Conical Shock Waves for Isentropic Euler System *†

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Abstract

Conical shock waves are generated as sharp conical solid projectiles fly supersonically in the air. We study such conical shock waves in steady supersonic flow using isentropic Euler system. The stability of such attached conical shock waves for non-symmetrical conical projectile and non-uniform incoming supersonic flow are established. Meanwhile, the existence of the solution to the Euler system with such attached conical shock as free boundary is also proved for solid projectile close to a regular solid cone.

1 Introduction

It is well-known that shock waves are produced as projectiles fly supersonically in the air. If the projectile has a blunt head, the shock front will be detached from the head of projectile. If the projectile has a sharp head, the shock front will be attached to the head. In particular, if the projectile has a sharp pointed conical head, a conical shock front will be generated which is attached to the vertex of the conical projectile, see [12,27].

Conical shock waves have been recently studied in the framework of isentropic irrotational flow [7,8,10,11]. The governing equation for isentropic irrotational flow is a second order quasi-linear wave equation for the velocity potential. In this paper, we study such

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conical shock wave in the framework of isentropic Euler system. This permits the incoming flow with non-zero rotations. The isentropic model is justified for weak shock waves, since across the shock front, the jump of entropy is small of third order with respect to the strength of the shock, which is measured by the jump, say, of density, see [12,16,28,29].

The Euler system for isentropic non-viscous flow in gas-dynamics is the quasi-linear hyperbolic system:

$$\begin{cases} \partial_t \rho + \sum_{j=1}^{3} \partial_{x_j}(\rho v_j) = 0, \\ \partial_t(\rho v_i) + \sum_{j=1}^{3} \partial_{x_j}(\rho v_i v_j + \delta_{ij} p) = 0, \quad i = 1, 2, 3. \end{cases} \tag{1.1}$$

In (1.1), $(\rho, \mathbf{v})$ are the density and the velocity of the gas particles, and the pressure $p = p(\rho, E)$ is a known function.

Shock waves are piecewise smooth solutions for (1.1) which have a jump discontinuity along a hyper-surface $\psi(t, x) = 0$. On this hyper-surface, the solutions for (1.1) must satisfy the following Rankine-Hugoniot conditions, (see [12,13,28])

$$\psi_t \begin{bmatrix} \rho \\ \rho v_1 \\ \rho v_2 \\ \rho v_3 \end{bmatrix} + \psi_{x_1} \begin{bmatrix} \rho v_1 \\ \rho v_1^2 + p \\ \rho v_1 v_2 \\ \rho v_1 v_3 \end{bmatrix} + \psi_{x_2} \begin{bmatrix} \rho v_2 \\ \rho v_1 v_2 \\ \rho v_2^2 + p \\ \rho v_2 v_3 \end{bmatrix} + \psi_{x_3} \begin{bmatrix} \rho v_3 \\ \rho v_1 v_3 \\ \rho v_2 v_3 \\ \rho v_3^2 + p \end{bmatrix} = 0. \tag{1.2}$$

Here $[f] = f_+ - f_-$ denotes the jump difference of f across the hyper-surface $\psi(t, x) = 0$ (shock front discontinuity). In this paper, we use subscript "$_+$" to denote the status on the upstream side (or, ahead) of the shock front and subscript "$_-$" to denote the status on the downstream side (or, behind).

It is well-known that the Rankine-Hugoniot condition (1.2) admits many non-physical solutions to (1.1). Entropy condition or Lax' shock inequality is needed to guarantee the solution to be physical, see [12,13,16,28]. In the case of high space dimension, it is shown that Lax' shock inequality also implies the linear stability of the shock front under multi-dimensional perturbation for isentropic gas, and extra conditions are needed for general non-isentropic flow, see [12,24,28].

For steady shock waves, the time-derivatives in (1.1) and (1.2) all vanish and we obtain

$$\begin{cases} \sum_{j=1}^{3} \partial_{x_j}(\rho v_j) = 0, \\ \sum_{j=1}^{3} \partial_{x_j}(\rho v_i v_j + \delta_{ij} p) = 0, \quad i = 1, 2, 3. \end{cases} \tag{1.3}$$

$$\psi_{x_1}\begin{bmatrix}\rho v_1\\ \rho v_1^2+p\\ \rho v_1 v_2\\ \rho v_1 v_3\end{bmatrix}+\psi_{x_2}\begin{bmatrix}\rho v_2\\ \rho v_1 v_2\\ \rho v_2^2+p\\ \rho v_2 v_3\end{bmatrix}+\psi_{x_3}\begin{bmatrix}\rho v_3\\ \rho v_1 v_3\\ \rho v_2 v_3\\ \rho v_3^2+p\end{bmatrix}=0. \tag{1.4}$$

In addition to (1.3) and (1.4), we need to add the boundary condition on solid surface $\Theta(x_1, x_2, x_3) = 0$. The flow should be tangential to the solid surface and the normal velocity is zero, i.e.,

$$v_1\Theta_{x_1} + v_2\Theta_{x_2} + v_3\Theta_{x_3} = 0. \tag{1.5}$$

For the irrotational flow, there exists velocity potential φ such that $\mathbf{v} = \nabla\varphi$ and (1.3) can be rewritten equivalently as the following second order equation for φ (see [12]), noticing that for sound speed a, we have $a^2 = p'_\rho = H/H'$,

$$\begin{aligned}&\left(v_1^2-a^2\right)\varphi_{x_1x_1}+\left(v_2^2-a^2\right)\varphi_{x_2x_2}+\left(v_3^2-a^2\right)\varphi_{x_3x_3}\\ &\qquad+2v_1v_2\varphi_{x_1x_2}+2v_1v_3\varphi_{x_1x_3}+2v_2v_3\varphi_{x_2x_3}=0,\end{aligned} \tag{1.6}$$

where $v_i = \varphi_{x_i}$, $(i = 1, 2, 3)$ and the sound speed a is a known function of $\nabla\varphi$.

Under the irrotational and isentropic assumption, the conical shock waves have been recently studied extensively by using the governing equation (1.6), see [7,8,10,11,23]. In this paper, we discuss the conical shock wave without the irrotational assumption, using directly system (1.3). In particular, the result also covers the case when the incoming flow is irrotational. Since the flow now is described by a differential system with four equations rather than a second order differential equation, more complicated computations are needed to prove the stability and existence of shock front. The conical shock wave for general Euler system has been studied in [9] for the situation when the cone is symmetric and is generated by a curved generator. The case in [9] can be reduced to a 2-dimensional problem. In this paper, we will allow genuine 3-dimensional perturbation of the solid conic object, as well as in the incoming flow.

As multi-dimensional shock waves, all these shock waves should satisfy the Lax' shock inequality mentioned above [16,28]. However, in the study of steady oblique or conical shock waves, the issue is the stability of shock waves with respect to the small perturbation in the incoming supersonic flow or the solid surface. In the recent interesting work of [4], the stability and existence of multidimensional transonic shocks are studied using irrotational and isentropic model. The stability in this paper is in the similar sense as in [4], and is different from the time-stability studied in [24], and also different from the study of other unsteady flow, see [2,25].

