will also indicate that the reformulation of the problem in Section 3 does not introduce any new difficulty in establishing the whole tame estimate.

Please cite this article in press as: S. Chen, D. Li, Cauchy problem with general discontinuous initial data along a smooth curve for 2-d Euler system, J. Differential Equations (2014), http://dx.doi.org/10.1016/j.jde.2014.05.027

JID:YJDEQ AID:7503 /FLA [m1+; v 1.192; Prn:28/05/2014; 13:14] P.33 (1-50)

Firstly, the formulation of the problem involving rarefaction wave in [1] is identical to the formulation in this paper after the transform (3.8), and consequently is equivalent to the formulation after the transform (3.10). In particular, the t-weighted norm near $t \sim 0$ is equivalent to the e^t-weighted norm near $t \sim -\infty$. Since the factor e^t is introduced only in the coefficients of tangential derivative terms or lower order terms, it has no effect upon the well-posedness of the rarefaction wave problem.

Secondly, for the case involving shock or contact discontinuity we also apply the blow up domain transformation (from $t > 0$ to $t > -\infty$), as well as the introduction of the small factor e^t near $t \sim -\infty$. Hence we need to address these differences as well as the possible difficulty caused by the localization process.

The domain change actually does not cause any difficulty. Indeed, in the discussion of both shock wave [17] and contact discontinuity [8], the estimate is always for the value between the unknown functions and the approximate solutions. Therefore, it is always assumed that the estimated quantity is identically zero in $t < 0$, and the discussion is carried out formally in $-\infty < t < T$ with $0 < T \ll 1$. In this paper, we will consider the formally same domain $-\infty < t < T$, but with $-T \gg 1$.

On the other hand, the introduction of the small factor e^t in our formulation does not cause any new difficulty in obtaining the estimate for linearized problem. For both shock wave and contact discontinuity, the linearized estimates are obtained by micro-local analysis on the cotangent bundle $s^2 + \omega^2 = 1$. In our formulation, notice that the factor e^t appears simultaneously in the coefficients of tangential derivative ∂_y terms, in both the interior equations and in the boundary conditions, so the factor e^t only increases the weight on s and the analysis can proceed as usual. On the other hand, the factor e^t in the lower order terms can only have a beneficial effect in obtaining the estimate since $e^t \ll 1$ near $t = -\infty$.

Finally, since all three types of wave are involved in the linear estimate, we have to resort to the localization. Now near different kinds of waves, the linearized estimates are different. We need to patch them together to obtain the general estimate for the whole wave structure discussed here. Here we will adopt the weakest estimate in three wave patterns. This means, even though we could have a standard estimate for the linearized shock wave, we will use only a watered-down weak version of "tame" estimate which would match the estimates available for the contact discontinuity. In this way, we can overcome the difficulty caused by the localization.

5.2. *The family of spaces*

In this paper, we will use the η-weighted norms both in the interior domains and on the boundaries. Such η-weighted norms are in form the same as the standard η-weighted norms usually used in the study of hyperbolic problems. However, our norms are defined in the region $-\infty < t < T$, in contrast to the standard region of $0 < t < T$. Indeed the norms we used here are equivalent to the t-weighted norms used in [1], see (3.13). Meanwhile, they have many similar properties, such as the Sobolev imbedding into continuous functions, Banach algebra property for index $s > [n/2] + k$, trace theorem, etc.

For a non-negative integer s, let $k = (k_0, k_1, k_2)$ be the multiple index with $|k| = k_0 + k_1 + k_2$. We define $H^s_\eta(\omega^T_j)$ to be the Sobolev space defined by the norm

$$\|U\|^2_{H^s_\eta(\omega^T_j)} = \sum_{0 \le |k|+2m \le s} \int_{\omega^T_j} \left|\partial_t^{k_0} D_x^{k_1} \partial_y^{k_2} \partial_x^m \left(e^{-\eta t} U(x,y,t)\right)\right|^2 dy\,dx\,dt, \tag{5.1}$$

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where η is a fixed sufficiently large constant, $D_x = x(x-1)(x-2)\partial_x$ is an operator tangential to the boundaries $x = 0, 1, 2$.

The norm defined in (5.1) is obviously equivalent to the following

$$\|U\|^2_{H^s_\eta(\omega^T_j)} = \sum_{0\le|k|+2m\le s} \int_{\omega^T_j} \eta^{2(k_0-k_0')} \left|e^{-\eta t}\partial_t^{k_0'} D_x^{k_1}\partial_y^{k_2}\partial_x^m U(x,y,t)\right|^2 dydxdt. \tag{5.2}$$

Let Γ^T_j $(j=-1,0,1,2)$ be the boundary

$$\Gamma^T_j = \{(t,x,y);\ -\infty < t < T,\ x = j,\ y \in \mathbb{R}\}. \tag{5.3}$$

And the Sobolev space on the boundary Γ^T_j can be defined similarly

$$|U|^2_{H^s_\eta(\Gamma^T_j)} = \sum_{0\le|k|\le s} \int_{\omega^T_j} \left|\partial_t^{k_0}\partial_y^{k_2}\left(e^{-\eta t}U(y,t)\right)_{x=j}\right|^2 dydt. \tag{5.4}$$

The Sobolev spaces $H^s_\eta(\omega^T_j)$ can be imbedded in the spaces of bounded and continuously differentiable functions

$$H^s_\eta(\omega^T_j) \subset C^m, \quad \text{for } s > 2+2m. \tag{5.5}$$

And also we have the trace theorem

$$s > 1, \qquad u \in H^s_\eta(\omega^T_j) \quad \Rightarrow \quad u|_{x=j-1} \in H^{s-1}_\eta(\Gamma_{j-1}), \qquad u|_{x=j} \in H^{s-1}_\eta(\Gamma_j), \tag{5.6}$$

and the corresponding inverse trace theorem.

5.3. The well-posedness of the linearized problem

The well-posedness of the linearized shock wave, contact discontinuity, and rarefaction wave has been discussed separately in [17,8,1]. In this paper, we will apply the results obtained therein to study the combination of such waves. For the well-posedness of the linearized waves, the preceding terms (U_n, ϕ_n) are always assumed to be near the background waves. Just as in the discussion of each separate wave, we will assume throughout the following discussion of the linearized problem that there exists a small constant $\kappa_0 > 0$, such that the values of (U_n, ϕ_n) in the coefficients of the linearized problem satisfy

$$\|U_n\|_{H^5_\eta(\omega^T)} + \|\phi_n\|_{H^5_\eta(\Gamma^T)} = \kappa \le \kappa_0. \tag{5.7}$$

The satisfaction of (5.7) guarantees the well-posedness of the linearized problems and the validity of the estimate for its solution, uniformly with respect to $\kappa \le \kappa_0$.

We formulated the three linearized boundary value problems of (4.13) with the boundary conditions (4.18), (4.21) or (4.29). To apply the estimates established in [17,8] or [1] for the single shock wave, contact discontinuity or rarefaction wave case, we have to derive three Cauchy

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problems, each of them only involving a single wave. To this end we use partition of unity. As before, let $\varphi(x) \in C^\infty(-\infty, \infty)$ satisfying

$$\varphi(x) = \begin{cases} 0 & x < 1/3, \\ 1 & x > 2/3. \end{cases}$$

Define

$$\zeta_1(x) = \varphi(1+x)\big(1-\varphi(x)\big), \qquad \zeta_2(x) = \varphi(x)\big(1-\varphi(x-2)\big),$$
$$\zeta_0(x) = \zeta_1(1+x). \tag{5.8}$$

Noticing that $\zeta_j(x) = 1$ as $x = j-1$, we consider the boundary value problems satisfied by $(\zeta_j(x)\dot{V}_n^{(j)}, \phi_n^{(j)})$ with $j = 0, 1, 2$ and derive the estimates for them respectively. Since

$$\operatorname{supp}\zeta_0 \subset (-5/3, -1/3), \qquad \operatorname{supp}\zeta_1 \subset (-2/3, 2/3), \qquad \operatorname{supp}\zeta_2 \subset (1/3, 8/3),$$

we can make the zero extension into $x \in \mathbb{R}$, and the problems for $(\zeta_j(x)\dot{V}_n^{(j)}, \phi_n^{(j)})$ become the same problem discussed in [1,8,17]. This fact allows us to apply the estimates established in [1,8,17] to our problem.

