Global Solutions for Coupled Kuramoto-Sivashinsky-KdV System *†

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Abstract

We study the global smooth solution for coupled Kuramoto-Sivanshinsky-KdV system in two-dimensional space. The model is proposed to describe the surface waves on multi-layered liquid films. The global solution is obtained for general initial data, using an a priori estimate for the nonlinear system, and the smoothness of such solution is established in $t > 0$.

1 Introduction

In the study of surface waves on multi-layered liquid films, the following coupled Kuramoto-Sivashinsky-Korteweg-de Vries equations are introduced, see [9] and also [6] for the 2-dimensional version:

$$\begin{cases} u_t + uu_x + \Delta u_x = -\alpha u_{xx} - \gamma \Delta^2 u + \epsilon_1 v_x, \\ v_t + a_1 v_x = \Gamma \Delta v + \epsilon_2 u_x. \end{cases} \tag{1.1}$$

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Here, $\Delta = \partial_{xx} + \partial_{yy}$ as usual. The coefficients $\gamma, \Gamma, \alpha, a_1, \epsilon_1, \epsilon_2$ in (1.1) are all positive constants.

The mixed Kuramoto-Sivashinsky -Korteweg-de Vries (KS-KdV) equation finds various applications in plasma physics, hydrodynamics and other fields, see [1, 3, 4]. The system (1.1) is a mixed KS-KdV equation, linearly coupled with an additional linear dissipative equation for an extra real wave field $v(x, y, t)$.

The one-dimensional version of (1.1) was proposed in [9] based on the KS-Kdv equation for a real wave field $u(x, t)$, which is linearly coupled to an additional linear dissipative equation for an extra real wave field $v(x, t)$. The two-dimensional version is proposed in [6] in the study of cylindrical solitary pulses. One immediately notices that the two space variables (x, y) in (1.1) are not symmetric. This is because of the underlying non-symmetric physics, see [6].

The previous research by [3, 4, 5, 6, 9] on system (1.1) mainly studied the stability of harmonic wave mode, for example, the stability of steady-state soliton solutions is analyzed by perturbation theory and wave mode in [6]. In [2], the linear stability is analyzed in the context of energy estimate and local solution is established for Cauchy problem to (1.1). So far, no global existence of solution for such system is given. On the other hand, the existence of global solution for mixed system usually require a stronger dissipative term, or require the initial data to be sufficiently small [10].

In this paper, we take advantage of the special form of the nonlinear term to derive the global estimate for a weak solution in η-weighted norms (see Theorem 3.1) and obtain smooth global solutions without the usual smallness constraints on the initial data (see Theorem 4.3) . This method can also be used to establish the global existence of a class of more general systems.

Specifically, we study in this paper the global (in time) solution of (1.1) with the initial condition

$$u(x, y, 0) = u_0(x, y), \quad v(x, y, 0) = v_0(x, y). \tag{1.2}$$

First, introduce some notations. Let $(\cdot, \cdot)$ denote the L^2 inner product in R^2, and $H^k(R^2)$ be the usual Sobolev space defined by the norm

$$\|f\|_k^2 = \int_{R^2} \sum_{|j| \le k} |D^j f|^2 dxdy$$

with $H^0(R^2) = L^2(R^2)$ and $\|f\| = \|f\|_0$ and $H^\infty(R^2) = \bigcap_{k \ge 0} H^k(R^2)$.

Let Π_T^k be the Banach space for (u,v): $(u,v) \in \Pi_T^k$ if

$$\begin{array}{l} u \in C([0,T], H^{k+2}(R^2)) \cap L^2([0,T], H^{k+4}(R^2)) \\ v \in C([0,T], H^{k+1}(R^2)) \cap L^2([0,T], H^{k+2}(R^2)). \end{array} \tag{1.3}$$

Also, define P_T be the Banach space for (u,v):

$$\begin{array}{l} u \in C([0,T], H^{0}(R^2)) \cap L^2([0,T], H^{2}(R^2)) \\ v \in C([0,T], H^{0}(R^2)) \cap L^2([0,T], H^{1}(R^2)). \end{array} \tag{1.4}$$

The corresponding norm will be denoted as

$$|||(u,v)|||_T^2 \equiv \sup_{0 \le t \le T} (\|u(t)\|_0^2 + \|v(t)\|_0^2) + \int_0^T (\|u(s)\|_2^2 + \|v(s)\|_1^2) ds.$$

The main result of this paper is the following theorem.

Theorem 1.1 Consider the initial value problem (1.1) and (1.2).

- If intitial data $(u_0, v_0) \in H^{k+2}(R^2) \times H^{k+1}(R^2)$ $(k = 0, 1, 2, \dots)$, then the initial value problem (1.1)(1.2) has a unique global solution $(u,v) \in \Pi_T^k$ for all $T > 0$.
- For any k, in particular for $k = 0$, the solution (u,v) is C^∞ in $t > 0$.

The paper is arranged as follows. In section 2, we derive the energy estimate for linearized system and establish the local existence of solution for (1.1)(1.2). In section 3, we establish an a priori η-weighted estimate for (1.1) and (1.2). Section 4, we prove the global existence of a weak solution. In section 5, we show the global smooth solution with improved initial date. Section 6 shows that the weak solution is indeed C^∞ in $t > 0$.

2 A Priori Linear Estimate and Local Existence

For the problem (1.1)(1.2), we will derive the a priori estimate for linearized problem and then establish the local (in time) existence of the solution.