Based upon the Kreiss' condition in [15], sufficient (and necessary for uniform flow) conditions are derived in [22] for the linear stability of oblique shock waves for Euler system (1.3). It was shown in [22] that for polytropic gas, this stability condition is the supersonic flow downstream of the shock front, in addition to the usual Lax' shock inequality. Therefore we will always assume that the downstream flow behind the conical shock front is always uniformly supersonic.

The main result of this paper is the following theorem.

Theorem 1.1 *For a three-dimensional steady conical shock wave in supersonic isentropic flow, assume that*

1. *The solid conical projectile is sufficiently close (in the sense of (2.5)) to a regular symmetrical cone with vertex angle smaller than the critical angle given in [12];*
2. *The usual entropy condition is satisfied across the shock front;*
3. *The flow is supersonic uniformly in the region behind the shock front;*
4. *The shock strength $(\rho_1 - \rho_0)/\rho_0$ satisfies*
$$\frac{\rho_1 - \rho_0}{\rho_0} < \csc^2 \delta, \tag{1.7}$$
where δ is the angle between the shock line and the downstream flow direction $\mathbf{v}$ immediately behind the shock front. (1.7) can also be written in the form
$$\left(\frac{v_n}{|\mathbf{v}|}\right)^2 \left(\frac{\rho_1}{\rho_0} - 1\right) < 1, \tag{1.8}$$
where v_n is the normal flow velocity behind the shock front and $|\mathbf{v}|$ is the supersonic flow behind the shock front.

Then

1. *The conical shock wave is linearly stable with respect to the three dimensional perturbation in the incoming supersonic flow and in the sharp solid surface.*
2. *If the incoming supersonic flow is a small perturbation of uniform flow and the sharp solid surface is a small perturbation of a regular conical surface, a perturbed conical shock wave exists near the vertex of the solid conical projectile.*

Here the linear stability in the Theorem 1.1 means that the solution for corresponding linearized boundary value problem at the given conical shock wave satisfies an energy estimate defined in Section 3.

The paper is arranged as follows. Spherical coordinates are introduced and mathematical problem is formulated in Section 2. Section 3 establishes the energy estimate for the linearized problem, and hence the linear stability. The existence of conical shock waves is proved in Section 4, using a modified nonlinear iteration scheme.

2 Transform and Formulation

We assume the incoming supersonic flow be a small perturbation of the steady uniform flow in the positive direction of x_1-axis. Let b_0 be the half-angle of the symmetric solid cone and

$$r = \sqrt{x_2^2 + x_3^2} = x_1 \tan b_0. \tag{2.1}$$

be the equation for this cone. Then there is a critical value $\tilde{b}_0 > 0$ such that for $b_0 < \tilde{b}_0$ and the uniform incoming flow, there is a symmetric conical shock wave which has a symmetric regular conical shock front (see [12])

$$r = \sqrt{x_2^2 + x_3^2} = x_1 \tan s_0. \tag{2.2}$$

Here in (2.2), $s_0 > b_0$.

Because of the conical shape of the shock front, it is convenient to introduce spherical coordinates (see also [3]) (x, ϕ, θ):

$$x = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad \tan\phi = \frac{\sqrt{x_2^2 + x_3^2}}{x_1}, \quad \tan\theta = \frac{x_3}{x_2}. \tag{2.3}$$

Remark 2.1 *A conical coordinate system has been previously used in the study of conical shock waves for irrotational flow, see [8,9,10,11]. In these papers, it was assumed that the flow in x_1 direction is supersonic, i.e., $v_1 > a$, which is obviously a stronger condition than the downstream supersonic flow in Theorem 1.1. From the discussion in [22], the downstream supersonic flow should be the optimal condition for conical shock waves. The spherical coordinates (2.3) is introduced to achieve the optimal result.*

In the spherical coordinates (2.3), the perturbed non-symmetrical solid surface is given by

$$\phi = b(x, \theta), \tag{2.4}$$

with $b(x, \theta)$ sufficiently smooth in $x \geq 0$ and periodic of 2π in θ. (2.4) is a small perturbation of (2.1) in the sense that

$$b(x, \theta) - b_0 = O(x^N) \tag{2.5}$$

for sufficiently large N.

Similarly, the perturbed non-symmetrical conical shock front in the conical coordinates (x, ϕ, θ) is described by

$$\phi = s(x, \theta). \tag{2.6}$$

The domain between the solid and shock front boundaries becomes

$$b(x, \theta) < \phi < s(x, \theta). \tag{2.7}$$

For the spherical coordinates (2.3), we have

$$\begin{cases} dx = \cos\phi\; dx_1 + \sin\phi\cos\theta\; dx_2 + \sin\phi\sin\theta\; dx_3, \\ d\phi = \dfrac{1}{x}\left(-\sin\phi\; dx_1 + \cos\phi\cos\theta\; dx_2 + \cos\phi\sin\theta\; dx_3\right), \\ d\theta = \dfrac{1}{x\sin\phi}\left(-\sin\theta\; dx_2 + \cos\theta\; dx_3\right). \end{cases} \tag{2.8}$$

The system (1.3) can be written as a symmetric system for the unknown vector function $U = (\rho, v_1, v_2, v_3)^T$:

$$A_1\partial_{x_1}U + A_2\partial_{x_2}U + A_3\partial_{x_3}U = 0 \tag{2.9}$$

where

$$A_1 = \begin{pmatrix} a^2\rho^{-1}v_1 & a^2 & 0 & 0 \\ a^2 & \rho v_1 & 0 & 0 \\ 0 & 0 & \rho v_1 & 0 \\ 0 & 0 & 0 & \rho v_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} a^2\rho^{-1}v_2 & 0 & a^2 & 0 \\ 0 & \rho v_2 & 0 & 0 \\ a^2 & 0 & \rho v_2 & 0 \\ 0 & 0 & 0 & \rho v_2 \end{pmatrix}$$
$$A_3 = \begin{pmatrix} a^2\rho^{-1}v_3 & 0 & 0 & a^2 \\ 0 & \rho v_3 & 0 & 0 \\ 0 & 0 & \rho v_3 & 0 \\ a^2 & 0 & 0 & \rho v_3 \end{pmatrix}. \tag{2.10}$$

From (2.8), we have

$$\begin{cases} \partial_{x_1} = \cos\phi\partial_x - \dfrac{\sin\phi}{x}\partial_\phi, \\ \partial_{x_2} = \sin\phi\cos\theta\partial_x + \dfrac{\cos\phi\cos\theta}{x}\partial_\phi - \dfrac{\sin\theta}{x\sin\phi}\partial_\theta, \\ \partial_{x_3} = \sin\phi\sin\theta\partial_x + \dfrac{\cos\phi\sin\theta}{x}\partial_\phi + \dfrac{\cos\theta}{x\sin\phi}\partial_\theta. \end{cases} \tag{2.11}$$

Therefore in spherical coordinates (x, ϕ, θ), (2.9) becomes

$$x\tilde{A}_1\partial_x U + \tilde{A}_2\partial_\phi U + \tilde{A}_3\partial_\theta U = 0 \tag{2.12}$$

with

$$\begin{cases} \tilde{A}_1 = A_1\cos\phi + A_2\sin\phi\cos\theta + A_3\sin\phi\sin\theta, \\ \tilde{A}_2 = \left(-A_1\sin\phi + A_2\cos\phi\cos\theta + A_3\cos\phi\sin\theta\right), \\ \tilde{A}_3 = \csc\phi\left(A_3\cos\theta - A_2\sin\theta\right). \end{cases} \tag{2.13}$$