5.3.1. The linearized problem near $x = -1$

Near $x = -1$, we have the linearized problem:

$$\begin{cases} \mathbb{L}_a^{(0)}(U_n^{(0)}, \phi_n^{(0)})(\zeta_0\dot{V}_n^{(0)}, \dot{\phi}_n^{(0)}) = \zeta_0\dot{F}_n^{(0)} + \big(\mathscr{L}_a'(U_n^{(0)}, \phi_n^{(0)})\zeta_0\big)\dot{V}_n^{(0)}, & x > -1, \\ \mathbb{B}_a^{(-1)}(U_n^{(0)}, \phi_n^{(0)})(\dot{V}_n^{(0)}, \dot{\phi}_n^{(0)}) = \dot{G}_n^{(-1)}, & x = -1. \end{cases} \tag{5.9}$$

To establish the estimates for the solution of (5.9), we apply the results in [17]. In [17], the n-dimensional Cauchy problem of a general quasilinear hyperbolic system with a single shock wave was studied, and the solution includes the shock front location and status on both sides of the shock front. In our case with left-propagating shock, the status to the left of the shock is known by the property of the finite propagating speed. Therefore, the only unknowns are the location of the shock front and the status to the right of the shock. Such situation permits some simplification of the estimates in [17], and the estimates involve only the unknowns $\zeta_0\dot{V}_n^{(0)}$ (we also notice that $\zeta_0\dot{V}_n^{(0)} = \dot{V}_n^{(0)}$ on the boundary $x = -1$) and $\phi_n^{(0)}$. Consequently, we have the following lemma. (In all the lemmas in this section, the conditions in Theorem 1.2 and the conditions (3.16)–(3.23) are always assumed.)

Lemma 5.1. *Let the integer $s \geq s_0 > 4$ and η be sufficiently large, $\dot{F}_n^{(0)} \in H_\eta^s(\omega_0^T)$ and $\dot{G}_n^{(-1)} \in H_\eta^s(\gamma_{-1}^T)$. Then the solution $(\zeta_0\dot{V}_n^{(0)}, \dot{\phi}_n^{(0)})$ of (5.9) satisfies*

$$\begin{aligned} &\eta\big\|\zeta_0\dot{V}_n^{(0)}\big\|^2_{H_\eta^s(\omega_0^T)} + \big\|\zeta_0\dot{V}_n^{(0)}\big\|^2_{H_\eta^s(\Gamma_{-1}^T)} \\ &\quad + \eta\big\|e^{-t}\dot{\phi}_n^{(0)}\big\|^2_{H_\eta^s(\Gamma_{-1}^T)} + \big\|e^{-t}D_t\dot{\phi}_n^{(0)}\big\|^2_{H_\eta^s(\Gamma_{-1}^T)} + \big\|D_y\dot{\phi}_n^{(0)}\big\|^2_{H_\eta^s(\Gamma_{-1}^T)} \\ &\leq C_s\Big[\big\|\dot{F}_n^{(0)}\big\|^2_{H_\eta^s(\omega_0^T)} + \big\|\big(\mathscr{L}_a'(U_n^{(0)}, \phi_n^{(0)})\zeta_0\big)\dot{V}_n^{(0)}\big\|_{H_\eta^s(\omega_0^T)} + \big\|\dot{G}_n^{(-1)}\big\|^2_{H_\eta^s(\Gamma_{-1}^T)} \end{aligned}$$

$$+ \big(\| \dot{F}_n^{(0)} \|^2_{H^{s_0}_\eta(\omega_0^T)} + \big\| \big(\mathscr{L}'_a(U_n^{(0)}, \phi_n^{(0)}) \zeta_0 \big) \dot{V}_n^{(0)} \big\|_{H^{s_0}_\eta(\omega_0^T)} + \| \dot{G}_n^{(-1)} \|^2_{H^{s_0}_\eta(\Gamma_{-1}^T)} \big) \cdot \big(1 + \| \mathrm{coeff}^{(-1)} \|_s^2 \big) \big]. \tag{5.10}$$

Here the constant C_s depends only on κ_0, while the notation $\|\mathrm{coeff}\|_s$ represents the terms

$$\| \mathrm{coeff}^{(-1)} \|_s = \| U_n^{(0)} \|_{H^s_\eta(\omega_0^T)} + \| U_n^{(0)} \|_{H^s_\eta(\Gamma_{-1}^T)} + \| \phi_n^{(-1)} \|_{H^{s+1}_\eta(\Gamma_{-1}^T)}.$$

5.3.2. The linearized problem near $x = 0$

Near $x = 0$, we have the linearized equation (4.13) with the contact discontinuity linearized boundary conditions (4.18):

$$\begin{cases} \mathbb{L}_a^{(0)}(U_n^{(0)}, \phi_n^{(0)})(\zeta_1 \dot{V}_n^{(0)}, \dot{\phi}_n^{(0)}) = \zeta_1 \dot{F}_n^{(0)} + \big(\mathscr{L}'_a(U_n^{(0)}, \phi_n^{(0)}) \zeta_1 \big) \dot{V}_n^{(0)}, & x < 0, \\ \mathbb{L}_a^{(1)}(U_n^{(1)}, \phi_n^{(1)})(\zeta_1 \dot{V}_n^{(1)}, \dot{\phi}_n^{(1)}) = \zeta_1 \dot{F}_n^{(1)} + \big(\mathscr{L}'_a(U_n^{(1)}, \phi_n^{(1)}) \zeta_1 \big) \dot{V}_n^{(1)}, & x > 0, \\ \mathbb{B}_a^{(0)}(U_n^{(0,1)}, \phi_n^{(0)})(\dot{V}_n^{(0)}, \dot{\phi}_n^{(0)}) = \dot{G}_n^{(0)}, \quad x = 0. \end{cases} \tag{5.11}$$

Since ζ_1 vanishes outside $(-2/3, 2/3)$, then by extending $\zeta_1 \dot{V}_n^{(0,1)}$ as zero outside $(-2/3, 2/3)$, the problem (5.11) becomes the same problem studied in [8]. It is established in [8] the corresponding tame estimate (Proposition 6) as follows:

Lemma 5.2. *Let an integer $s \ge s_0 > 4$ and η be sufficiently large, $\dot{F}_n^{(0,1)} \in H^{s+1}_\eta(\omega_{0,1}^T)$ and $\dot{G}_n^{(0)} \in H^s_\eta(\Gamma_0^T)$. Then (5.11) has a unique solution $(\zeta_1 \dot{V}_n^{(0)}, \dot{\phi}_n^{(0)})$, $(\zeta_1 \dot{V}_n^{(1)}, \dot{\phi}_n^{(1)})$ (or simply denoted by $(\zeta_1 \dot{V}_n^{(0,1)}, \dot{\phi}_n^{(0,1)})$ with noticing $\dot{\phi}_n^{(0)} = \dot{\phi}_n^{(1)}$ on $x = 0$), satisfying*

$$\begin{aligned} &\eta \| \zeta_1 \dot{V}_n^{(0,1)} \|_{H^s_\eta(\omega_{0,1}^T)} + \| \mathbb{P}^{(0)} \zeta_1 \dot{V}_n^{(0,1)} \|_{H^s_\eta(\Gamma_0^T)} \\ &\quad + \eta \| e^{-t} \dot{\phi}_n^{(0)} \|_{H^s_\eta(\Gamma_0^T)} + \| e^{-t} D_t \dot{\phi}_n^{(0)} \|_{H^s_\eta(\Gamma_0^T)} + \| D_y \dot{\phi}_n^{(0)} \|_{H^s_\eta(\Gamma_0^T)} \\ &\le C_s \big[\| \zeta_1 \dot{F}_n^{(0,1)} \|_{H^{s+1}_\eta(\omega_{0,1}^T)} + \big\| \big(\mathscr{L}'_a(U_n^{(0,1)}, \phi_n^{(0)}) \zeta_1 \big) \dot{V}_n^{(0,1)} \big\|_{H^s_\eta(\omega_{(0,1)}^T)} + \| \dot{G}_n^{(-1)} \|_{H^{s+1}_\eta(\Gamma_0^T)} \\ &\quad + \big(\| \dot{F}_n^{(0,1)} \|_{H^{s_0}_\eta(\omega_0^T)} + \big\| \big(\mathscr{L}'_a(U_n^{(0,1)}, \phi_n^{(0)}) \zeta_1 \big) \dot{V}_n^{(0,1)} \big\|_{H^{s_0}_\eta(\omega_{(0,1)}^T)} + \| \dot{G}_n^{(-1)} \|_{H^{s_0}_\eta(\Gamma_0^T)} \big) \\ &\quad \cdot \big(1 + \| \mathrm{coeff}^{(0)} \|_{s+3} \big) \big]. \end{aligned} \tag{5.12}$$

Here $\mathbb{P}^{(0)}$ is the projection operator onto the non-degenerate components of vector $\dot{V}_n^{(0,1)}$ at the boundary $x = 0$. And $\|\mathrm{coeff}^{(0)}\|_{s+3}$ represents the terms

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$$\| \mathrm{coeff}^{(0)} \|_{s+3} = \| U_n^{(0,1)} \|_{H^{s+3}_\eta(\omega_{0,1}^T)} + \| U_n^{(0,1)} \|_{H^{s+3}_\eta(\Gamma_0^T)} + \| \phi_n^{(0)} \|_{H^{s+3}_\eta(\Gamma_0^T)}.$$

Remark 5.1. Proposition 6 in [8] also requires that the integer $s \le 2\mu - 1$, with μ being the compatibility order of the initial data. Here we do not need this condition, because our approximate solution has infinite compatibility by Theorem 2.2.