Consider the following linearized problem for (1.1)(1.2):

$$\begin{cases} u_t + w u_x + \Delta u_x = -\alpha u_{xx} - \gamma \Delta^2 u + \epsilon_1 v_x + f, \\ v_t + a_1 v_x = \Gamma \Delta v + \epsilon_2 u_x + g, \\ u(x,y,0) = u_0(x,y), \quad v(x,y,0) = v_0(x,y). \end{cases} \tag{2.1}$$

We have the following:

Theorem 2.1 Let $k \geq 0$ be an integer, and assume

- $u_0 \in H^{k+2}(R^2)$, $v_0(x,y) \in H^{k+1}(R^2)$;
- $f \in L^2([0,T], H^k(R^2))$, $g \in L^2([0,T], H^k(R^2))$;
- $w \in C([0,T], H^k(R^2)) \cap L^2([0,T], H^{k+2}(R^2))$.

Then problem (2.1) admits a unique solution (u,v) in the space Π_T^k satisfying the energy estimate

$$\begin{aligned}&\sup_{0\leq t\leq T}(\|u(t)\|_{k+2}^2 + \|v(t)\|_{k+1}^2)\\ &\quad + \int_0^T(\|u(s)\|_{k+4}^2 + \|v(s)\|_{k+2}^2)ds\\ &\leq C_k(\|u_0\|_{k+2}^2 + \|v_0\|_{k+1}^2 + \int_0^T(\|f(s)\|_k^2 + \|g(s)\|_k^2)ds).\end{aligned} \tag{2.2}$$

Here C_k is a constant depending on T. It depends on w only in its larger norm in the spaces $C([0,T], H^k(R^2))$ and $L^2([0,T], H^{k+2}(R^2))$.

Proof: To establish estimate (2.2), we need only to consider smooth functions (u,v).

For $k=0$, we take $L^2(R^2)$ inner product of the equations in (2.1) with (u,v) and integrate by parts in (x,y)-direction. A straightforward computation yields

$$\begin{aligned}&\partial_t(\|u(t)\|^2 + \|v(t)\|^2) + \|u(t)\|_2^2 + \|v(t)\|_1^2\\ &\leq C(\|u(t)\|^2 + \|v(t)\|^2 + \|f(t)\|^2 + \|g(t)\|^2),\end{aligned} \quad \forall t \in [0,T] \tag{2.3}$$

Applying Gronwall inequality to (2.3), we obtain further

$$\begin{aligned}&\sup_{0\leq t\leq T}(\|u(t)\|^2 + \|v(t)\|^2) + \int_0^T(\|u(s)\|_2^2 + \|v(s)\|_1^2)ds\\ &\leq C(\|u_0\|^2 + \|v_0\|^2 + \int_0^T(\|f(s)\|^2 + \|g(s)\|^2)ds),\end{aligned} \tag{2.4}$$

where C always denotes a constant depending only on T and coefficients of (2.1).

Then we take $L^2(R^2)$ inner product of the equations in (2.1) with $(\Delta^2 u, \Delta v)$, and integrate by parts in (x,y)-direction. Similarly as above and also taking into account the estimates (2.3) and (2.4), we obtain

$$\begin{aligned}&\partial_t(\|u(t)\|_1^2 + \|\Delta u\|^2 + \|v(t)\|_1^2) + \|u(t)\|_4^2 + \|v(t)\|_2^2\\ &\leq C(\|u(t)\|_2^2 + \|v(t)\|_1^2 + \|f(t)\|^2 + \|g(t)\|^2),\end{aligned} \quad \forall t \in [0,T] \tag{2.5}$$

$$\begin{aligned}&\sup_{0\le t\le T}(\|u(t)\|_2^2+\|v(t)\|_1^2)+\textstyle\int_0^T(\|u(s)\|_4^2+\|v(s)\|_2^2)ds\\&\le C(\|u_0\|_2^2+\|v_0\|_1^2+\textstyle\int_0^T(\|f(s)\|^2+\|g(s)\|^2)ds).\end{aligned}\tag{2.6}$$

(2.6) is the $k=0$ case for (2.2). For the case $k>0$, the estimate (2.2) can be obtained by applying $\nabla^k_{x,y}$ to (2.1) and then apply the result for $k=0$ to the expanded system.

The existence of the solution can be obtained by continuation argument. Consider a family $(0\le\lambda\le 1)$ of problems:

$$\begin{cases}u_t+\lambda\,[wu_x+\Delta u_x+\alpha u_{xx}+\gamma\Delta^2u-\epsilon_1v_x]+(1-\lambda)\Delta^2u=f,\\ v_t+\lambda\,[a_1v_x-\Gamma\Delta v-\epsilon_2u_x]-(1-\lambda)\Delta v=g,\\ u(x,y,0)=u_0(x,y),\ \ v(x,y,0)=v_0(x,y).\end{cases}\tag{2.7}$$

Problem (2.1) is the case $\lambda=1$ in (2.7), while the case $\lambda=0$ in (2.7) is the well-known parabolic problem. It remains to show that the set $\mathscr{B}\subset[0,1]$ of all λ for which (2.7) has solution is both open and closed. This can be achieved by the fact that the solution of (2.7) satisfies (2.2) uniformly for $0\le\lambda\le 1$.

- $\mathscr{B}$ is closed in $[0,1]$.

  Let $\lambda_j\in\mathscr{B}$ and $\lambda_j\to\lambda_0$. Let (u_j,v_j) be the solution of the following initial value problem

$$\begin{cases}{u_j}_t+\lambda_j\mathscr{L}_1(u_j,v_j)+(1-\lambda_j)\Delta^2u_j=f,\\ {v_j}_t+\lambda_j\mathscr{L}_2(u_j,v_j)-(1-\lambda_j)\Delta v_j=g,\\ u_j(x,y,0)=u_0(x,y),\ \ v_j(x,y,0)=v_0(x,y).\end{cases}\tag{2.8}$$

  By (2.2), (u_j,v_j) is uniformly bounded in Π^k_T. Let $(\bar u_j,\bar v_j)=(u_j-u_{j-1},v_j-v_{j-1})$ $(j=2,3,\cdots)$. Apply (2.2) to $(\bar u_j,\bar v_j)$, we have

$$\begin{aligned}&\sup_{0\le t\le T}(\|\bar u_j(t)\|^2_{k+2}+\|\bar v_j(t)\|^2_{k+1})\\&\qquad+\textstyle\int_0^T(\|\bar u_j(s)\|^2_{k+4}+\|\bar v_j(s)\|^2_{k+2})ds\\&\le C_k|(\lambda_j-\lambda_{j-1})|^2\textstyle\int_0^T(\|u_{j-1}(s)\|^2_{k+4}+\|v_{j-1}(s)\|^2_{k+2})ds\\&\le C_k|(\lambda_j-\lambda_{j-1})|^2K.\end{aligned}\tag{2.9}$$

  Since $\lambda_j\to\lambda_0$, it follows that (u_j,v_j) is a Cauchy sequence in Π^k_T and its limit (u,v) is the solution of (3.3) for λ_0. Hence $\mathscr{B}$ is closed in [0,1].