From the assumption in Theorem 1.1 that the flow is uniformly supersonic, matrix $\tilde{A}_1$ is positively definite near a regular symmetrical conical shock wave. This can be checked readily by noticing that

$$v_1 \cos\phi + v_2 \sin\phi\cos\theta + v_3 \sin\phi\sin\theta = |\mathbf{v}|. \tag{2.14}$$

Hence we have

$$\tilde{A}_1 = \begin{pmatrix} a^2\rho^{-1}|\mathbf{v}| & a^2\cos\phi & a^2\sin\phi\cos\theta & a^2\sin\phi\sin\theta \\ a^2\cos\phi & \rho|\mathbf{v}| & 0 & 0 \\ a^2\sin\phi\cos\theta & 0 & \rho|\mathbf{v}| & 0 \\ a^2\sin\phi\sin\theta & 0 & 0 & \rho|\mathbf{v}| \end{pmatrix} \tag{2.15}$$

which is positively definite if and only if

$$|\mathbf{v}| > a. \tag{2.16}$$

This explains the optimal condition achieved by the choice of spherical coordinates mentioned in Remark 2.1.

For $\tilde{A}_1 > 0$, (2.12) is a hyperbolic symmetric system away from $x = 0$ with x being the time-like direction. It degenerates at $x = 0$ because of the geometrical singularity of the cone.

In spherical coordinates (x, ϕ, θ), the zero normal velocity condition (1.5) on solid boundary $\Theta(x_1, x_2, x_3) = 0$ becomes

$$\begin{aligned} &v_1(xb'_x\cos b + \sin b) - v_2\left(\cos b\cos\theta - xb'_x\sin b\cos\theta + b'_\theta\csc b\sin\theta\right) \\ &\quad -v_3\left(\cos b\sin\theta - xb'_x\sin b\sin\theta - b'_\theta\csc b\cos\theta\right) = 0, \end{aligned} \tag{2.17}$$

on solid surface $\phi = b(x, \theta)$.

The Rankine-Hugoniot conditions (1.4) on shock front (2.6) becomes

$$\begin{aligned} &(xs'_x\cos s + \sin s)\begin{bmatrix} \rho v_1 \\ \rho v_1^2 + p \\ \rho v_1 v_2 \\ \rho v_1 v_3 \end{bmatrix} \\ &\quad - \left(\cos s\cos\theta - xs'_x\sin s\cos\theta + s'_\theta\csc s\sin\theta\right)\begin{bmatrix} \rho v_2 \\ \rho v_1 v_2 \\ \rho v_2^2 + p \\ \rho v_2 v_3 \end{bmatrix} \\ &\quad - \left(\cos s\sin\theta - xs'_x\sin s\sin\theta - s'_\theta\csc s\cos\theta\right)\begin{bmatrix} \rho v_3 \\ \rho v_1 v_3 \\ \rho v_2 v_3 \\ \rho v_3^2 + p \end{bmatrix} = 0. \end{aligned} \tag{2.18}$$

Next we perform the transformation to fix the free boundary $\phi = s(x, \theta)$ and also formally remove the singularity at $x = 0$. Let $\xi(\phi)$ be a smooth cut-off function in ϕ such that

$$\xi(\phi) = \begin{cases} 1 \ \text{near} \quad \phi = b_0, \\ 0 \quad \text{near} \quad \phi = s_0. \end{cases} \tag{2.19}$$

We introduce a coordinate transform involving the unknown function $s(x, \theta)$:

$$\begin{cases} t = \ln x, \\ y = \xi(\phi)(\phi - b(x, \theta) + b_0) + (1 - \xi)(\phi - s(x, \theta) + s_0), \\ \theta = \theta \end{cases} \tag{2.20}$$

with

$$\begin{cases} dt = \dfrac{1}{x} dx, \\ dy = -[\xi b'_x + (1 - \xi) s'_x] dx + [1 + \xi'(b_0 - b + s - s_0)] d\phi \\ \qquad\qquad -[\xi b'_\theta + (1 - \xi) s'_\theta] d\theta, \\ d\theta = d\theta. \end{cases} \tag{2.21}$$

For simplicity of notation, we use the same U, s and b to denote the functions in the new coordinates (t, y, θ) as in the original coordinates (x, ϕ, θ).

In the coordinates (t, y, θ), the domain (2.7) becomes a domain with fixed boundaries

$$b_0 < y < s_0, \quad -\infty < t < \infty, \quad 0 \leq \theta < 2\pi \tag{2.22}$$

and the system (2.12) becomes:

$$L(U, s) \equiv \bar{A}_1 \partial_t U + \bar{A}_2 \partial_y U + \bar{A}_3 \partial_\theta U = 0 \tag{2.23}$$

with

$$\bar{A}_2 = \tilde{A}_2(1 + \xi'(b_0 - b + s - s_0)) - \xi(\tilde{A}_1 b'_t + \tilde{A}_3 b'_\theta) - (1 - \xi)(\tilde{A}_1 s'_t + \tilde{A}_3 s'_\theta)$$

and $\bar{A}_1 = \tilde{A}_1$, $\bar{A}_3 = \tilde{A}_3$ in (t, y, θ).

The boundary condition (2.17) is now defined on $y = b_0$:

$$\begin{aligned} \ell_1(U) \equiv\ & v_1(b'_t \cos b + \sin b) - v_2 \left(\cos b \cos\theta - b'_t \sin b \cos\theta + b'_\theta \csc b \sin\theta\right) \\ & -v_3 \left(\cos b \sin\theta - b'_t \sin b \sin\theta - b'_\theta \csc b \cos\theta\right) = 0. \end{aligned} \tag{2.24}$$

The Rankine-Hugoniot boundary condition (2.18) on shock front becomes

$$\begin{aligned}\ell_2(U,s) \equiv (s_t'\cos s + \sin s)\begin{bmatrix}\rho v_1\\ \rho v_1^2+p\\ \rho v_1 v_2\\ \rho v_1 v_3\end{bmatrix}\\ -(\cos s\cos\theta - s_t'\sin s\cos\theta + s_\theta'\csc s\sin\theta)\begin{bmatrix}\rho v_2\\ \rho v_1 v_2\\ \rho v_2^2+p\\ \rho v_2 v_3\end{bmatrix}\\ -(\cos s\sin\theta - s_t'\sin s\sin\theta - s_\theta'\csc s\cos\theta)\begin{bmatrix}\rho v_3\\ \rho v_1 v_3\\ \rho v_2 v_3\\ \rho v_3^2+p\end{bmatrix} = 0.\end{aligned} \tag{2.25}$$

defined on the fixed boundary $y = s_0$.

In the next two sections, we will study conical shock waves using the formulation (2.23)-(2.25). Section 3 studies the linearization of (2.23)-(2.25) and derives the energy estimate for linearized problem. This provides the linear stability of the conical shock wave. Section 4 establishes the existence of conical shock waves near $t = -\infty$ (or $x_1 = 0$ in the original coordinates) by a simplified nonlinear iteration.