5.3.3. The linearized problem near $1 \le x \le 2$

Near $1 \le x \le 2$, we have the linearized equation (4.13) with the rarefaction wave linearized boundary conditions (4.29):

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$$\begin{cases} \mathbb{L}_a^{(1)}\big(U_n^{(1)},\phi_n^{(1)}\big)\big(\zeta_2\dot{V}_n^{(1)},\dot{\phi}_n^{(1)}\big) = \zeta_2\dot{F}_n^{(1)} + \big(\mathcal{L}_a'\big(U_n^{(1)},\phi_n^{(1)}\big)\zeta_2\big)\dot{V}_n^{(1)}, & x<1, \\ \mathbb{L}_a^{(2)}\big(U_n^{(2)},\phi_n^{(2)}\big)\big(\zeta_2\dot{V}_n^{(2)},\dot{\phi}_n^{(2)}\big) = \dot{F}_n^{(2)}, \quad 1<x<2, \\ \mathbb{B}_a^{(1)}\big(U_n^{(1)},\phi_n^{(1)}\big)\big(\dot{V}_n^{(1)},\dot{\phi}_n^{(1)}\big) = \dot{G}_n^{(1)}, \\ \mathbb{B}_a^{(2)}\big(U_n^{(2)},\phi_n^{(2)}\big)\big(\dot{V}_n^{(2)},\dot{\phi}_n^{(2)}\big) = \dot{G}_n^{(2)}. \end{cases} \tag{5.13}$$

Since $\zeta_2 = 1$ in the domain $1 < x < 2$, the function $\zeta_2\dot{V}_n^{(2)}$ equals $\dot{V}_n^{(2)}$ in the second equation of (5.13). Since ζ_2 vanishes as $x < 1/3$, we can extend $\zeta_2\dot{V}_n^{(1)}$ as zero into $-\infty < x < 1/3$. Besides, from the property of the finite propagation speed, the right-propagating rarefaction wave is known, and the solution to the right of the characteristics is also known. Therefore in applying the result in [1], we don't need to list the equation in the domain $x > 2$. Hence the well-posedness result of the boundary value problem (5.13) studied in [1] gives us the following:

Lemma 5.3. *Let integer $s \ge s_0 \ge 6$ and let η be sufficiently large. If $\dot{F}_n^{(1,2)} \in H_\eta^s(\omega_{1,2}^T)$ and $G_n^{(1,2)} \in H_\eta^{s+1}(\Gamma_{1,2}^T)$, then the boundary value problem (5.13) has a unique solution $(\zeta_2\dot{V}_n^{(1,2)},\dot{\phi}_n^{(1,2)})$, satisfying the following estimate*

$$\begin{aligned} &\big\|\zeta_2\dot{V}_n^{(1,2)}\big\|_{H_\eta^s(\omega_{1,2}^T)} + \big\|\dot{\phi}_n^{(1,2)}\big\|_{H_\eta^{s-1}(\omega_{1,2}^T)} \\ &\quad \le C_s\Big[\big\|e^t\zeta_2\dot{F}_n^{(1,2)}\big\|_{H_\eta^s(\omega_{1,2}^T)} + \big\|\big(\mathcal{L}_a'\big(U_n^{(1,2)},\phi_n^{(1)}\big)\zeta_2\big)\dot{V}_n^{(1,2)}\big\|_{H_\eta^s(\omega_{1,2}^T)} + \big\|\dot{G}_n^{(1,2)}\big\|_{H_\eta^{s+1}(\Gamma_{1,2}^T)} \\ &\qquad + \Big(\big\|e^t\dot{F}_n^{(1,2)}\big\|_{H_\eta^{s_0}(\omega_{1,2}^T)} + \big\|\big(\mathcal{L}_a'\big(U_n^{(1,2)},\phi_n^{(1)}\big)\zeta_2\big)\dot{V}_n^{(1,2)}\big\|_{H_\eta^{s_0}(\omega_{1,2}^T)} + \big\|\dot{G}_n^{(1,2)}\big\|_{H_\eta^{s_0}(\Gamma_{1,2}^T)}\Big) \\ &\qquad \cdot\big(1+\big\|\mathrm{coeff}^{(1,2)}\big\|_s\big)\Big]. \end{aligned} \tag{5.14}$$

Here, the term $\|\mathrm{coeff}^{(1,2)}\|_s$ is defined similarly as above.

5.4. Summary of the linear estimate

To simplify the notation, we introduce the following notations:

$$\begin{aligned} &\mathbb{L}_a \equiv \big(\mathbb{L}_a^{(0)},\mathbb{L}_a^{(1)},\mathbb{L}_a^{(2)}\big), \\ &\mathbb{B}_a \equiv \big(\mathbb{B}_a^{(-1)},\mathbb{B}_a^{(0)},\mathbb{B}_a^{(1)},\mathbb{B}_a^{(2)}\big), \\ &U_n \equiv \big(U_n^{(0)},U_n^{(1)},U_n^{(2)}\big), \qquad \phi_n \equiv \big(\phi_n^{(0)},\phi_n^{(1)},\phi_n^{(2)}\big), \\ &\dot{U}_n \equiv \big(\dot{U}_n^{(0)},\dot{U}_n^{(1)},\dot{U}_n^{(2)}\big), \qquad \dot{V}_n \equiv \big(\dot{V}_n^{(0)},\dot{V}_n^{(1)},\dot{V}_n^{(2)}\big), \qquad \dot{\phi}_n \equiv \big(\dot{\phi}_n^{(0)},\dot{\phi}_n^{(1)},\dot{\phi}_n^{(2)}\big). \end{aligned}$$

Then the linearized problem (5.10), (5.12) and (5.14) at $x = -1, 0, 1$ and 2 respectively can be briefly written as follows

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$$\begin{cases} \mathbb{L}_a(U_n,\phi_n)(\dot V_n,\dot\phi_n)=\dot F_n, \\ \mathbb{B}_a(U_n,\phi_n)(\dot V_n,\dot\phi_n)=\dot G_n. \end{cases} \tag{5.15}$$

To combine the estimates obtained in Section 5.3 we notice

$$\zeta_0(x)+\zeta_1(x)=1 \quad \text{in } -1\le x\le 0,$$
$$\zeta_1(x)+\zeta_2(x)=1 \quad \text{in } 0\le x\le 1,$$

and

$$\zeta_2(x)=1 \quad \text{in } 1\le x\le 2.$$

Then it is easy to have

$$\|\dot V_n^{(0)}\|_{H^s_\eta(\omega_0^T)}\le \|\zeta_0\dot V_n^{(0)}\|_{H^s_\eta(\omega_0^T)}+\|\zeta_1\dot V_n^{(0)}\|_{H^s_\eta(\omega_0^T)}, \tag{5.16}$$
$$\|\dot V_n^{(1)}\|_{H^s_\eta(\omega_0^T)}\le \|\zeta_1\dot V_n^{(1)}\|_{H^s_\eta(\omega_1^T)}+\|\zeta_2\dot V_n^{(1)}\|_{H^s_\eta(\omega_1^T)}. \tag{5.17}$$

Besides, in view of (5.7) and the boundedness of all derivatives of ζ_j we can sum up the estimates (5.10), (5.12), (5.14) to obtain the following

Theorem 5.2. *For the complete linearized shock-contact-rarefaction wave problem (5.15), assume*

- *(5.7) is satisfied;*
- *Integer $s_0\ge 6$ and even integer $s\ge s_0$;*
- *$-T\gg 1$;*
- *$\dot F_n\in H^s_\eta(\omega^T)$ and $\dot G_n\in H^{s+1}_\eta(\Gamma^T)$.*

Then the solution of (5.15) satisfies the following estimate

$$\|\dot V_n\|_{H^s_\eta(\omega^T)}+\|\dot\phi_n\|_{H^{s-1}_\eta(\Gamma^T)}\le C_s\big[\|\dot F_n\|_{H^s_\eta(\omega^T)}+\|\dot G_n\|_{H^{s+1}_\eta(\Gamma^T)}+ \\ +\big(\|\dot F_n\|_{H^{s_0}_\eta(\omega^T)}+\|\dot G_n\|_{H^{s_0}_\eta(\Gamma^T)}\big)\big(1+\|\mathrm{coeff}\|_{s+3}\big)\big]. \tag{5.18}$$

Remark 5.3. The estimate (5.18) is a weak combination of the estimates (5.10), (5.12) and (5.14). As the orders of both the left-hand side terms and right-hand side terms in these estimates are different from wave to wave, we simply adopt the weaker version among the three estimates.