- $\mathscr{B}$ is open in $[0,1]$.

  Let $\lambda_0 \in \mathscr{B}$, and $\lambda \in [0,1]$ with $|\lambda - \lambda_0| \le \epsilon$.

  Let (u_1, v_1) be the solution of the following problem:

$$\begin{cases} u_{1t} + \lambda_0 \mathscr{L}_1(u_1, v_1) + (1-\lambda_0)\Delta^2 u_1 = f, \\ v_{1t} + \lambda_0 \mathscr{L}_2(u_1, v_1) - (1-\lambda_0)\Delta v_1 = g, \\ u_1(x,y,0) = u_0(x,y), \;\; v_1(x,y,0) = v_0(x,y). \end{cases} \tag{2.10}$$

  Let (u_j, v_j) $(j = 2, 3, \cdots)$ be the solution of the following problem

$$\begin{cases} u_{j_t} + \lambda_0 \mathscr{L}_1(u_j, v_j) + (1-\lambda_0)\Delta^2 u_j \\ \qquad = f + (\lambda_0 - \lambda)(\mathscr{L}_1(u_{j-1}, v_{j-1}) - \Delta^2 u_{j-1}), \\ v_{j_t} + \lambda \mathscr{L}_2(u_j, v_j) - (1-\lambda_0)\Delta v_j \\ \qquad = g + (\lambda_0 - \lambda)(\mathscr{L}_2(u_{j-1}, v_{j-1}) - \Delta v_{j-1}), \\ u_j(x,y,0) = u_0(x,y), \;\; v_j(x,y,0) = v_0(x,y). \end{cases} \tag{2.11}$$

  It is easy to show that for $\epsilon \ll 1$, (u_j, v_j) is a Cauchy sequence with limit (u, v) being the solution of (2.7) for λ. Hence $\mathscr{B}$ is open.

For the local solution of the nonlinear system (1.1)(1.2), we have the following

Theorem 2.2 $\forall$ integer $k \ge 0$ and $(u_0, v_0) \in H^{k+2}(R^2) \times H^{k+1}(R^2)$: there is a $T > 0$ such that (1.1)(1.2) has a unique solution $(u, v) \in C([0,T]; H^{k+2}(R^2) \times H^{k+1}(R^2))$ satisfying

$$\begin{aligned} &\sup_{0 \le t \le T} (\|u(t)\|_{k+2}^2 + \|v(t)\|_{k+1}^2) \\ &\qquad + \int_0^T (\|u(s)\|_{k+4}^2 + \|v(s)\|_{k+2}^2) ds \\ &\le C_k (\|u_0\|_{k+2}^2 + \|v_0\|_{k+1}^2). \end{aligned} \tag{2.12}$$

In addition, the existence time span $[0,T]$ depends upon (u_0, v_0) only in its norm $(\|u_0\|_{k+2}^2 + \|v_0\|_{k+1}^2)$.

Proof: The theorem is proved by linear iteration. First we construct an approximate solution (u_a, v_a) by solving

$$\begin{cases} u_t + \Delta u_x + \alpha u_{xx} + \gamma \Delta^2 u - \epsilon_1 v_x = 0, \\ v_t + a_1 v_x - \Gamma \Delta v - \epsilon_2 u_x = 0, \\ u(x,y,0) = u_0(x,y), \;\; v(x,y,0) = v_0(x,y). \end{cases} \tag{2.13}$$

We look for the solution of (1.1)(1.2) in the form of $(u, v) = (u_a + \dot{u}, v_a + \dot{v})$. The solution $(\dot{u}, \dot{v})$ is obtained by the following linear iteration ($j = 1, 2, \dots$):

$$\begin{cases} \dot{u}_{jt} + (u_a + \dot{u}_{j-1})\dot{u}_{jx} + u_{ax}\dot{u}_j + \Delta\dot{u}_{jx} + \alpha\dot{u}_{jxx} + \gamma\Delta^2\dot{u}_j - \epsilon_1\dot{v}_{jx} = -u_a u_{ax}, \\ \dot{v}_{jt} + a_1\dot{v}_{jx} - \Gamma\Delta\dot{v}_j - \epsilon_2\dot{u}_{jx} = 0, \\ \dot{u}_j(x, y, 0) = 0, \quad \dot{v}_j(x, y, 0) = 0. \end{cases} \tag{2.14}$$

By Theorem 2.1, the approximate solution $(u_a, v_a) \in \Pi_T^k$ and satisfies (2.2) with $(f, g) = 0$. Since (u_a, v_a) is thus fixed, and $u_a u_{ax} \in L^2([0, T], H^k(R^2))$, it follows from (2.2) that the $(\dot{u_j}, \dot{v_j})$ is uniformly bounded in the space Π_T^k. $(\dot{u_j}, \dot{v_j})$ is a Cauchy sequence from the choice of sufficiently small T_0. Such choice of T_0 depends only upon the Π_T^k norms of (u_a, v_a) which in turn depends only on the corresponding norm of (u_0, v_0). This concludes the proof of Theorem 2.2.