3 Linearized Problem and Energy Estimate

To study the linear stability of conical shock waves, we consider the linearization of (2.23)-(2.25) at a given status described by solution (U, s). Let $(\dot U, \dot s)$ be a small perturbation of (U, s). The linearization of (2.23) is the following linear system for $(\dot U, \dot s)$.

$$\bar A_1(U)\partial_t\dot U + \bar A_2(U)\partial_y\dot U + \bar A_3(U)\partial_\theta\dot U + \mathbf{a_4}(U)\dot s = f. \tag{3.1}$$

In (3.1), the matrix $\mathbf{a}_4(U)$ is a vector of first order differential operators in (t, θ) which vanishes near the boundary $y = b_0$. The explicit form of $\mathbf{a_4}$ is of no consequence in deriving energy estimate later in this section.

The boundary condition (2.24) on $y = b_0$ is linear, hence the perturbation $(\dot U, \dot s)$ satisfies the same equation as (U, s):

$$\begin{aligned}\dot v_1(b_t'\cos b + \sin b) - \dot v_2\,(\cos b\cos\theta - b_t'\sin b\cos\theta + b_\theta'\csc b\sin\theta)\\ -\dot v_3\,(\cos b\sin\theta - b_t'\sin b\sin\theta - b_\theta'\csc b\cos\theta) = 0.\end{aligned} \tag{3.2}$$

The linearization of (2.25) on the boundary $y = s_0$ can be written as follows.

$$\mathbf{a_0}\dot{s}'_t + \mathbf{a_1}\dot{s}'_\theta + \mathbf{a_2}\dot{s} + B(U,s)\dot{U} = g. \tag{3.3}$$

In (3.3), $\mathbf{a_0}$ and $\mathbf{a_1}$ are vectors defined by

$$\begin{aligned}
\mathbf{a_0} &= \cos s \begin{bmatrix} \rho v_1 \\ \rho v_1^2 + p \\ \rho v_1 v_2 \\ \rho v_1 v_3 \end{bmatrix} + \sin s \begin{bmatrix} \rho v_r \\ \rho v_1 v_r \\ \rho v_2 v_r + p\cos\theta \\ \rho v_3 v_r + p \sin\theta \end{bmatrix} \\
\mathbf{a_1} &= \csc s \begin{bmatrix} \rho v_\theta \\ \rho v_1 v_\theta \\ \rho v_2 v_\theta - p\sin\theta \\ \rho v_3 v_\theta + p\cos\theta \end{bmatrix}
\end{aligned} \tag{3.4}$$

with

$$v_r = v_2\cos\theta + v_3\sin\theta, \quad v_\theta = v_3\cos\theta - v_2\sin\theta. \tag{3.5}$$

The explicit form of $\mathbf{a_2}$ is of no consequence in the following discussion.

$B(U,s)$ is a 4×4 matrix which results from the linearization of (2.25) at $U = (\rho, v_1, v_2, v_3)$. Specifically for regular symmetric conical shock waves at $\theta = 0$, we have $v_3 = 0$, $s = s_0$ and

$$B(U,s) = \sin s_0 \begin{pmatrix} v_1 & \rho & 0 & 0 \\ v_1^2 + a^2 & 2\rho v_1 & 0 & 0 \\ v_1 v_2 & \rho v_2 & \rho v_1 & 0 \\ 0 & 0 & 0 & \rho v_1 \end{pmatrix} - \cos s_0 \begin{pmatrix} v_2 & 0 & \rho & \\ v_1 v_2 & \rho v_2 & \rho v_1 & 0 \\ v_2^2 + a^2 & 0 & 2\rho v_2 & 0 \\ 0 & 0 & 0 & \rho v_2 \end{pmatrix} \tag{3.6}$$

For $\eta > 0$, let $\|u\|_{\eta,T}$ be the standard hyperbolic η-weighted norm (see [14,24]):

$$\|u\|^2_{\eta,T} = \int_{-\infty}^{T}\int_{b_0}^{s_0}\int_0^{2\pi} e^{-2\eta t}|u(t,y,\theta)|^2 d\theta\, dy\, dt. \tag{3.7}$$

The boundary $y = b_0$ is characteristic for hyperbolic system (3.1), one can not derive estimate for the same higher order derivatives in y-direction near $y = b_0$ as in t or θ-direction. We need a partially degenerate norm at the boundary $y = b_0$. For any integer $k \geq 0$, we introduce in the interior region $b_0 < y < s_0$ the hyperbolic η-weighted norm

$$\|u\|^2_{2k,\eta,T} = \sum_{i_0+i_1+2i_2-j_2+i_3\leq 2k,\ j_2\leq i_2} \eta^{2i_0}\|(y-b_0)^{j_2}\partial_t^{i_1}\partial_y^{i_2}\partial_\theta^{i_3} u\|^2_{\eta,T}. \tag{3.8}$$

Such partially degenerate norms as (3.8) have been used for treating characteristic boundaries, see [4,5,19,26]. When $k = 0$, (3.8) is simply the usual hyperbolic η-weighted norm (3.5).

Denote E_{2k} be the Hilbert space consisting of all u with finite norm (3.8). Then E_{2k} can be imbedded into classical continuous function space C^r (see [1]):

$$E_{2k} \subset C^r \text{ for } 2k > 3 + r.$$

Near the boundary $y = s_0$, we will use the standard hyperbolic η-weighted norm

$$\|u\|_{k,\eta,T} = \left(\sum_{i_0+i_1+i_2 \le k} \eta^{2i_0} \int_{-\infty}^{T} \int_0^{2\pi} e^{-2\eta t} |\partial_t^{i_1} \partial_\theta^{i_2} u(t,\theta)|^2 d\theta \, dt \right)^{\frac{1}{2}}. \tag{3.9}$$

The main theorem about the linear stability of conical shock waves is the following energy estimate for the solution $(\dot{U}, \dot{s})$ of (3.1)-(3.3).

Theorem 3.1 *For conical shock waves formulated by (2.23)-(2.25), assume that*

- *the downstream flow behind the conical shock front is uniformly supersonic;*
- *the strength of the shock satisfies*

$$\sin^2 \delta \left(\frac{\rho_1}{\rho_0} - 1 \right) < 1 \tag{3.10}$$

Then for any integer $k \ge 0$, there exist $\eta_0 > 0$ and $C_k > 0$ such that for all $\eta \ge \eta_0$, the solutions $(\dot{U}, \dot{s})$ of the linearized problem (3.1)-(3.3) satisfy the following a priori estimate

$$|||(\dot{U}, \dot{s})|||^2_{2k,\eta,T} \equiv \|\dot{U}\|^2_{2k,\eta,T} + |\dot{U}|^2_{2k,\eta,T} + |\dot{s}|^2_{2k+1,\eta,T} \le C_k \left(\frac{1}{\eta} \|f\|^2_{2k,\eta,T} + |g|^2_{2k,\eta,T} \right). \tag{3.11}$$

Remark 3.1 *When $k = 0$ in (3.11), we obtain*

$$\|\dot{U}\|^2_{\eta,T} + |\dot{U}|^2_{\eta,T} + |\dot{s}|^2_{1,\eta,T} \le C_0 \left(\frac{1}{\eta} \|f\|^2_{\eta,T} + |g|^2_{\eta,T} \right). \tag{3.12}$$

(3.12) provides the linear stability result for conical shock waves claimed in Theorem 1.1.