6. Nash–Moser iteration and convergence

Using the energy estimates obtained in Section 5 for the solution of the linearized problem, we will now perform the Nash–Moser iteration to establish the existence of the solution for (3.16)–(3.23).

The existence of shock wave, rarefaction wave, and contact discontinuity has already been established separately in [17,1,8]. We will establish the existence of solutions containing all

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three different waves. We are going to show that it is possible to use the Nash–Moser iteration scheme to produce a convergent sequence of approximate solutions.

Let (U_n, ϕ_n) $(n = 0, 1, 2, \dots)$ be the sequence of the approximate solutions in (4.3) with (U^a, ϕ^a) being the C^∞ approximate solution for (3.16)–(3.23) established in Theorem 2.2.

Next we will introduce the recurrence hypotheses which include a family of estimates for $(\dot{U}_k, \dot{\phi}_k)$ as well as for the differential operators $\mathscr{L}(U_k, \phi_k)$ in the interior domain with the boundary operators $\mathscr{B}(U_k, \phi_k)$. The recurrence hypotheses are slightly different from those used in [1] and [8]. Meanwhile, we notice here that to obtain a unified estimate to proceed with the iteration scheme, we need to have the same estimate in the overlapping interior domain, while the boundary estimates need only to match the corresponding interior estimate for each separate wave.

On the other hand, we always have better estimate for the solutions of linearized shock waves, compared with the rarefaction wave and contact discontinuity. We have the same order of estimate for the boundary value as for the interior and without any loss of regularity for the solution [17]. Indeed, we can establish the convergence of the sequence of approximate solutions without using Nash–Moser type iteration. Since the Nash–Moser iteration also works for the shock wave as well, as indicated in [15], we can simply adopt the same form of estimate for shock wave as for contact discontinuity. And we will always do so in the following.

Consequently, the main issue here is for the combination of rarefaction wave and contact discontinuity, i.e., we should focus on the domain ω_1^T lying between rarefaction wave and contact discontinuity.

Let $(\mathscr{H}_n)$ be the following recurrence hypotheses:

Q15

$$\left\|(\dot{U}_k, \dot{\phi}_k)\right\|_{H^s_\eta(\omega^T)} + \|\dot{\phi}_k\|_{H^{s+1}_\eta(\Gamma^T)} \le \delta\theta_k^{s-\alpha-1}, \quad 0 \le k \le n,\ s_0 \le s \le s_+, \tag{6.3}$$

$$\left\|\mathscr{L}_a(U_k, \phi_k)\right\|_{H^s_\eta(\omega^T)} \le \delta\theta_k^{s-\alpha}, \quad 0 \le k \le n,\ s_0 \le s \le s_+ - 2, \tag{6.4}$$

$$\left\|\mathscr{B}_a(U_k, \phi_k)\right\|_{H^s_\eta(\Gamma^T)} \le \delta\theta_k^{s-\alpha}, \quad 0 \le k \le n,\ s_0 \le s \le s_+ - 1. \tag{6.5}$$

The success of Nash–Moser iteration depends upon the appropriate choice of constants α, δ and the integer $s_+ \ge s_0$ such that $(\mathscr{H}_0)$ is true and $(\mathscr{H}_{n-1})$ implies $(\mathscr{H}_n)$.

In the proof of the existence for rarefaction wave [1] and for contact discontinuity [8], in addition to the choice of α, δ, s_+, there was also an extra requirement on the compatibility order for the initial data. Fortunately, the requirement is automatically satisfied in this paper because of the existence of infinite approximate solution by Theorem 2.2.

Once it is shown that $(\mathscr{H}_n)$ is true for all n, it follows readily that the sequence of approximate (U_n, ϕ_n) converges in the space $H^s_\eta(\omega^T) \times H^s_\eta(\Gamma^T)$ with $s < \alpha$, because of the choice in (4.1): $\theta_n \sim \sqrt{n}$. This implies the existence of the solution $(U, \phi) \in H^{\alpha-1}_\eta(\omega^T) \times H^{\alpha-1}_\eta(\Gamma^T)$.

Our main effort in this section is to prove the following:

Theorem 6.1. *The assumptions $(\mathscr{H}_n)$ are true for all $n \ge 0$ under the following choice of parameters*

$$\delta \ll 1; \qquad s_0 = 6 \quad \left(> \frac{2+1}{2} + 2\right); \qquad \alpha > s_0 + 6 = 12; \qquad s_+ = 2\alpha - s_0 \ge \alpha + 6. \tag{6.6}$$

Here, the parameter α is chosen and fixed, while the parameter δ will be determined later.

Q16 **Remark.** Following Theorem **??** and Theorem 2.2, since there is no restriction from above on the index s_+, one can choose the index α larger than any given integer k. Hence we obtain the existence of the solution in $H^k_\eta(\omega^T) \times H^k_\eta(\Gamma^T)$. Since k is arbitrary, this implies the existence of C^∞ solution.

Q17 We will prove Theorem **??** by combining the estimates obtained for rarefaction wave in [1], for contact discontinuity in [8], and for shock wave in [17] and [15]. We begin with the proof that $(\mathcal{H}_{n-1}) \Rightarrow (\mathcal{H}_n)$, and then we choose parameter δ to satisfy $(\mathcal{H}_0)$.

First, let's recall some important properties for the mollifier $\mathcal{S}_k \equiv \mathcal{S}_{\theta_k}$ which were used in [1, Proposition 4.2] and in an improved form used in [8, Lemma 4]. The same notation $\mathcal{S}_k$ will be used for the mollifiers in all the domain ω^T, as well as on the boundary Γ^T.

Proposition 6.2. *For $M \in \mathbb{N}$ and $M \ge 4$, $\alpha, \beta \in \mathbb{N}$ and $1 \le \alpha, \beta \le M$, the mollifier operator $\mathcal{S}_k$ in the spaces $H^\alpha_\eta(\omega^T)$ has the following properties:*

(1) $\|\mathcal{S}_k u\|_{H^\beta_\eta(\omega^T)} \le C\theta_k^{(\beta-\alpha)_+} \|u\|_{H^\alpha_\eta(\omega^T)}$;

(2) $\|\mathcal{S}_k u - u\|_{H^\beta_\eta(\omega^T)} \le C\theta_k^{\beta-\alpha} \|u\|_{H^\alpha_\eta(\omega^T)}$, $\beta \le \alpha$;

(3) $\|(\frac{d}{d\theta}\mathcal{S}_\theta)|_{\theta=\theta_k} u\|_{H^\beta_\eta(\omega^T)} \le C\theta_k^{\beta-\alpha-1} \|u\|_{H^\alpha_\eta(\omega^T)}$.

Here $\sigma_+ = \max(\sigma, 0)$ and $C = C_M$.

(1)–(3) *are also true if $H^\alpha_\eta(\omega^T)$ is replaced by $H^\alpha_\eta(\Gamma^T)$.*

In addition, the mollifier $\mathcal{S}_k$ keeps the trace on Γ^T of function $u \in H^\alpha_\eta(\omega^T)$ in the following sense

$$\left\|\left(\mathcal{S}_k u^{(j)} - \mathcal{S}_k u^{(j+1)}\right)\big|_{\Gamma^T}\right\|_{H^\beta_\eta(\Gamma^T)} \le C\theta_k^{\beta-\alpha+1} \left\|\left(u^{(j)} - u^{(j+1)}\right)\big|_{\Gamma^T}\right\|_{H^\alpha_\eta(\Gamma^T)}, \quad j = 0, 1.$$

6.1. $(\mathcal{H}_{n-1}) \Rightarrow (\mathcal{H}_n)$ – 1: estimate for (U_n, ϕ_n) and solvability of linearized problem

By definition (4.3), we have

$$(U_n, \phi_n) = \sum_{k=0}^{n-1} (\dot{U}_k, \dot{\phi}_k)\Delta_k.$$

From the property of mollifier $\mathcal{S}_k$ in Proposition 6.2, it is easy to obtain from (6.3)–(6.5) that for any fixed $\epsilon > 0$ and $0 \le k \le n$ we have

- For (U_k, ϕ_k):

$$\begin{aligned}\left\|(U_k, \phi_k)\right\|_{H^s_\eta(\omega^T)} &\le \sum_{k=0}^{n-1} \left\|(\dot{U}_k, \dot{\phi}_k)\right\|_{H^s_\eta(\omega^T)} \Delta_k \\ &\le C\delta \sum_{k=1}^{n-1} \theta_k^{s-\alpha-1} \frac{1}{\theta_k} = C\delta \sum_{k=1}^{n-1} \theta_k^{s-\alpha-2}\end{aligned}$$

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then

$$\begin{cases} \|(U_k,\phi_k)\|_{H^s_\eta(\omega^T)} \le \delta\theta_k^{(s-\alpha)_+}, & s_0\le s\le s_+,\ s\ne\alpha, \\ \|(U_k,\phi_k)\|_{H^\alpha_\eta(\omega^T)} \le \delta\log\theta_k; \end{cases} \tag{6.7}$$

- For the mollification $(\mathscr{S}_kU_k, \mathscr{S}_k\phi_k)$:

$$\|(\mathscr{S}_kU_k,\mathscr{S}_k\phi_k)\|_{H^s_\eta(\omega^T)} \le C\delta\theta_k^{\epsilon+(s-\alpha)_+}, \quad s\ge s_0,\ (\epsilon=0 \text{ if } s\ne\alpha);$$
$$\|(U_k-\mathscr{S}_kU_k,\phi_k-\mathscr{S}_k\phi_k)\|_{H^s_\eta(\omega^T)} \le C\delta\theta_k^{s-\alpha}, \quad s_0\le s\le s_+. \tag{6.8}$$

The estimates for $(\mathscr{S}_kU_k, \mathscr{S}_k\phi_k)$ in (6.8) follow readily from $\mathscr{H}_n$ and Proposition 6.2.