To prepare for the study of global weak solution in P_T (see (1.4)), we need to study the linear problem under a weaker assumption on the term f.

Consider the following linear initial value problem:

$$\begin{cases} u_t + \Delta u_x + \alpha u_{xx} + \gamma\Delta^2 u - \epsilon_1 v_x = f_1, \\ v_t + a_1 v_x - \Gamma\Delta v - \epsilon_2 u_x = 0, \\ u(x, y, 0) = u_0(x, y), \quad v(x, y, 0) = v_0(x, y). \end{cases} \tag{2.15}$$

Theorem 2.3 If $(u_0, v_0) \in L^2(R^2) \times L^2(R^2)$ and $f_1 \in L^1([0, T]; L^2(R^2))$, then (2.15) has a unique solution $(u, v) \in P_T$.

Proof: Taking $L^2(R^2)$ inner product of the two equations in (2.15) with (u, v), we have

$$\begin{aligned} &(u, u_t) = \tfrac{1}{2}\partial_t(\|u(t)\|_0^2), \quad (v, v_t) = \tfrac{1}{2}\partial_t(\|v(t)\|_0^2); \\ &(u, \gamma\Delta^2 u) = \gamma\|\Delta u\|_0^2; \quad (v, -\Gamma\Delta v) = \gamma\|\nabla v\|_0^2; \\ &(u, \Delta u_x + \alpha u_{xx}) \le \|u\|_1\|u\|_2 \le \delta\|u\|_2^2 + C_\delta\|u\|_1^2; \\ &(u, \epsilon_1 v_x) \le \delta\|v\|_1^2 + C_\delta\|u\|_0^2; \\ &(v, a_1 v_x) \le \delta\|v\|_1^2 + C_\delta\|v\|_0^2; \\ &(v, \epsilon_2 u_x) \le \|v\|_0^2 + C\|u\|_1^2. \end{aligned}$$

Noticing $\|u\|_2 \sim \|u\|_0 + \|\Delta u\|_0$, $\|v\|_1 \sim \|v\|_0 + \|\nabla v\|_0$, and $\|u\|_1^2 \le \delta\|u\|_2^2 + C_\delta\|u\|_0^2$, we obtain by taking $\delta \ll 1$,

$$\begin{aligned} &\partial_t(\|u(t)\|_0^2 + \|v(t)\|_0^2) + \|u\|_2^2 + \|v\|_1^2 \\ &\qquad \le C\left(\|u(t)\|_0^2 + \|v(t)\|_0^2 + \|f_1\|_0\|u(t)\|_0\right) \end{aligned} \tag{2.16}$$

As in Gronwall's inequality, multiply (2.16) by e^{-Ct}:

$$\partial_t[e^{-Ct}(\|u(t)\|_0^2+\|v(t)\|_0^2)]+e^{-Ct}(\|u\|_2^2+\|v\|_1^2)\le Ce^{-Ct}\left(\|f_1\|_0\|u(t)\|_0\right),$$

Integrating the above in t yields

$$\begin{aligned}&(\|u(t)\|_0^2+\|v(t)\|_0^2)]+\int_0^t\left(\|u\|_2^2+\|v\|_1^2\right)ds\\&\le C\left(\|u_0\|_0^2+\|v_0\|_0^2+\int_0^t\|f_1\|_0\|u(s)\|_0ds\right).\end{aligned}\tag{2.17}$$

Since $\|f_1(t)\|_0$ is only in $L^1(0,T)$, we have

$$\begin{aligned}&\int_0^t\|f_1\|_0\|u(s)\|_0ds\le\left(\sup_{0\le s\le t}\|u(s)\|_0\right)\int_0^t\|f_1\|_0ds\\&\le\delta\left(\sup_{0\le s\le t}\|u(s)\|_0\right)^2+C_\delta\left(\int_0^t\|f_1\|_0ds\right)^2,\end{aligned}$$

for $\delta\ll 1$, we obtain from (2.17)

$$|\!|\!|(u,v)|\!|\!|_T^2\le C_{T_0}\left(\|u_0\|_0^2+\|v_0\|_0^2+(\int_0^T\|f_1\|_0ds)^2\right).\tag{2.18}$$

(2.18) is the a priori estimate for solution (u,v) of (2.15). The constant C_{T_0} in (2.18) depends on T_0, but is uniform for all $T\le T_0$.

The existence of the solutions can be obtained from Theorem 2.1 and (2.18) as follows. First construct a sequence of $(u_{0k},v_{0k})\in H^{k+2}(R^2)\times H^{k+1}(R^2)$ and $f_{1k}\in L^2([0,T],H^k(R^2))$ such that $(u_{0k},v_{0k})\to(u_0,v_0)$ in $H^{k+2}(R^2)\times H^{k+1}(R^2)$ and $f_{1k}\to f_1$ in $L^1([0,T];L^2(R^2))$. With data (u_{0k},v_{0k}) and f_{1k}, (2.15) has a unique solution $(u_k,v_k)\in\Pi_T^k$ by Theorem 2.1. (2.18) implies that (u_k,v_k) is a Cauchy sequence in Π_T^k, and its limit (u,v) is the required solution. This finishes the proof of Theorem 2.3.

3 Solution in P_T

First we derive a global estimate for the solutions of (1.1)(1.2) in an η-weighted norm which plays a crucial role in the proof of the global solution.

Theorem 3.1 There is an $\eta_0>0$ such that for any $T>0$, the solution $(u,v)\in C^1([0,T],S(R^2))$ of (1.1)(1.2) satisfies the estimate

$$\begin{aligned}&\sup_{0\le t\le T}(\|e^{-\eta t}u(t)\|_0^2+\|e^{-\eta t}v(t)\|_0^2)\\&\qquad+\int_0^T(\|e^{-\eta s}u(s)\|_2^2+\|e^{-\eta s}v(s)\|_1^2)ds\\&\le C_\eta(\|u_0\|_0^2+\|v_0\|_0^2),\end{aligned}\tag{3.1}$$

$\forall \eta \geq \eta_0$. the constant C_η in (3.1) depends only on η_0 and is independent of T.