Remark 3.2 *The condition (3.10) in Theorem 3.1 comes from (3.24) (or equivalently (3.25) or (3.26)) of [22], and it is automatically satisfied for polytropic gas if the downstream flow is supersonic.*

From the energy estimate (3.11) in Theorem 3.1, we can prove the following existence of solutions for (3.1)-(3.3) by standard dual argument or continuation method.

Theorem 3.2 *Under the assumption of Theorem 3.1, if (f, g) satisfies*

$$\|f\|_{2k,\eta,T}^2 + |g|_{2k,\eta,T}^2 < \infty, \tag{3.13}$$

Then there exists a unique solution $(\dot{U}, \dot{s})$ for the linearized problem (3.1)-(3.3) with

$$|||(\dot{U}, \dot{s})|||_{2k,\eta,T}^2 = \|\dot{U}\|_{2k,\eta,T}^2 + |\dot{U}|_{2k,\eta,T}^2 + |\dot{s}|_{2k+1,\eta,T}^2 < \infty \tag{3.14}$$

and satisfying the (3.11).

The main task in this section is to establish (3.11). By standard localization, the energy estimate (3.11) can be derived near boundaries $y = b_0$ and $y = s_0$ separately. Also because the boundary conditions in (3.2) and (3.3) are all local, we need only to consider the problem at $\theta = 0$ without loss of generality.

3.1 Estimate for linear problem near $y = b_0$

Since $\xi = 1$ near boundary $y = b_0$, the interior equation (3.1) becomes

$$\bar{A}_1 \partial_t \dot{U} + \bar{A}_2 \partial_y \dot{U} + \bar{A}_3 \partial_\theta \dot{U} = f. \tag{3.15}$$

At $\theta = 0$, the matrix $\bar{A}_2$ becomes

$$\bar{A}_2 = (\tilde{A}_2 - A_1 b_t' - \tilde{A}_3\, b_\theta') = -A_1(\sin\phi + \cos\phi\, b_t') + A_2(\cos\phi - \sin\phi\, b_t') - \csc\phi A_3\, b_\theta' \tag{3.16}$$

and the boundary condition (3.2) becomes

$$\dot{v}_1(\cos b\, b_t' + \sin b) - \dot{v}_2(\cos b - \sin b\, b_t') + \dot{v}_3 \csc b\, b_\theta' = 0. \tag{3.17}$$

From (2.21), $y = b_0$ is equivalent to $\phi = b$. Using boundary condition (2.24) at $\theta = 0$, we find matrix $\bar{A}_2$ in (3.16) degenerate on the boundary $y = b_0$. Indeed, $\bar{A}_2$ can be computed explicitly:

$$\bar{A}_2 = \begin{pmatrix} 0 & -a^2(\sin b + \cos b\, b_t') & a^2(\cos b - \sin b\, b_t') & -a^2 \csc b\, b_\theta' \\ -a^2(\sin b + \cos b\, b_t') & 0 & 0 & 0 \\ a^2(\cos b - \sin b\, b_t') & 0 & 0 & 0 \\ -a^2 \csc b\, b_\theta' & 0 & 0 & 0 \end{pmatrix} \tag{3.18}$$

For $\dot{U}$ satisfying (3.16), we always have on the boundary $y = b_0$

$$\dot{U}^T \bar{A}_2 \dot{U} = 0. \tag{3.19}$$

Therefore, the system (3.14) with boundary condition (3.17) consists a symmetric positive system with characteristic boundary of constant multiplicity [14,17,26]. The boundary condition (3.16) is admissible in the sense of [14].

Integrating by parts the following inner product over the domain $-\infty < t < \infty$, $b_0 < y$

$$(e^{-\eta t}\dot{U}, \bar{A}_1(\eta+\partial_t)e^{-\eta t}\dot{U} + \bar{A}_2\partial_y e^{-\eta t}\dot{U} + \bar{A}_3\partial_\theta e^{-\eta t}\dot{U} - e^{-\eta t}f) = 0, \tag{3.20}$$

we obtain the zero-order estimate by standard procedure, see [14]

$$\|\dot{U}\|^2_{0,\eta,T} \leq \frac{C_0}{\eta}\|f\|^2_{0,\eta,T}. \tag{3.21}$$

Let D_σ be the complete set of differential operators $(\partial_t, (y-b_0)\partial_y, \partial_\theta)$ which are tangent to the boundary $y = b_0$. We can use the same methods in [26] to obtain the estimate for tangential derivatives $\|D^k_\sigma \dot{U}\|$ near $y = b_0$. The procedure is outlined as follows.

First of all, we perform a linear transform $\dot{V} = E\dot{U}$ so that (3.15) becomes

$$E^*\bar{A}_1 E\partial_t\dot{V} + E^*\bar{A}_2 E\partial_y\dot{V} + E^*\bar{A}_3 E\partial_\theta\dot{V} + A_4\dot{V} = E^* f. \tag{3.22}$$

In (3.22), the boundary matrix is block-diagonal at $y = b_0$:

$$E^*\bar{A}_2 E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & a_{34} & a_{44} \end{pmatrix} \tag{3.23}$$

with nonsingular sub-matrix:

$$\det\begin{pmatrix} a_{33} & a_{34} \\ a_{34} & a_{44} \end{pmatrix} \neq 0. \tag{3.24}$$

In addition, the boundary condition (3.17) can be expressed as

$$\dot{V}_4 = 0. \tag{3.25}$$

Secondly, we take tangential operator D_σ of the (3.22) and (3.25). Obviously, $D_\sigma\dot{V}$ satisfies the first order extension system of (3.22) with the same boundary condition as (3.25):

$$D_\sigma\dot{V}_4 = 0. \tag{3.26}$$

Consequently we obtain the estimate for the first order tangential derivatives $D_\sigma\dot{V}$. Repeating the same procedure, we obtain the estimate for k-th order tangential derivatives

$$\sum_{i_0+i_1+i_2+i_3\leq k} \eta^{2i_0}\|(y-b_0)^{i_2}\partial_t^{i_1}\partial_y^{i_2}\partial_\theta^{i_3}\dot{U}\|^2_{0,\eta,T} \leq \frac{C'_k}{\eta}\|f\|^2_{k,\eta,T}. \tag{3.27}$$

Indeed, (3.27) is different from the result in [26] only in η-weighted norms.

Finally, to obtain the partial estimate in the normal derivatives to the boundary $y = b_0$, we use the approach used in [1,6,7,19].