- For the regularization $(\bar{U}_k, \bar{\phi}_k)$:
Q12 In the contact discontinuity case the regularization $(\bar{U}_k, \bar{\phi}_k)$ is the solution of a boundary value problem of the constraint equation (3.23) with the boundary value as $(\hat{U}_k, \hat{\phi}_k)$ ($(\bar{U}_k, \bar{\phi}_k)$ is denoted as $(U_{k+\frac{1}{2}}, \phi_{k+\frac{1}{2}})$) in [8, Proposition 7]. Hence the estimates for $(\bar{U}_k, \bar{\phi}_k)$ can be Q13 obtained by this fact and the estimate for $(\hat{U}_k, \hat{\phi}_k)$.

$$\|(\bar{U}_k,\bar{\phi}_k)\|_{H^s_\eta(\omega^T)} \le C\delta\theta_k^{\epsilon+(s-\alpha)_+}, \quad s\ge s_0\ (\epsilon=0 \text{ if } s\ne\alpha);$$
$$\|(U_k-\bar{U}_k,\phi_k-\bar{\phi}_k)\|_{H^s_\eta(\omega^T)} \le C\delta\theta_k^{s-\alpha}, \quad s_0\le s\le s_+. \tag{6.9}$$

In particular, in order that the iteration could proceed infinitely, we will require the linearized problem to be well-posed at each step. Since the linear problem is well-posed at (U_0, ϕ_0) and stable under a small perturbation of the coefficients, the linearized problem with uniform characteristic boundaries at $\Gamma^T_{0,1,2}$ remains well-posed for

$$|(\bar{U}_k-U_0,\bar{\phi}_k-\phi_0)|_{C^1} \ll 1.$$

Here, $|\cdot|_{C^1}$ denotes the uniform C^1 norm. This is true if $\alpha > s_0 > \frac{3}{2}+2$ and $\delta \ll 1$, by Sobolev imbedding.

6.2. $(\mathscr{H}_{n-1}) \Rightarrow (\mathscr{H}_n)$ – 2: estimate for error term (e_k, d_k) $(k \le n-1)$

Having established the feasibility of each iteration, we next estimate the error terms (e_k, d_k) $(k \le n-1)$. Noticing the form of energy estimate in Section 5 for the linearized problem, we need to estimate

$$\|e_k\|_{H^s_\eta(\omega^T)} \quad \text{and} \quad \|d_k\|_{H^s_\eta(\Gamma^T)}.$$

- First, let's look at the shock front $x=-1$. As shown in Section 4, in the neighborhood of shock front, we have $\hat{U}_k = \mathscr{S}_kU_k$, and the error $e_k = e_{k1}+e_{k2}$ with e_{k1} being the standard Nash–Moser error (quadratic linearized error plus the smoothing error) and e_{k2} being the error incurred by the introduction of new variable $\dot{V}_k$:

$$\begin{aligned} e_{k1} \equiv e'_{k1} + e''_{k1} &\equiv \{\ell_a(U_k, \phi_k)(\dot U_k, \dot\phi_k) - \ell_a(\hat U_k, \hat\phi_k)(\dot U_k, \dot\phi_k)\} \\ &\quad + \{\mathscr{L}_a(U_{k+1}, \phi_{k+1}) - \mathscr{L}_a(U_k, \phi_k) - \ell_a(U_k, \phi_k)(\dot U_k, \dot\phi_k)\}, \end{aligned} \tag{6.10}$$

$$e_{k2} = \frac{\dot\phi_k}{\phi^a_x + \hat\phi_{kx}} \partial_x\big(\mathscr{L}_a(\hat U_k, \hat\phi_k)\big). \tag{6.11}$$

The error $d_k^{(-1)} \equiv d_{k1}^{(-1)} + d_{k2}^{(-1)}$ has only the standard Nash–Moser part coming from regularization and linearization

$$d_{k1}^{(-1)} \equiv \mathbb{B}_a^{(-1)}\big(U_k^{(0)}, \phi_k^{(0)}\big)\big(\dot V_k^{(0)}, \dot\phi_k^{(0)}\big) - \mathbb{B}_a^{(-1)}\big(\hat U_k^{(0)}, \hat\phi_k^{(0)}\big)\big(\dot V_k^{(0)}, \dot\phi_k^{(0)}\big), \tag{6.12}$$

$$d_{k2}^{(-1)} \equiv \mathscr{B}_a\big(U_{k+1}^{(0)}, \phi_{k+1}^{(0)}\big) - \mathscr{B}_a\big(U_k^{(0)}, \phi_k^{(0)}\big) - \mathbb{B}_a^{(-1)}\big(U_k^{(0)}, \phi_k^{(0)}\big)\big(\dot U_k^{(0)}, \dot\phi_k^{(0)}\big), \tag{6.13}$$

The first term e'_{k1} in (6.10) and the term $d_{k1}^{(-1)}$ in (6.12) are the errors caused by smoothing the coefficients.
By the mean value theorem

$$e'_{k1} = \left[\int_0^1 \ell'_a\big(\hat U_k + \tau(U_k - \hat U_k), \hat\phi_k + \tau(\phi_k - \hat\phi_k)\big)d\tau\right](\dot U_k, \dot\phi_k)(U_k - \hat U_k, \phi_k - \bar\phi_k),$$

we have

$$\begin{aligned} &\|e'_{k1}\|_{H^s_\eta(\omega^T)} \\ &\quad \le C\big\|\big[1 + \partial(\hat U_k, U_k, \hat\phi_k, \phi_k)\big]\big[\partial(\dot U_k, \dot\phi_k)\big]\big[\partial(U_k - \hat U_k, \phi_k - \hat\phi_k)\big]\big\|_{H^s_\eta(\omega^T)}. \end{aligned} \tag{6.14}$$

From the inequality

$$\|uv\|_{H^s_\eta(\omega^T)} \le C\big(\|u\|_{H^s_\eta(\omega^T)}\|v\|_{H^{s_0}_\eta(\omega^T)} + \|u\|_{H^{s_0}_\eta(\omega^T)}\|v\|_{H^s_\eta(\omega^T)}\big) \tag{6.15}$$

and (6.3), (6.7), (6.9), and for $s_0 \le s \le s_+ - 2$ and $\alpha > s_0 + 6$, we obtain

$$\begin{aligned} \|e'_{k1}\|_{H^s_\eta(\omega^T)} &\le C\delta^2\big(\theta_k^{\epsilon + (s+1-\alpha)_+}\theta_k^{2s_0 - 2\alpha + 1} + \theta_k^{\epsilon + (s_0 + 1 - \alpha)_+}\theta_k^{s + s_0 + 1 - 2\alpha}\big) \\ &\le C\delta^2\theta_k^{s + s_0 + 2 - 2\alpha}. \end{aligned} \tag{6.16}$$

The error e''_{k1} has the similar form as e'_{k1}:

$$e''_{k1} = \left[\int_0^1 \ell'_a(U_k + \tau\dot U_k, \phi_k + \tau\dot\phi_k)d\tau\right](\dot U_k, \dot\phi_k)(\dot U_k, \dot\phi_k).$$

Therefore e''_{k1} can be estimated in the same way as e'_{k1} except that we need to replace the estimates of $(U_k - \hat U_k, \phi_k - \hat\phi_k)$ and $\partial(U_k - \hat U_k, \phi_k - \hat\phi_k)$ by the estimates of $(\dot U_k, \dot\phi_k)$ and $\partial(\dot U_k, \dot\phi_k)$.