Proof: Let $(\tilde{u}, \tilde{v}) = (e^{-\eta t}u, e^{-\eta t}v)$, then $(\tilde{u}, \tilde{v})$ satisfies the following

$$\begin{cases} \tilde{u}_t + e^{\eta t}\tilde{u}\tilde{u}_x + \Delta \tilde{u}_x + \eta \tilde{u} = -\alpha \tilde{u}_{xx} - \gamma \Delta^2 \tilde{u} + \epsilon_1 \tilde{v}_x, \\ \tilde{v}_t + a_1 \tilde{v}_x + \eta \tilde{v} = \Gamma \Delta \tilde{v} + \epsilon_2 \tilde{u}_x, \\ \tilde{u}(x, y, 0) = u_0(x, y), \quad \tilde{v}(x, y, 0) = v_0(x, y). \end{cases} \tag{3.2}$$

Take inner product of the two equations in (3.2) with $(\tilde{u}, \tilde{v})$. Since the only nonlinear term

$$(e^{\eta t}\tilde{u}\tilde{u}_x, \tilde{u}) = 0, \tag{3.3}$$

and we have on the left-hand-side of the estimate the terms $\eta\|\tilde{u}\|^2 + \eta\|\tilde{v}\|^2$, hence we can estimate all the remaining terms as in the case of linear problem by choosing $\eta_0 \gg 1$.

Remark 3.1 For functions $(u, v) \in P_T$, since the dual of uu_x with u is well-defined, we conclude that the estimate (3.1) is also valid for solution $(u, v) \in P_T$.

The following theorem establishes the uniqueness of solution $(u, v) \in P_T$ for (1.1)(1.2).

Theorem 3.2 The solution $(u, v) \in P_T$ of (1.1)(1.2) is unique.

Proof: Let (u_1, v_1) and (u_2, v_2) be two solutions in P_T for (1.1)(1.2). Since

$$u_1 u_{1x} - u_2 u_{2x} = u_{1x}\hat{u} + u_2 \hat{u}_x, \tag{3.4}$$

hence $(\hat{u}, \hat{v}) \equiv (u_1 - u_2, v_1 - v_2)$ satisfies the following linear problem

$$\begin{cases} \hat{u}_t + u_{1x}\hat{u} + u_2\hat{u}_x + \Delta \hat{u}_x = -\alpha \hat{u}_{xx} - \gamma \Delta^2 \hat{u} + \epsilon_1 \hat{v}_x, \\ \hat{v}_t + a_1 \hat{v}_x = \Gamma \Delta \hat{v} + \epsilon_2 \hat{u}_x, \\ \hat{u}(x, y, 0) = 0, \quad \hat{v}(x, y, 0) = 0. \end{cases} \tag{3.5}$$

Because $H^2(R^2)$ is a Banach algebra,

$$(u_{1x}\hat{u}, \hat{u}) \leq \|u_{1x}\| \|\hat{u}^2\|_0 \leq C\|u_1\|_1 \|\hat{u}\|_0^2. \tag{3.6}$$

Also by Sobolev imbedding theorem, we have

$$(u_2\hat{u}_x, \hat{u}) \leq \sup_{x,y} |u_2| \|\hat{u}\|_1^2 \leq C\|u_2\|_2 \|\hat{u}\|_1^2 \leq \epsilon\|\hat{u}\|_2^2 + C(\epsilon)\|\hat{u}\|_0^2. \tag{3.7}$$

Since u_1, u_2 are fixed functions in P_T, we can choose $\epsilon \ll 1$ such that the solution for linear problem (3.5) satisfies the estimate (2.10) in Theorem 2.3. Since $f_1 = 0$ and $(\hat{u}(0), \hat{v}(0)) = 0$, we have $(\hat{u}, \hat{v}) = 0$. This concludes the proof of Theorem 3.2.

The following local existence theorem is similar to Theorem 2.2, except that the regularity condition on the initial data (u_0, v_0) is weaker and the solution (u, v) is also weaker in regularity.

Theorem 3.3 Let $(u_0, v_0) \in L^2(R^2) \times L^2(R^2)$. Then there is a $T_0 > 0$ such that (1.1)(1.2) has a unique solution $(u, v) \in P_{T_0}$ satisfying

$$\begin{aligned} &\sup_{0 \le t \le T} (\|u(t)\|_0^2 + \|v(t)\|_0^2) + \int_0^T (\|u(s)\|_2^2 + \|v(s)\|_1^2) ds \\ &\qquad \le C_T(\|u_0\|_0^2 + \|v_0\|_0^2). \end{aligned} \tag{3.8}$$

In particular, the existence time span $[0, T_0]$ depends upon (u_0, v_0) only in its norm $(\|u_0\|_0^2 + \|v_0\|_0^2)$.

Proof: As in the proof of Theorem 2.2, we first construct an approximate solution (u_a, v_a) by solving (2.5). And (u_a, v_a) satisfies the estimate

$$\begin{aligned} &\sup_{0 \le t \le T} (\|u_a(t)\|_0^2 + \|v_a(t)\|_0^2) + \int_0^T (\|u_a(s)\|_2^2 + \|v_a(s)\|_1^2) ds \\ &\qquad \le C_T(\|u_0\|_0^2 + \|v_0\|_0^2). \end{aligned} \tag{3.9}$$

In particular, we notice that $u_a u_{ax} \in L^1((0,T); L^2(R^2))$ with

$$\int_0^T \|u_a u_{ax}\|_0 ds \le \int_0^T \|u_a(s)\|_2^2 ds \le C_T(\|u_0\|_0^2 + \|v_0\|_0^2)^2. \tag{3.10}$$