By (3.23), $\partial_y D_\sigma^{2k-1}\dot{V}_3$ and $\partial_y D_\sigma^{2k-1}\dot{V}_4$ can be estimated by $D_\sigma^{2k}\dot{V}$. Then applying ∂_y to the first two equations of (3.22) and treating $\dot{V}_{34}$ as given, we obtain a system for $\partial_y \dot{V}_1$ and $\partial_y \dot{V}_2$ without boundary condition. Using (3.27) on $\partial_y \dot{V}_1$ and $\partial_y \dot{V}_2$, we obtain

$$\|\partial_y D_\sigma^{2k-2}\dot{V}_{1,2}\|_{0,\eta,T} \leq C\left(\|f\|_{2k,\eta,T} + \|\partial_y D_\sigma^{2k-1}\dot{V}_{3,4}\|_{0,\eta,T}\right). \tag{3.28}$$

From (3.28), we can further obtain the estimate of $\partial_y^2 D_\sigma^{2k-3}\dot{V}_{3,4}$, and consequently $\partial_y^2 D_\sigma^{2k-4}\dot{V}_{1,2}$. Repeating the procedure, we obtain

$$\|\dot{U}\|_{2k,\eta,T}^2 \leq \frac{C_k}{\eta}\|f\|_{2k,\eta,T}^2. \tag{3.29}$$

This is the part of energy estimate (3.11) near the boundary $y = b_0$.

3.2 Estimate for linear problem near $y = s_0$

Near boundary $y = b_0$, $\xi = 0$ and the interior equation (3.1) becomes

$$\bar{A}_1(U)\partial_t \dot{U} + \bar{A}_2(U)\partial_y \dot{U} + \bar{A}_3(U)\partial_\theta \dot{U} + \mathbf{a_4}(U)\dot{s} = f. \tag{3.30}$$

with boundary condition on $y = s_0$:

$$\mathbf{a_0}\dot{s}_t' + \mathbf{a_1}\dot{s}_\theta' + \mathbf{a_2}\dot{s} + B(U,s)\dot{U} = g. \tag{3.31}$$

The energy estimate (3.11) for the problem (3.30)-(3.31) is obtained directly from the result in [22]. In [22], it is shown that for oblique shock waves, the linearized boundary value problem is uniformly Kreiss well-posed (see [15]) if (and only if for uniform flow) the downstream flow is supersonic and (3.10) is satisfied, in addition to the usual Lax' shock inequality. Since Kreiss condition is micro-local and stable under small perturbation of coefficients, we need only to examine boundary value problem (3.30)-(3.31) at regular symmetrical conical shock wave and at an arbitrary point, say, without loss of generality at $\theta = 0$.

At regular symmetric shock waves, $s = s_0$. We have in (3.31)

$$\begin{aligned} \bar{A}_1 &= A_1 \cos s_0 + A_2 \sin s_0, \\ \bar{A}_2 &= \tilde{A}_2 = -A_1 \sin s_0 + A_2 \cos s_0, \\ \bar{A}_3 &= \csc s_0 A_3. \end{aligned} \tag{3.32}$$

Noticing that $v_3 = 0$ and $v_\theta = 0$ for regular symmetric shock waves, we obtain in (3.3),

$$\mathbf{a_0} = \cos s_0 \begin{bmatrix} \rho v_1 \\ \rho v_1^2 + p \\ \rho v_1 v_2 \\ 0 \end{bmatrix} + \sin s_0 \begin{bmatrix} \rho v_2 \\ \rho v_1 v_2 \\ \rho v_2^2 + p \\ 0 \end{bmatrix}, \quad \mathbf{a_1} = \csc s_0 \begin{bmatrix} 0 \\ 0 \\ 0 \\ p \end{bmatrix}. \tag{3.33}$$

Let's compare (3.30)-(3.31) with the boundary value problem from oblique shock waves studied in [22]. (3.32) indicates that the (3.30) in coordinate (t, y) is obtained from a rotation of (x_1, x_2) by degree s_0 and θ obtained from multiplying x_3 by $\csc s_0$. Similarly from (3.33) and (3.6), the boundary condition (3.31) is obtained by the same rotation and multiplication.

Because Kreiss' condition is invariant under coordinates transform, by Remark 3.2 we conclude that under the assumption of Theorem 3.1, the boundary value problem (3.30)(3.31) satisfies uniform Kreiss' condition. Consequently, its solution $(\dot{U}, \dot{s})$ satisfies the following energy estimate

$$\|\dot{U}\|^2_{\eta,T} + |\dot{U}|^2_{\eta,T} + |\dot{s}|^2_{1,\eta,T} \le C_0 \left(\frac{1}{\eta} \|f\|^2_{\eta,T} + |g|^2_{\eta,T} \right). \tag{3.34}$$

(3.34) is the zero-order energy estimate in (3.11). As usual, we can further take tangential derivatives to derive the tangential estimate and then using the fact of non-characteristic boundary $y = s_0$ to obtain the estimate of derivatives in normal direction near the boundary $y = s_0$:

$$\|\dot{U}\|^2_{2k,\eta,T} + |\dot{U}|^2_{2k,\eta,T} + |\dot{s}|^2_{2k+1,\eta,T} \le C_k \left(\frac{1}{\eta} \|f\|^2_{2k,\eta,T} + |g|^2_{2k,\eta,T} \right). \tag{3.35}$$

Finally, combining the estimate (3.29) near $y = b_0$ and the estimate (3.35) near the boundary $y = s_0$, we obtain the energy estimate (3.11) for solutions $(\dot{U}, \dot{s})$ of the linearized problem (3.1),(3.2) and (3.3).

4 Existence of Conical Shock Waves

In this section we apply the result obtained in section 3 to prove the existence of conical shock waves for (1.3)(1.4) and (1.5) near the vertex of solid cone, by using a modified nonlinear iteration scheme. From the discussion in Section 2, it is equivalent to proving the existence of solution (U, s) for the boundary value problem (2.23)(2.24) and (2.25) in coordinate (t, y, θ) near $t = -\infty$.

The main theorem is the following

Theorem 4.1 *For the conical shock wave problem formulated in (2.23)-(2.25), assume*

(A1) *the solid conical surface* $y = b(t, \theta)$ *is a small perturbation of a regular symmetrical cone with straight line generator* $y = b_0$*:*

$$|b(t, \theta) - b_0| = O(e^{Nt}) \text{ near } t = -\infty \tag{4.1}$$

for sufficiently large N *and with* $b_0 < \tilde{b}_0$*, where* $\tilde{b}_0$ *is the critical value such that a regular symmetrical conical shock wave exists [12];*

(A2) *the incoming upstream flow* $(\rho_+, \mathbf{v}_+)$ *in front of shock wave is a small perturbation of a uniform constant flow* $(\rho_{0+}, \mathbf{v}_{0+})$ *in the sense that in the region* $y \geq b(t, \theta)$

$$|(\rho_+, \mathbf{v}_+) - (\rho_{0+}, \mathbf{v}_{0+})| = O(e^{Nt}) \text{ near } y = b(t, \theta), \;\; t = -\infty \tag{4.2}$$

(A3) *the regular symmetrical conical shock wave is compressive, i.e.,*

$$\rho_0 < \rho_1; \tag{4.3}$$

(A4) *the downstream flow behind the regular conical shock wave is supersonic;*

(A5) *the shock strength satisfies the relation (3.10) or equivalently (1.8).*

Then, there is a $T_0 \gg 1$ *such that in* $(-\infty, -T_0)$*, the boundary value problem (2.23)-(2.25) has a classical solution* $(U, s) \in E_{2k} \times H_{2k+1}$ $(k \geq 3)$ *near* $t = -\infty$ *which corresponds to the non-symmetrical conical shock wave.*

Remark 4.1 *The condition (A2) allows the incoming flow to be non-constant and having* $\nabla \times \mathbf{v} \neq 0$*. This was not permitted by the previous studies in [10,11] using irrotational model (1.6).*

Remark 4.2 *The condition (A3) is the usual Lax' shock inequality and can be replaced by any of its equivalent forms. (A4) and (A5) are the linear stability conditions for oblique shock waves derived in [22]. As was shown in [22], (A4) implies (A5) for polytropic gas with* $p = A\rho^\gamma$*.*

Theorem 4.1 will be proved by a modified nonlinear iteration scheme. Such iteration scheme has been used before in [18], and later also in [11,20,21]. The iteration scheme simplifies the convergence proof of iteration sequence. It is generally useful in the proof of existence of solution for evolutionary equations near an approximate solution if the linearized problem has an a priori estimate with no loss of regularity. Otherwise, the more powerful Nash-Moser iteration scheme is needed, as in the study of rarefaction waves, see [1,19].