Noticing that the estimates (6.3) and (6.9) have the same form, we find that the estimates for $(\dot{U}_k, \dot{\phi}_k)$ have an extra factor θ_k^{-1} than the estimate for $(U_k - \hat{U}_k, \phi_k - \hat{\phi}_k)$. Hence for $\theta_0 \gg 1$, e''_{k1} is negligible compared with e'_{k1}. Therefore, we have for $s_0 \le s \le s_+ - 2$, $\alpha > s_0 + 6$:

$$\|e_{k1}\|_{H^s_\eta(\omega^T)} \le C\delta^2\theta_k^{s+s_0+2-2\alpha}. \tag{6.17}$$

For the error e_{k2}, noticing (4.3), we have

$$\begin{aligned}\|e_{k2}\|_{H^s_\eta(\omega^T)} \le & \left\| \frac{\dot{\phi}_k}{\phi^a_x + \hat{\phi}_{kx}} \partial_x\big[\mathscr{L}_a(U_k, \phi_k)\big] \right\|_{H^s_\eta(\omega^T)} \\ & + \left\| \frac{\dot{\phi}_k}{\phi^a_x + \hat{\phi}_{kx}} \partial_x\big[\mathscr{L}_a(\hat{U}_k, \hat{\phi}_k) - \mathscr{L}_a(U_k, \phi_k)\big] \right\|_{H^s_\eta(\omega^T)}. \end{aligned} \tag{6.18}$$

For the first term in (6.18),

$$\left\| \frac{\dot{\phi}_k}{\phi^a_x + \hat{\phi}_{kx}} \partial_x \mathscr{L}_a(U_k, \phi_k) \right\|_{H^s_\eta(\omega^T)} \le C \big\| \dot{\phi}_k\big[\phi^a_x + \hat{\phi}_{kx}\big]\big[\partial_x \mathscr{L}_a(U_k, \phi_k)\big] \big\|_{H^s_\eta(\omega^T)},$$

we consider two cases: $s_0 \le s \le s_+ - 5$ or $s_+ - 5 < s \le s_+ - 2$.
If $s_0 \le s \le s_+ - 5$, then from (6.3), (6.4), (6.7) and (6.8), we have

$$\begin{aligned} & \left\| \frac{\dot{\phi}_k}{\phi^a_x + \hat{\phi}_{kx}} \partial_x \mathscr{L}_a(U_k, \phi_k) \right\|_{H^s_\eta(\omega^T)} \\ & \quad \le C\delta^2 \theta_k^{\epsilon + (s_0+1-\alpha)_+} \theta_k^{s+s_0-2\alpha} + \theta_k^{\epsilon+(s+1-\alpha)_+} \theta_k^{2s_0-2\alpha}. \end{aligned} \tag{6.19}$$

Noticing $\alpha > s_0 + 6$, hence for $s > \alpha - 1$, we have

$$\begin{aligned} \epsilon + (s+1-\alpha)_+ + (2s_0 - 2\alpha) &= \epsilon + s + 1 - \alpha + 2s_0 - 2\alpha \\ &= \epsilon + s + s_0 - 2\alpha + (s_0 + 1 - \alpha) < s + s_0 - 2\alpha. \end{aligned}$$

For $s \le \alpha - 1$, we have

$$\epsilon + (s+1-\alpha)_+ + (2s_0 - 2\alpha) \le 1 + 2s_0 - 2\alpha \le s + s_0 + 1 - 2\alpha.$$

Therefore for $s_0 \le s \le s_+ - 5$,

$$\left\| \frac{\dot{\phi}_k}{\phi^a_x + \hat{\phi}_{kx}} \partial_x \mathscr{L}_a(U_k, \phi_k) \right\|_{H^s_\eta(\omega^T)} \le C\delta^2\theta_k^{s+s_0+2-2\alpha}. \tag{6.20}$$

If $s_+ - 5 < s \le s_+ - 2$, we need only to consider the term

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$$\begin{aligned}
&\|\dot\phi_k\|_{H^{s_0}_\eta(\omega^T)}\big\|\phi^a_x+\hat\phi_{kx}\big\|_{H^{s_0}_\eta(\omega^T)}\big\|\partial_x\mathscr{L}_a(U_k,\phi_k)\big\|_{H^s_\eta(\omega^T)}\\
&\quad\le C\delta^2\theta_k^{s_0-\alpha-1}\theta_k^{\epsilon+(s_0+1-\alpha)_+}\theta_k^{\epsilon+(s+2-\alpha)_+}\\
&\quad= C\delta^2\theta_k^{s_0-\alpha}\theta_k^{\epsilon+(s+2-\alpha)_+}.
\end{aligned}$$

Since by (6.6), $\alpha\le s_+-6$, then

$$s-\alpha+2>s_+-5-\alpha+2=s_+-3-\alpha>0,$$

and the same estimate (6.20) can be obtained by estimating $\partial_x\mathscr{L}(U_k,\phi_k)$ directly from (6.7) without using (6.4).

For the second term in (6.18), from (6.3), (6.7) and (6.9) and applying the mean value formula, we obtain for $s_0\le s\le s_+-2$

$$\begin{aligned}
&\left\|\frac{\dot\phi_k}{\phi^a_x+\hat\phi_{kx}}\partial_x\big[\mathscr{L}_a(\hat U_k,\hat\phi_k)-\mathscr{L}_a(U_k,\phi_k)\big]\right\|_{H^s_\eta(\omega^T)}\\
&\quad\le C\big\|\dot\phi_k\big[1+\partial^2(\hat U_k,U_k,\hat\phi_k,\phi_k)\big]\big[\partial^2(U_k-\hat U_k,\phi_k-\hat\phi_k)\big]\big\|_{H^s_\eta(\omega^T)}\\
&\quad\le C\delta^2\theta_k^{s+s_0-2\alpha+2}.
\end{aligned}\tag{6.21}$$

The boundary errors d_{k1} and d_{k2} have exactly the same form as e'_{k1} and e''_{k1} and can be estimated similarly. The main difference here is that instead of the interior norms for (U_k,ϕ_k) $(\hat U_k,\hat\phi_k)$ and $(\dot U_k,\dot\phi_k)$, we should use their boundary norms on $x=-1$. But the latter have the same estimates as the interior estimates at the shock front, as pointed out at the beginning of the section. Also noticing that on the boundary $x=-1$, the operator involves at most the first order derivative, we obtain

$$\|d_k\|_{H^s_\eta(\Gamma^T)}\le C\delta\theta_k^{s+s_0+2-2\alpha}.\tag{6.22}$$

Combining (6.17)–(6.22), we obtain the estimates for the error terms e_k and d_k $(k\le n-1)$ near the shock front $x=-1$:

$$\|e_k\|_{H^s_\eta(\omega^T)}+\|d_k\|_{H^s_\eta(\Gamma^T)}\le C\delta^2\theta_k^{s+s_0+3-2\alpha}\tag{6.23}$$

with $s_0\le s\le s_+-2$.

- Near the rarefaction wave $1\le x\le 2$, such estimates are already obtained in [1] in the form of equivalent t-weighted norms.
  Indeed, the interior error estimates of e_{k1} and e_{k2} can be obtained similarly as in the case near shock front $x=-1$.
  In particular, the estimate of e''_{k1} here is denoted as e''_k in [1]. It was required in [1] certain appropriate choice of ϵ_0 to obtain the estimate of e''_{k1}. It is readily checked as shown above that ϵ_0 can be simply chosen as $\epsilon_0=2$.
  The error e_{k3} comes from replacing the operator L by $\bar{\bar{L}}$, the estimate is obtained in [1, Propositions 6.4.1] as follows

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$$\|e_{k3}\|_{H^s_\eta(\omega^T)} \le C\delta^2\theta_k^{s+s_0+3-2\alpha}. \tag{6.24}$$

Again following [1, Proposition 6.4.2], we have the estimates for the boundary error d_k. Hence near the rarefaction wave $1 \le x \le 2$, we also have the estimate

$$\|e_k\|_{H^s_\eta(\omega^T)} + \|d_k\|_{H^s_\eta(\Gamma^T)} \le C\delta^2\theta_k^{s+s_0+3-2\alpha} \tag{6.25}$$

with $s_0 \le s \le s_+ - 4$.