We again look for the solution of (1.1)(1.2) in the form of $(u, v) = (\dot{u}, \dot{v})$. Obviously, (u, v) is a solution of (1.1)(1.2) if and only if $(u_a + \dot{u}, v_a + \dot{v})$ satisfies

$$\begin{cases} \dot{u}_t + (u_a + \dot{u})\dot{u}_x + u_{ax}\dot{u} + \Delta \dot{u}_x + \alpha \dot{u}_{xx} + \gamma \Delta^2 \dot{u} - \epsilon_1 \dot{v}_x = -u_a u_{ax}, \\ \dot{v}_t + a_1 \dot{v}_x - \Gamma \Delta \dot{v} - \epsilon_2 \dot{u}_x = 0, \\ \dot{u}(x, y, 0) = 0, \quad \dot{v}(x, y, 0) = 0, \end{cases} \tag{3.11}$$

The solution $(\dot{u},\dot{v})$ of (3.11) is obtained by the linear iteration ($j=1,2,\dots$):

$$\begin{cases} \dot{u}_{jt}+(u_a+\dot{u}_{j-1})\dot{u}_{jx}+u_{ax}\dot{u}_j+\Delta\dot{u}_{jx}+\alpha\dot{u}_{jxx}+\gamma\Delta^2\dot{u}_j-\epsilon_1\dot{v}_{jx} \\ \qquad =-u_a u_{ax}, \\ \dot{v}_{jt}+a_1\dot{v}_{jx}-\Gamma\Delta\dot{v}_j-\epsilon_2\dot{u}_{jx}=0, \\ \dot{u}_j(x,y,0)=0,\ \ \dot{v}_j(x,y,0)=0, \end{cases} \tag{3.12}$$

with $(\dot{u}_0,\dot{v}_0)=(0,0)$.

We are going to show that

1. For any $\kappa>0$, we can choose $T_0\ll 1$ such that for all j:

$$|||(\dot{u}_j,\dot{v}_j)|||_{T_0}^2\le\kappa. \tag{3.13}$$

2. There is a $T_1\le T_0$ such that for all $j\ge 1$:

$$|||(\dot{u}_j,\dot{v}_j)|||_{T_1}\le\frac{1}{2}|||(\dot{u}_{j-1},\dot{v}_{j-1})|||_{T_1}. \tag{3.14}$$

Obviously, (3.13) and (3.14) implies the existence of a local solution $(u,v)\in P_{T_1}$, as claimed in Theorem 3.3.

- Prove **(3.13)**:

  Assume $|||(\dot{u}_{j-1},\dot{v}_{j-1})|||_T\le\kappa$ and consider the energy estimate for $(\dot{u}_j,\dot{v}_j)$. We have

$$\begin{cases} (u_a\dot{u}_{jx},\dot{u}_j)\le\|\dot{u}_j\|_0\|u_a\|_2\|\dot{u}_j\|_2\le C_\delta\|\dot{u}_j(t)\|_0^2+\delta\|\dot{u}_j\|_2^2; \\ (u_{ax}\dot{u}_j,\dot{u}_j)\le\|\dot{u}_j\|_0\|u_a\|_2\|\dot{u}_j\|_2\le C_\delta\|\dot{u}_j(t)\|_0^2+\delta\|\dot{u}_j\|_2^2; \\ (\dot{u}_{j-1}\dot{u}_{jx},\dot{u}_j)\le\|\dot{u}_j\|_0\|\dot{u}_{j-1}\|_2\|\dot{u}_j\|_2\le C_\delta\|\dot{u}_j(t)\|_0^2+\delta\|\dot{u}_j\|_2^2. \end{cases} \tag{3.15}$$

  In particular, the constant C_δ in (3.15) depends only on κ and $|||(u_a,v_a)|||_T$, and is independent of specific $\dot{u}_{j-1}$.

  Choosing δ in (3.14) sufficiently small, we can obtain energy estimate for $(\dot{u}_j,\dot{v}_j)$ similar to (2.10):

$$|||(\dot{u}_j,\dot{v}_j)|||_T^2\le C_T\left(\int_0^T\|u_a u_{ax}\|_0 ds\right)^2. \tag{3.16}$$

Since the constant C_T in (3.16) is uniform for $T < 1$, we can choose $T_0 \ll 1$ so that

$$C_T \left(\int_0^{T_0} \|u_a u_{ax}\|_0 ds \right)^2 \leq \kappa. \tag{3.17}$$

Here, T_0 depends only upon $L^1(0,T)$ norm of $\|u_a u_{ax}\|_0$ which in turn depends on $\|u_0\|_0 + \|v_0\|_0$ by (3.10). This concludes the proof of (3.13).

- Prove **(3.14)**

  For T_0 chosen above and $\forall T \leq T_0$, let $(\tilde{u}_j, \tilde{v}_j) = (\dot{u}_j - \dot{u}_{j-1}, \dot{v}_j - \dot{v}_{j-1})$. Then from (3.12), $(\tilde{u}_j, \tilde{v}_j)$ satisfies

$$\begin{cases} \tilde{u}_{jt} + \Delta \tilde{u}_{jx} + \alpha \tilde{u}_{jxx} + \gamma \Delta^2 \tilde{u}_j - \epsilon_1 \tilde{v}_{jx} \\ \qquad + u_a \tilde{u}_{jx} + u_{ax} \tilde{u}_j = \dot{u}_{j-2} \dot{u}_{(j-1)x} - \dot{u}_{j-1} \dot{u}_{jx}, \\ \tilde{v}_{jt} + a_1 \tilde{v}_{jx} - \Gamma \Delta \tilde{v}_j - \epsilon_2 \tilde{u}_{jx} = 0, \\ \tilde{u}_j(x,y,0) = 0, \ \ \tilde{v}_j(x,y,0) = 0. \end{cases} \tag{3.18}$$