The idea of the modified nonlinear iteration scheme used in this paper can be illustrated by looking at a simple equation

$$G(x) = 0 \tag{4.4}$$

where G is a smooth function. To solve (4.4) near $\tilde{x}$, we first rewrite $G(x)$ into the following form

$$G(\tilde{x} + \dot{x}) = G(\tilde{x}) + G_1(\tilde{x}, \dot{x})\dot{x}, \tag{4.5}$$

where $x = \tilde{x} + \dot{x}$ and $\tilde{x}$ is an approximate solution. Indeed, the expression of $G_1(\tilde{x}, \dot{x})$ can be obtained by Taylor's expansion. In particular, for the case of $x \in R^1$ the function $G_1(\tilde{x}, \dot{x})$ is simply defined by

$$G_1(\tilde{x}, \dot{x}) \equiv \frac{G(\tilde{x} + \dot{x}) - G(\tilde{x})}{\dot{x}} \tag{4.6}$$

for $\dot{x} \neq 0$ and

$$G_1(\tilde{x}, 0) = G'(\tilde{x}). \tag{4.7}$$

To solve x for the equation $G(x) = 0$ is then equivalent to solving $\dot{x}$ for the equation

$$G_1(\tilde{x}, \dot{x})\dot{x} = -G(\tilde{x}). \tag{4.8}$$

Based upon the equation (4.8), the solution sequence $\{\dot{x}_k\}$ can be obtained by taking $\dot{x}_0 = 0$ and solving $\dot{x}_{k+1}$ from the following linear equation with given $\dot{x}_k$,

$$G_1(\tilde{x}, \dot{x}_k)\dot{x}_{k+1} = -G(\tilde{x}). \tag{4.9}$$

In the following, first we construct an approximate solution $(\tilde{U}, \tilde{s})$ for (2.23)-(2.25). At this approximate solution, we then use the iteration scheme as in (4.9). The energy inequality obtained in section 3 for linear problem (3.1)-(3.3) will be applied to the linearized problem (4.9) to derive the estimate for $\dot{x}_k$. Finally as usual, the existence of the solution follows readily from the bounded-ness of higher order norm and the convergence of lower order norm for the iteration sequence.

Proof of Theorem 4.1

Step 1: Construction of approximate solution.

The iteration will be performed near an N-order approximate solution. $(\tilde{U}, \tilde{s})$ is called an N-order approximate solution for (2.23)-(2.25) near $t - \infty$ if

$$L(\tilde{U}, \tilde{s}) = O(e^{Nt}) \text{ in } b_0 < y < s_0, \tag{4.10}$$

$$\ell_1(\tilde{U}) = 0 \text{ on } y = b_0, \tag{4.11}$$

$$\ell_2(\tilde{U}, \tilde{s}) = O(e^{Nt}) \ \text{ on } \ y = s_0. \tag{4.12}$$

Here, L, ℓ_1, ℓ_2 are the operators defined in (2.23),(2.24) and (2.25).

In particular, (4.11) is a linear equation and is satisfied accurately. This is necessary to obtain the estimate in section 3.1 near uniform characteristic boundary $y = b_0$.

In the study of conical shock waves in irrotational isentropic flow, the regular symmetrical conical shock wave solution (U^*, s^*) can be adopted as an approximate solution, see [11]. However, U^* does not satisfy (4.11) accurately. Here we will choose $(\tilde{U}, \tilde{s})$ instead as a modification of (U^*, s^*).

Let $\tilde{U}(t, y, \theta)$ be the projection of $U^*(t, y, \theta)$ onto the unit normal direction $\mathbf{n}$ of the surface $y = b(t, \theta)$:

$$\tilde{U}(t, y, \theta) = (U^*(t, y, \theta) \cdot \mathbf{n})\mathbf{n}. \tag{4.13}$$

Obviously the velocity $\tilde{U}(t, y, \theta)$ defined by (4.13) is normal to the surface $y = b$ and hence satisfies accurately (4.11). Since $|b(t, \theta) - b_0| = O(e^{Nt})$, so

$$|\tilde{U}(t, y, \theta) - U^*(t, y, \theta)| = O(e^{Nt}) \ \text{ near } \ t = -\infty. \tag{4.14}$$

Let

$$\tilde{s} = s^*. \tag{4.15}$$

$(\tilde{U}, \tilde{s})$ defined by (4.13) and (4.15) satisfies (4.10)-(4.12) if (U^*, s^*) satisfies (4.10) and (4.12). Therefore $(\tilde{U}, \tilde{s})$ is the required N-order approximate solution.

Step 2: Iteration scheme.

We are looking for the solution (U, s) of (2.23)-(2.25) near the approximate solution $(\tilde{U}, \tilde{s})$. Let

$$(U, s) = (\tilde{U} + \dot{U}, \tilde{s} + \dot{s}) \tag{4.16}$$

and expand (2.23)-(2.25) in the power of $(\dot{U}, \dot{s})$ as in (4.4). From (2.23) we obtain as in (4.8)

$$\begin{aligned}
&\mathcal{L}(\dot{U}, \dot{s})(\dot{U}, \dot{s}) \\
&\equiv \bar{A}_1(\dot{U})\partial_t \dot{U} + \bar{A}_2(\dot{U}, \dot{s})\partial_y \dot{U} + \bar{A}_3(\dot{U})\partial_\theta \dot{U} + \bar{A}_4(\dot{U})\dot{U} + \alpha_4(\dot{U}, \dot{s})\dot{s} \\
&= \dot{f},
\end{aligned} \tag{4.17}$$

where

$$\dot{f} = -L(\tilde{U}, \tilde{s}). \tag{4.18}$$

In (4.17) $\bar{A}_4$ is a 4×4 matrix with elements smooth in $(\dot{U}, \dot{s})$, and $\alpha_4(\dot{U}, \dot{s})$ is a vector of first order differential operators in (t, θ) with coefficients smooth in $(\dot{U}, \dot{s})$. The explicit forms of $\bar{A}_4$ and α_4 have no effect on the following discussion. We remark here that the notations of $\tilde{U}, \tilde{s}$ in the operators $\mathcal{L}$ and $\bar{A}_i$ are suppressed to simplify the expressions.