- Near the contact discontinuity $x = 0$, the estimates for the interior error terms e_{k1} and e_{k2} are obviously the same as near shock wave and rarefaction wave. For the error e_{k3}, we can use the result obtained in [8].
  In [8], the error e_{k3} is denoted as e_k''', which is introduced by replacing $(\hat{U}_k, \hat{\phi}_k)$ by $(\bar{U}_k, \bar{\phi}_k)$ (denoted as $(V_{k+1/2}, \Psi_{k+1/2})$ in [8])

$$\|e_{k3}\|_{H^s_\eta(\omega^T)} \le C\delta^2\big(\theta_k^{s+s_0+2-2\alpha} + \theta_k^{(s+1-\alpha)_+ + 2s_0+2-2\alpha}\big). \tag{6.26}$$

If $s + 1 - \alpha \ge 0$, then by (6.6)

$$\begin{aligned}(s+1-\alpha)_+ + 2s_0 + 2 - 2\alpha &= s + s_0 + 2 - 2\alpha + (s_0 - \alpha + 1)\\ &< s + s_0 + 2 - 2\alpha.\end{aligned}$$

If $s + 1 - \alpha < 0$, then by (6.6)

$$(s+1-\alpha)_+ + 2s_0 + 2 - 2\alpha = 2s_0 + 2 - 2\alpha \le s + s_0 + 2 - 2\alpha.$$

Therefore, we have

$$\|e_{k3}\|_{H^s_\eta(\omega^T)} \le C\delta^2\big(\theta_k^{s+s_0+3-2\alpha}\big). \tag{6.27}$$

For the estimates of the boundary error d_k, we have from [8] (denoted as $\tilde{e}_k'$ and $\tilde{e}_k''$ in [8, Lemma 8 and Lemma 9])

$$\|d_k\|_{H^s_\eta(\Gamma^T)} \le C\delta^2\theta_k^{m(s)-1},$$

where

$$m(s) \equiv \max\big\{(s+1-\alpha)_+ + 2s_0 - 2\alpha;\ s + s_0 + 2 - 2\alpha\big\} \le s + s_0 + 2 - 2\alpha. \tag{6.28}$$

Consequently we obtain

$$\|d_k\|_{H^s_\eta(\Gamma^T)} \le C\delta^2\theta_k^{s+s_0+1-2\alpha}. \tag{6.29}$$

Therefore we have near the contact discontinuity, for $s_0 \le s \le s_+ - 2$ (see Lemma 13, in [8])

$$\|e_k\|_{H^s_\eta(\omega^T)} + \|d_k\|_{H^s_\eta(\Gamma^T)} \le C\delta^2\theta_k^{s+s_0+3-2\alpha}. \tag{6.30}$$

Combining (6.23), (6.25) and (6.30), we have for $s_0 \le s \le s_+ - 4$,

$$\|e_k\|_{H^s_\eta(\omega^T)} + \|d_k\|_{H^s_\eta(\Gamma^T)} \le C\delta^2\theta_k^{s+s_0+3-2\alpha}. \tag{6.31}$$

In the domain $0 < x < 1$ the perturbation of the operator in (4.13) is the combination

$$\varphi(x)\bar{\bar{L}}(U_n, \phi_n) + \big(1 - \varphi(x)\big)\tilde{\ell}_a(\bar{U}_n, \bar{\phi}_n).$$

Obviously, the error term e_{k3} satisfies (6.24).

6.3. $(\mathscr{H}_{n-1}) \Rightarrow (\mathscr{H}_n)$ – 3: estimate for $(\dot{F}_n, \dot{G}_n)$

From (4.12), we have

$$\Delta_n \dot{F}_n = -(\mathscr{S}_n - \mathscr{S}_{n-1})\left(\mathscr{F}_T\mathscr{L}\big(U^a, \phi^a\big) + \sum_{k=0}^{n-2}\Delta_k e_k\right) - \mathscr{S}_n\Delta_{n-1}e_{n-1},$$

$$\Delta_n \dot{G}_n = -(\mathscr{S}_n - \mathscr{S}_{n-1})\left(\mathscr{F}_T\mathscr{B}\big(U^a, \phi^a\big) + \sum_{k=0}^{n-2}\Delta_k d_k\right) - \mathscr{S}_n\Delta_{n-1}d_{n-1}. \tag{6.32}$$

Q18 Notice that $\Delta_{n-1}/\Delta_n \sim 1$, then by the properties of $\mathscr{S}_k$ in Proposition ?? and from (6.31),

$$\left\|\frac{\Delta_{n-1}}{\Delta_n}\mathscr{S}_n e_{n-1}\right\|_{H^s_\eta(\omega^T)} \le C\delta^2\theta_n^{s+s_0+3-2\alpha}, \tag{6.33}$$

for $s_0 \le s \le s_+ - 4$. As for $s \ge s_+ - 4 \ge s_0$, we have

$$\|\mathscr{S}_n e_{n-1}\|_{H^s_\eta(\omega^T)} \le C\|\mathscr{S}_n e_{n-1}\|_{H^{s_+-4}}\theta_n^{s-(s_+-4)},$$
$$C\delta^2\theta_n^{s_+-4+s_0+3-2\alpha}\theta_n^{s-(s_+-4)} \le C\delta^2\theta_n^{s+s_0+3-2\alpha}.$$

On the other hand, by (6.31)

$$\left\|\sum_{k=0}^{n-2}\Delta_k e_k\right\|_{H^{s_+-4}_\eta(\omega^T)} \le C\delta^2\sum_{k=0}^{n-2}\Delta_k\theta_k^{(s_+-4)+s_0+3-2\alpha} \le C\delta^2\theta_n^{s_++s_0-2\alpha}, \tag{6.34}$$

therefore from the item (2) of Proposition 6.2 we have for all $s \ge s_0$,

$$\begin{aligned}\frac{1}{\Delta_n}\left\|(\mathscr{S}_n - \mathscr{S}_{n-1})\sum_{k=0}^{n-2}\Delta_k e_k\right\|_{H^s_\eta(\omega^T)} &\le C\theta_n^{s-(s_+-4)-1}\left\|\sum_{k=0}^{n-2}\Delta_k e_k\right\|_{s_+-4}\\ &\le C\delta^2\theta_n^{s-(s_+-4)-1}\theta_n^{s_++s_0-2\alpha}\\ &\le C\delta^2\theta_n^{s+s_0+3-2\alpha}.\end{aligned} \tag{6.35}$$

Finally, we have

$$\frac{1}{\Delta_n}\big\|(\mathscr{S}_n-\mathscr{S}_{n-1})\mathscr{F}_T\mathscr{L}\big(U^a,\phi^a\big)\big\|_{H^s_\eta(\omega^T)}\le C\theta_n^{s-\beta-1}\big\|\mathscr{L}\big(U^a,\phi^a\big)\big\|_{H^\beta_\eta(\omega^T)}. \tag{6.36}$$

In (6.36), let $\beta=2\alpha-s_0-4$. Also notice that (U^a,ϕ^a) is the C^∞ approximate solution we obtained in Theorem 2.2. Then for $-T\gg 1$ we can achieve

$$C\big\|\mathscr{L}\big(U^a,\phi^a\big)\big\|_{H^\beta_\eta(\omega^T)}\le\delta^2. \tag{6.37}$$

Therefore, we have

$$\frac{1}{\Delta_n}\big\|(\mathscr{S}_n-\mathscr{S}_{n-1})\mathscr{F}_T\mathscr{L}\big(U^a,\phi^a\big)\big\|_{H^s_\eta(\omega^T)}\le C\delta^2\theta_n^{s+s_0+3-2\alpha}. \tag{6.38}$$

Combining (6.33)–(6.38), we obtain that for all $s\ge s_0$,

$$\|\dot{F}_n\|_{H^s_\eta(\omega^T)}\le C\delta^2\theta_n^{s+s_0+3-2\alpha}. \tag{6.39}$$

Similarly, we can also obtain exactly the same estimate for the boundary term $\dot{G}_n$,

$$\|\dot{G}_n\|_{H^s_\eta(\Gamma^T)}\le C\delta^2\theta_n^{s+s_0+3-2\alpha}. \tag{6.40}$$

6.4. $(\mathscr{H}_{n-1})\Rightarrow(\mathscr{H}_n)$ – 4: estimate for $(\dot{U}_n,\dot{\phi}_n)$, $\mathscr{L}(U_k,\phi_k)$ and $\mathscr{B}(U_k,\phi_k)$

From the estimate (5.13) for the linearized problem in Theorem 5.2, and noticing the expression of $\dot{V}_n$, we have for all $s_0\le s\le s_+$

$$\begin{aligned}\big\|(\dot{U}_n,\dot{\phi}_n)\big\|_{H^s_\eta(\omega^T)}\le C_{s_+}\big[&\|\dot{F}_n\|_{H^{s+1}_\eta(\omega^T)}+\|\dot{G}_n\|_{H^{s+2}_\eta(\Gamma^T)}\\&+\big(\|\dot{F}_n\|_{H^4_\eta(\omega^T)}+\|\dot{G}_n\|_{H^5_\eta(\Gamma^T)}\big)\big(1+\big\|(\bar{U}_n,\bar{\phi}_n)\big\|_{H^{s+4}_\eta(\omega^T)}\big)\big].\end{aligned} \tag{6.41}$$