  Because

$$\dot{u}_{j-2} \dot{u}_{(j-1)x} - \dot{u}_{j-1} \dot{u}_{jx} = -\dot{u}_{j-1} \tilde{u}_{jx} - \dot{u}_{(j-1)x} \tilde{u}_{j-1},$$

  similar to (3.15), we have

$$\begin{cases} (\dot{u}_{j-1} \tilde{u}_{jx}, \tilde{u}_j) \leq \kappa \|\tilde{u}_j\|_2^2; \\ (\dot{u}_{(j-1)x} \tilde{u}_{j-1}, \tilde{u}_j) \leq \kappa (\|\tilde{u}_{j-1}\|_0^2 + \|\tilde{u}_j\|_2^2). \end{cases} \tag{3.19}$$

  Then as in (3.16), we obtain

$$\begin{aligned} &|||(\tilde{u}_j, \tilde{v}_j)|||_T^2 \leq C_T \left(\int_0^T \|\dot{u}_{j-1} \tilde{u}_{jx}\|_0 ds + \int_0^T \|\dot{u}_{(j-1)x} \tilde{u}_{j-1}\|_0 ds \right)^2 \\ &\leq C_T \left(\kappa \int_0^T \|\tilde{u}_j\|_2^2 ds + \kappa \int_0^T \|\tilde{u}_{j-1}\|_2^2 ds \right). \end{aligned} \tag{3.20}$$

  Notice C_T is uniform for $T \leq T_0$, we can choose $T_0 \ll 1$ (by (3.13)) so that $\kappa \ll 1$ such that $C_T \kappa \leq \frac{1}{4}$. For such chosen T_1, (3.14) follows readily from (3.20).

This concludes the proof of Theorem 3.3.

Theorem 3.4 [Global existence of P_T solution] Let $(u_0, v_0) \in L^2(R^2) \times L^2(R^2)$. Then $\forall T > 0$, (1.1)(1.2) has a unique solution $(u, v) \in P_T$ satisfying

$$\begin{aligned}&\sup_{0\le t\le T}(\|u(t)\|_0^2 + \|v(t)\|_0^2) + \int_0^T (\|u(s)\|_2^2 + \|v(s)\|_1^2)ds \\ &\quad \le C_T(\|u_0\|_0^2 + \|v_0\|_0^2).\end{aligned} \tag{3.21}$$

Proof: Let Λ be the subset of all $T \ge 0$ such that in $[0, T]$ (1.1)(1.2) has a unique solution in P_τ. Theorem 3.3 guarantees that $\Lambda \ne \emptyset$. Since $\forall T \in \Lambda$, $(u(T), v(T)) \in L^2(R^2)$, Theorem 3.3 also implies that the set Λ must be open. Let $\{T_k\}$ be a monotone increasing sequence in Λ such that $T_k \to T_\tau$. By Theorem 3.1, $(u(T_\tau), v(T_\tau)) \in L^2(R^2)$, then there must be a solution in $[0, T_\tau + \epsilon]$ by Theorem 3.3. Hence Λ must be open. Consequently $\Lambda = [0, \infty)$ and this concludes the proof of Theorem 3.4.

4 Global Smooth Solutions

Let $(u, v) \in P_T$ be the global solution obtained in Theorem 3.4. Define $f_1 = -uu_x$ then $f_1 \in L^1([0, T]; H^1(R^2))$ and

$$\int_0^T \|f_1\|_1 dt \le \int_0^T \|u\|_2^2 dt \le C_T(\|u_0\|_0^2 + \|v_0\|_0^2). \tag{4.1}$$

Consider the following linear initial value problem:

$$\begin{cases} u_t + \Delta u_x + \alpha u_{xx} + \gamma\Delta^2 u - \epsilon_1 v_x = f_1, \\ v_t + a_1 v_x - \Gamma\Delta v - \epsilon_2 u_x = 0, \\ u(x, y, 0) = u_0(x, y), \quad v(x, y, 0) = v_0(x, y). \end{cases} \tag{4.2}$$

Then we have the following improved version of Theorem 2.3.

Theorem 4.1 If $(u_0, v_0) \in H^2(R^2) \times H^1(R^2)$ and $f_1 \in L^1([0, T]; H^1(R^2))$, then (4.2) has a unique solution $(u, v) \in \Pi_T^0$.

Proof: Taking L^2 inner product over R^2 of the equations in (4.2) with $(\Delta u, \Delta v)$ and noticing $(\Delta u_x, \Delta u) = 0$, we obtain readily the following

$$\begin{aligned}&\partial_t(\|\nabla u\|^2 + \|\nabla v\|^2) + \gamma\|\nabla\Delta u\|^2 + \Gamma\|\nabla v\|^2 \\ &\quad \le C(\|u\|_2^2 + \|v\|_1^2 + |(\nabla f_1, \nabla u)|).\end{aligned} \tag{4.3}$$

Integrating (4.3) from 0 to T and noticing $(u, v) \in P_T$ and satisfies (3.21), we have

$$\begin{aligned}&\sup_{0\le t\le T}(\|u(t)\|_1^2 + \|v(t)\|_1^2) + \textstyle\int_0^T (\|u(s)\|_3^2 + \|v(s)\|_2^2)ds \\ &\quad \le C(\|u_0\|_1^2 + \|v_0\|_1^2 + \textstyle\int_0^T |(\nabla f_1, \nabla u)|ds).\end{aligned} \tag{4.4}$$

Since

$$\begin{aligned}&\textstyle\int_0^T |(\nabla f_1, \nabla u)|ds \le \int_0^T \|\nabla f_1\|_0 \|\nabla u(s)\|_0 ds \\ &\quad \le \sup_{0\le t\le T} \|u(s)\|_1 \textstyle\int_0^T \|f_1\|_1 ds \\ &\quad \le \delta \sup_{0\le t\le T} \|u(s)\|_1^2 + C_\delta(\textstyle\int_0^T \|f_1\|_1 ds)^2.\end{aligned} \tag{4.5}$$