The linear relation (2.24) remains unchanged:

$$\ell_1(\dot{U}) = 0. \tag{4.19}$$

From (2.25), we obtain for $(\dot{U}, \dot{s})$ as in (4.8):

$$\beta_2(\dot{U}, \dot{s})(\dot{U}, \dot{s}) \equiv \mathbf{a_0}(\dot{U}, \dot{s})\dot{s}'_t + \mathbf{a_1}(\dot{U}, \dot{s})\dot{s}'_\theta + \alpha_{\mathbf{2}}(\dot{U}, \dot{s})\dot{s} + \mathcal{B}(\dot{U}, \dot{s})\dot{U} = \dot{g}. \tag{4.20}$$

Here in (4.20),

$$\dot{g} = -\ell_2(\tilde{U}, \tilde{s}) \tag{4.21}$$

and $\alpha_{\mathbf{2}}(\dot{U}, \dot{s})$ is a vector with components smooth in $\dot{U}, \dot{s}$, the explicit form of which has no effect in the discussion. The matrix $\mathcal{B}(\dot{U}, \dot{s})$ in (4.20) is obtained from the same expansion as in (4.8). In particular, we notice that $\mathcal{B}(0, 0) = B(\tilde{U}, \tilde{s})$.

From the choice of approximate solution $(\tilde{U}, \tilde{s})$, we have

$$\dot{f} = O(e^{Nt}), \quad \dot{g} = O(e^{Nt}) \text{ near } t = -\infty. \tag{4.22}$$

We solve the problem (4.17)(4.19)(4.20) by linear iteration as in (4.8). Choose $(\dot{U}_0, \dot{s}_0) = (0, 0)$ and define $(\dot{U}_j, \dot{s}_j)$ to be the solution of following linear boundary value problem

$$\begin{cases} \mathcal{L}(\dot{U}_{j-1}, \dot{s}_{j-1})(\dot{U}_j, \dot{s}_j) = \dot{f} \text{ in } b_0 < y < s_0, \\ \ell_1(\dot{U}_j) = 0 \text{ on } y = b_0, \\ \beta_2(\dot{U}_{j-1}, \dot{s}_{j-1})(\dot{U}_j, \dot{s}_j)\dot{g} \text{ on } y = s_0. \end{cases} \tag{4.23}$$

Step 3: Bounded-ness of $(\dot{U}_j, \dot{s}_j)$ in higher norm.

From Theorem 3.1 and from the stability of Kreiss boundary condition with respect to small perturbation of the coefficients, we obtain that there is $\delta_0 > 0$ such that for

$$|\dot{U}_{j-1}| + |\dot{s}_{j-1}| + |\partial_{t,\theta}\dot{s}_{j-1}| < \delta_0, \tag{4.24}$$

the solution $(\dot{U}_j, \dot{s}_j)$ of (4.23) satisfies the energy estimate

$$|||(\dot{U}_j, \dot{s}_j)|||^2_{2k,\eta,T} \le C_k \left(\frac{1}{\eta}\|\dot{f}\|^2_{2k,\eta,T} + |\dot{g}|^2_{2k,\eta,T}\right). \tag{4.25}$$

Because of the continuous imbedding property of Sobolev space E_{2k} with $k > 2$, we conclude that (4.25) is satisfied if

$$|||(\dot{U}_{j-1}, \dot{s}_{j-1})|||^2_{2k,\eta,T} < \delta_1 \tag{4.26}$$

and the constant C_k depends only on δ_1.

Making use of the fact that the righthand side of (4.23) is independent of j, and by (4.22) we can choose $-T \gg 1$ such that

$$C_k \left(\frac{1}{\eta} \|\dot{f}\|^2_{2k,\eta,T} + |\dot{g}|^2_{2k,\eta,T} \right) \leq \delta_1. \tag{4.27}$$

Consequently with such choice of T, we have for all j

$$|||(\dot{U}_j, \dot{s}_j)|||^2_{2k,\eta,T} < \delta_1. \tag{4.28}$$

Step 4: Convergence of solution sequence $\{(\dot{U}_j, \dot{s}_j)\}$ in lower norm.

Let

$$(\dot{V}_j, \dot{\sigma}_j) = (\dot{U}_j - \dot{U}_{j-1}, \dot{s}_j - \dot{s}_{j-1}) \ \ j = 1, 2, \cdots. \tag{4.29}$$

Then $(\dot{V}_j, \dot{\sigma}_j)$ satisfies the boundary value problem:

$$\begin{cases} \mathcal{L}(\dot{U}_j, \dot{s}_j)(\dot{V}_j, \dot{\sigma}_j) = -[\mathcal{L}(\dot{U}_j, \dot{s}_j) - \mathcal{L}(\dot{U}_{j-1}, \dot{s}_{j-1})](\dot{U}_j, \dot{s}_j) \ \text{ in } \ b_0 < y < s_0, \\ \ell_1(\dot{V}_j) = 0 \ \text{ on } \ y = b_0, \\ \beta_2(\dot{U}_j, \dot{s}_j)(\dot{V}_j, \dot{\sigma}_j) = -[\beta_2(\dot{U}_j, \dot{s}_j) - \beta_2(\dot{U}_{j-1}, \dot{s}_{j-1})](\dot{U}_j, \dot{s}_j) \ \text{ on } \ y = s_0. \end{cases} \tag{4.30}$$

By the Gagliardo-Nirenberg inequality for the norms of product and composition [1], we have

$$\begin{aligned} &\|[\mathcal{L}(\dot{U}_j, \dot{s}_j) - \mathcal{L}(\dot{U}_{j-1}, \dot{s}_{j-1})](\dot{U}_j, \dot{s}_j)\|_{2k-2,\eta,T} \\ &\qquad \leq C \left(\|\dot{V}_{j-1}\|_{2k-2,\eta,T} + |\dot{\sigma}_{j-1}|_{2k-2,\eta,T} \right) \left(\|\dot{U}_j\|_{2k-1,\eta,T} + |\dot{s}_j|_{2k-1,\eta,T} \right), \\ &|[\beta_2(\dot{U}_j, \dot{s}_j) - \beta_2(\dot{U}_{j-1}, \dot{s}_{j-1})](\dot{U}_j, \dot{s}_j)|_{2k-2,\eta,T} \\ &\qquad \leq C \left(|\dot{V}_{j-1}|_{2k-2,\eta,T} + |\dot{\sigma}_{j-1}|_{2k-1,\eta,T} \right) \left(|\dot{U}_j|_{2k-2,\eta,T} + |\dot{s}_j|_{2k-1,\eta,T} \right). \end{aligned} \tag{4.31}$$

Apply the $(k-1)$-order energy estimate (3.17) to (4.30), we have

$$|||(\dot{V}_j, \dot{\sigma}_j)|||^2_{2k-2,\eta,T} \leq C_{k-1} |||(\dot{V}_{j-1}, \dot{\sigma}_{j-1})|||^2_{2k-2,\eta,T} |||(\dot{U}_j, \dot{s}_j)|||^2_{2k,\eta,T}. \tag{4.32}$$

From (4.28), for sufficiently small δ_1 such that

$$C_{k-1}\delta_1 < 1,$$

the mapping from $(\dot{V}_{j-1}, \dot{\sigma}_{j-1})$ to $(\dot{V}_j, \dot{\sigma}_j)$ is a contraction in the Sobolev space E_{2k-2}. This concludes the proof of Theorem 4.1 on the existence of conical shock wave solution.

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