By (6.39), (6.40) and (6.8), we obtain

$$\big\|(\dot{U}_n,\dot{\phi}_n)\big\|_{H^s_\eta(\omega^T)}\le C_{s_+}\delta^2\big[\theta_n^{s+s_0+5-2\alpha}+\theta_n^{8+s_0-2\alpha}\theta_n^{\epsilon+(s+4-\alpha)_+}\big]. \tag{6.42}$$

By the choice of s_0 and α in (6.6), we have

$$s+s_0+5-2\alpha=s-\alpha-1+(s_0+6-\alpha)\le s-\alpha-1. \tag{6.43}$$

For $8+s_0-2\alpha+(s+4-\alpha)_+$, we have

$$\begin{cases}\text{if } s+4-\alpha\ge 0:\\ 8+s_0-2\alpha+\epsilon+(s+4-\alpha)_+=s-\alpha-1+(s_0+12-2\alpha)\le s-\alpha-1;\\ \text{if } s+4-\alpha<0:\\ 8+s_0-2\alpha+(s+4-\alpha)_+=s-\alpha-1+(9-\alpha)\le s-\alpha-1.\end{cases} \tag{6.44}$$

Combining (6.42)–(6.44) and choosing $\delta \ll 1$ such that $\delta C_{s_+} \le 1$, we obtain (6.3) for $k = n$:

$$\|(\dot{U}_n, \dot{\phi}_n)\|_{H^s_\eta(\omega^T)} \le \delta\theta_n^{s-\alpha-1}. \tag{6.45}$$

The estimate for $\|\dot{\phi}_n\|_{H^{s+1}_\eta(\Gamma^T)}$ can be obtained similarly.

Next consider (6.4) for $\mathscr{L}_a(U_k, \phi_k)$. From (4.11) and the choice of $\dot{F}_n$ in (4.12), we have

$$\begin{aligned}\mathscr{L}_a(U_n, \phi_n) &= \mathscr{F}_T\mathscr{L}\big(U^a, \phi^a\big) + \sum_{k=0}^{n-1}\dot{F}_k\Delta_k + \sum_{k=0}^{n-1}e_k\Delta_k \\ &= (1-\mathscr{S}_{n-1})\left[\mathscr{F}_T\mathscr{L}\big(U^a, \phi^a\big) + \sum_{k=0}^{n-2}e_k\Delta_k\right] + e_{n-1}\Delta_{n-1}.\end{aligned} \tag{6.46}$$

By Proposition 6.2, we obtain

$$\|(1-\mathscr{S}_n)\mathscr{F}_T\mathscr{L}\big(U^a, \phi^a\big)\|_{H^s_\eta(\omega^T)} \le C\theta_n^{s-\alpha}\|\mathscr{L}\big(U^a, \phi^a\big)\|_{H^\alpha_\eta(\omega^T)}, \tag{6.47}$$

and combining (6.31), we have for $s_0 \le s \le s_+ - 4$

$$\begin{aligned}\left\|(1-\mathscr{S}_n)\sum_{k=0}^{n-2}e_k\Delta_k\right\|_s &\le C\theta_n^{s-(s_+-4)}\delta^2\sum_{k=0}^{n-2}\theta_k^{(s_+-4)+s_0+3-2\alpha} \\ &\le C\delta^2\theta_n^{s+s_0+3-2\alpha} \le C\delta^2\theta_n^{s-\alpha}.\end{aligned} \tag{6.48}$$

In (6.47) and (6.48), choose $-T \gg 1$ such that $C\|\mathscr{L}(U^a, \phi^a)\|_{H^\alpha_\eta(\omega^T)} \le \frac{1}{2}\delta$, and choose $\delta \ll 1$ such that $C\delta \le \frac{1}{2}$, we obtain (6.4) for $k = n$.

The estimate for $\mathscr{B}_a(U_n, \phi_n)$ in (6.5) can be proven exactly in the same way.

This finishes the proof that $(\mathscr{H}_{n-1})$ implies $(\mathscr{H}_n)$.

6.5. *Proof of* $(\mathscr{H}_0)$

For $n = 0$

$$\mathscr{L}_a(U_0, \phi_0) = \mathscr{L}\big(U^a, \phi^a\big), \qquad \mathscr{B}_a(U_0, \phi_0) = \mathscr{B}\big(U^a, \phi^a\big).$$

If $\alpha + 4 \le s \le s_+ + 2$, we choose $\theta_0 \gg 1$ such that

$$\|\mathscr{L}\big(U^a, \phi^a\big)\|_{H^{s_++2}_\eta(\omega^T)} + \|\mathscr{B}\big(U^a, \phi^a\big)\|_{H^{s_++2}_\eta(\Gamma^T)} \le \frac{\delta}{2(1+C_{s_+})}\theta_0, \tag{6.49}$$

and therefore

$$\begin{aligned}&\|\mathscr{L}\big(U^a, \phi^a\big)\|_{H^s_\eta(\omega^T)} + \|\mathscr{B}\big(U^a, \phi^a\big)\|_{H^s_\eta(\Gamma^T)} \\ &\le \|\mathscr{L}\big(U^a, \phi^a\big)\|_{H^{s_++2}_\eta(\omega^T)} + \|\mathscr{B}\big(U^a, \phi^a\big)\|_{H^{s_++2}_\eta(\Gamma^T)} \\ &\le \frac{\delta}{(1+C_{s_+})}\theta_0^{s-\alpha-3} \le \delta\theta_0^{s-\alpha}.\end{aligned} \tag{6.50}$$

Please cite this article in press as: S. Chen, D. Li, Cauchy problem with general discontinuous initial data along a smooth curve for 2-d Euler system, J. Differential Equations (2014), http://dx.doi.org/10.1016/j.jde.2014.05.027

If $s_0 \le s < \alpha + 4$, then we choose $-T \gg 1$ such that

$$\left\|\mathscr{L}\left(U^a, \phi^a\right)\right\|_{H_\eta^{\alpha+4}(\omega^T)} + \left\|\mathscr{B}\left(U^a, \phi^a\right)\right\|_{H_\eta^{\alpha+4}(\Gamma^T)} \le \frac{\delta}{2(1 + C_{s_+})}\theta_0^{s_0-\alpha-3}, \tag{6.51}$$

and therefore

$$\begin{aligned} &\left\|\mathscr{L}\left(U^a, \phi^a\right)\right\|_{H_\eta^{s}(\omega^T)} + \left\|\mathscr{B}\left(U^a, \phi^a\right)\right\|_{H_\eta^{s}(\Gamma^T)} \\ &\quad \le \left\|\mathscr{L}\left(U^a, \phi^a\right)\right\|_{H_\eta^{\alpha+4}(\omega^T)} + \left\|\mathscr{B}\left(U^a, \phi^a\right)\right\|_{H_\eta^{\alpha+4}(\Gamma^T)} \\ &\quad \le \frac{\delta}{(1 + C_{s_+})}\theta_0^{s_0-\alpha-3} \le \delta\theta_0^{s-\alpha}. \end{aligned} \tag{6.52}$$

These are (6.4) and (6.5) for $n = 0$.

From the expressions for $(\dot{F}_0, \dot{G}_0)$,

$$\Delta_0\dot{F}_0 = -\mathscr{S}_0\mathscr{F}_T\mathscr{L}\left(U^a, \phi^a\right), \qquad \Delta_0\dot{G}_0 = -\mathscr{S}_0\mathscr{F}_T\mathscr{B}\left(U^a, \phi^a\right),$$

and the estimate (5.13) for solutions of linearized problem, we obtain similarly as (6.41)

$$\begin{aligned} \left\|(\dot{U}_0, \dot{\phi}_0)\right\|_{H_\eta^{s}(\omega^T)} &\le C_{s_+}\left[\|\dot{F}_0\|_{H_\eta^{s+1}(\omega^T)} + \|\dot{G}_0\|_{H_\eta^{s+2}(\Gamma^T)}\right] \\ &\le C_{s_+}\left[\left\|\mathscr{L}\left(U^a, \phi^a\right)\right\|_{H_\eta^{s+2}(\omega^T)} + \left\|\mathscr{B}\left(U^a, \phi^a\right)\right\|_{H_\eta^{s+2}(\Gamma^T)}\right]. \end{aligned} \tag{6.53}$$

From (6.49)–(6.52), we have for $s_0 \le s \le s_+ + 2$

$$\left\|\mathscr{L}\left(U^a, \phi^a\right)\right\|_{H_\eta^{s}(\omega^T)} + \left\|\mathscr{B}\left(U^a, \phi^a\right)\right\|_{H_\eta^{s}(\Gamma^T)} \le \frac{\delta}{(1 + C_{s_+})}\theta_0^{s-\alpha-3}. \tag{6.54}$$

Combining (6.53) and (6.54) gives (6.3) for $n = 0$.

This completes the proof of the convergence of the iteration scheme.

Uncited references

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