By (4.1), $(\int_0^T \|f_1\|_1 ds)$ is bounded by $\|u_0\|_0^2 + \|v_0\|_0^2$. Combining (4.4) and (4.5), we obtain

$$\begin{aligned}&\sup_{0\le t\le T}(\|u(t)\|_1^2 + \|v(t)\|_1^2) + \textstyle\int_0^T (\|u(s)\|_3^2 + \|v(s)\|_2^2)ds \\ &\quad \le C_T(\|u_0\|_1^2 + \|v_0\|_1^2 + (\textstyle\int_0^T \|f_1\|_1 ds)^2).\end{aligned} \tag{4.6}$$

(4.6) implies that for $(u_0, v_0) \in H^1(R^2)$, the solution (u, v) in Theorem 3.4 is actually in the space

$$\begin{aligned}&u \in C([0,T]; H^1) \bigcap L^2([0,T]; H^3), \\ &v \in C([0,T]; H^1) \bigcap L^2([0,T]; H^2).\end{aligned} \tag{4.7}$$

A further consequence of (4.7) is that $f_1 = uu_x \in L^2((0,T) \times R^2)$.

Taking L^2 inner product over R^2 of the first equation in (4.2) with $\Delta^2 u$, we further obtain

$$\partial_t \|\Delta u\|^2 + \gamma \|\Delta^2 u\|^2 \le C(\|u\|_3^2 + \|v\|_1^2 + \textstyle\int_0^T \|f_1\|_0^2 dt). \tag{4.8}$$

Combining (4.6) and (4.8), we obtain that the solution (u, v) in Theorem 3.4 is in Π_T^0. This completes the proof of Theorem 4.1.

For any integer $k > 0$, we have the following

Theorem 4.2 Assume that in (1.1)(1.2), $(u_0, v_0) \in H^{k+2}(R^2) \times H^{k+1}(R^2)$, then global solution (u, v) in Theorem 3.4 is actually $\in \Pi_T^k$.

Proof: Theorem 4.2 can be proved by induction. If the solution $(u, v) \in \Pi_T^{k-1}$, then $u \in C((0,T); H^{k+1}) \cap L^2((0,T); H^{k+3})$ and $\nabla^k(uu_x) \in L^2((0,T) \times R^2)$. Therefore we can apply ∇^k to (1.1)(1.2) and use the result obtained in Theorem 4.1 to $(\nabla^k u, \nabla^k v)$. And Theorem 4.2 follows readily.

Remark 4.1 Since integer k in Theorem 4.2 is arbitrary, Theorem 4.2 also implies that for smooth initial data (u_0, v_0), the unique global solution (u, v) is also smooth.

The global smooth solution in Theorem 4.2 is obtained under the assumption that the initial data (u_0, v_0) is sufficiently smooth. Actually we can drop such conditions on (u_0, v_0) and still obtain the smoothness of the solution (u, v) in $t > 0$.

Theorem 4.3 In the problem (1.1)(1.2), assume $(u_0, v_0) \in L^2(R^2) \times L^2(R^2)$, then the unique solution $(u, v) \in P_T$ obtained in Theorem 3.4 is actually in $C^\infty((0,T) \times R^2)$.

Proof: Let $\phi_\epsilon(t) \in C^\infty[0, \infty)$ such that

$$\phi_\epsilon(t) = \begin{cases} 0, \quad 0 \le t \le \epsilon; \\ \text{monnotone increasing}, \ \epsilon \le t \le 2\epsilon; \\ 1, \ 2\epsilon \le t. \end{cases} \tag{4.9}$$

Let

$$(u_\epsilon, v_\epsilon) = (u\phi_\epsilon(t), v\phi_\epsilon(t)). \tag{4.10}$$

Since (u, v) and (u_ϵ, v_ϵ) are identical in $t \ge 2\epsilon$, and ϵ is arbitrary, we need only to show that $(u_\epsilon, v_\epsilon) \in C^\infty((0,T) \times R^2)$.

(u_ϵ, v_ϵ) satisfies the following linear initial value problem:

$$\begin{cases} u_{\epsilon t} + \Delta u_{\epsilon x} + \alpha u_{\epsilon xx} + \gamma \Delta^2 u_\epsilon - \epsilon_1 v_{\epsilon x} = f, \\ v_{\epsilon t} + a_1 v_{\epsilon x} - \Gamma \Delta v_\epsilon - \epsilon_2 u_{\epsilon x} = g, \\ u_\epsilon(x, y, 0) = 0, \ \ v_\epsilon(x, y, 0) = 0. \end{cases} \tag{4.11}$$

Here in (4.11),

$$\begin{aligned} f &= -\phi_{\epsilon t} u - \phi_\epsilon u u_x \in L^1((0,T), H^1(R^2)) \\ g &= -\phi_{\epsilon t} v \in L^2((0,T), H^1(R^2)). \end{aligned} \tag{4.12}$$

Similar to the proof of Theorem 2.1 and Theorem 4.1, we can derive the estimate for the solution of (4.11) $(u_\epsilon, v_\epsilon) \in \Pi_T^0$. This would imply that $(f, g) \in L^2((0,T), H^1(R^2))$. Applying the result of Theorem 2.1 again, we have $(u_\epsilon, v_\epsilon) \in \Pi_T^1$ and hence $(f, g) \in L^2((0,T), H^2(R^2))$ and so on. Consequently we obtain that $\forall k$, $(u_\epsilon, v_\epsilon) \in \Pi_T^k$. This implies that $(u_\epsilon, v_\epsilon) \in L^2((0,T); C^\infty(R^2))$. From the fact that (u_ϵ, v_ϵ) satisfies the differential equation (4.11), we obtain the smoothness of (u_ϵ, v_ϵ) in the t-direction, hence $(u_\epsilon, v_\epsilon) \in C^\infty((0,T) \times R^2)$. This concludes the proof of Theorem 4.3.

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