

Analysis on Linear Stability of Oblique Shock Waves in Steady Supersonic Flow ^{*†}

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Abstract

An attached oblique shock wave is generated when a sharp solid projectile flies supersonically in the air. We study the linear stability of oblique shock waves in steady supersonic flow under three dimensional perturbation in the incoming flow. Euler system of equations for isentropic gas model is used. The linear stability is established for shock front with supersonic downstream flow, in addition to the usual entropy condition.

1 Introduction

The mathematical model for non-viscous flow in gas-dynamics is the quasi-linear hyperbolic system of Euler equations:

$$\begin{cases} \partial_t \rho + \sum_{j=1}^3 \partial_{x_j}(\rho v_j) = 0, \\ \partial_t(\rho v_i) + \sum_{j=1}^3 \partial_{x_j}(\rho v_i v_j + \delta_{ij} p) = 0, \quad i = 1, 2, 3 \\ \partial_t(\rho E) + \sum_{j=1}^3 \partial_{x_j}(\rho v_j E + p v_j) = 0. \end{cases} \quad (1.1)$$

In (1.1), (ρ, \mathbf{v}) are the density and the velocity of the gas particles, $E = e + \frac{1}{2}|\mathbf{v}|^2$ is the total energy, and the pressure $p = p(\rho, E)$ is a given convex function with sound speed $a > 0$ defined, as usual, by

$$a^2 = \frac{\partial p}{\partial \rho} > 0.$$

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Shock waves are piece-wise smooth solutions for (1.1) which have a jump discontinuity along a hyper-surface $\phi(t, x) = 0$. On this hyper-surface, the solutions for (1.1) must satisfy the following Rankine-Hugoniot conditions, see [6,16]

$$\phi_t \begin{bmatrix} \rho \\ \rho v_1 \\ \rho v_2 \\ \rho v_3 \\ \rho E \end{bmatrix} + \phi_{x_1} \begin{bmatrix} \rho v_1 \\ \rho v_1^2 + p \\ \rho v_1 v_2 \\ \rho v_1 v_3 \\ (\rho E + p)v_1 \end{bmatrix} + \phi_{x_2} \begin{bmatrix} \rho v_2 \\ \rho v_1 v_2 \\ \rho v_2^2 + p \\ \rho v_2 v_3 \\ (\rho E + p)v_2 \end{bmatrix} + \phi_{x_3} \begin{bmatrix} \rho v_3 \\ \rho v_1 v_3 \\ \rho v_2 v_3 \\ \rho v_3^2 + p \\ (\rho E + p)v_3 \end{bmatrix} = 0. \quad (1.2)$$

Here $[f]$ denotes the jump difference of f across the hyper-surface (shock front discontinuity) $\phi(t, x) = 0$. In this paper, we will also use subscript “+” to denote the status on the upstream side (or, ahead) of the shock front and subscript “-” to denote the status on the downstream side (or, behind).

It is well-known that the Rankine-Hugoniot condition (1.2) admits many non-physical solutions to (1.1). Extra conditions are needed to guarantee the solution to be physical. One of these conditions is the stability condition, which argues that for observable physical phenomena, the solution to mathematical model should be stable under small perturbation. In the case of one space dimension, this condition is provided by the famous Lax’ shock inequality, or entropy condition [9,16]. There are many equivalent forms for Lax’ shock inequality. One of them states that a shock wave is stable if and only if the flow in front of the shock front is supersonic and subsonic behind the shock front, see [16]. Here, the supersonic or subsonic refers to the normal velocity of the flow relative to the shock front.

In the case of high space dimension, it is shown that Lax’ shock inequality also implies the linear stability of the shock front under multi-dimensional perturbation for isentropic gas, and extra conditions are needed for general non-isentropic flow, see [10,15].

Shock waves are produced as solid object flying supersonically in the air. If the flying object is a long wing with sharp wedge front, a steady oblique shock wave will be generated. If the flying object is a conical projectile with sharp vertex, a conical shock wave will be produced [7]. The oblique shock wave produced by a three-dimensional wing was studied in [1,13]. And conical shock waves were studied in [2],[4] and [5] for irrotational isentropic flow. Paper [3] also studied the symmetrically curved conical shock in the framework of Euler system.

As multi-dimensional shock waves, all these shock waves should satisfy the Lax’ shock inequality mentioned above. However, the stability guaranteed by Lax’ shock inequality is the stability with respect to the time variable. In the case of steady oblique or conical shock waves, the issue is not the stability in time (indeed, time variable is eliminated for steady flow) but the stability of shock waves with respect to the small perturbation in the incoming supersonic flow or the solid surface. It is therefore different from the stability studied in [10] with respect to time. And it is by no means obvious that Lax’ shock

inequality will also guarantee such stability. The result of this paper provides the rigorous justification of the previous discussion in such shock waves.

Assume the air before the shock front to be steady. The study of steady oblique shock wave consists of determining the location of the shock front and the gas status behind the shock front. From Lax' shock inequality, the normal component of flow velocity relative to the shock front behind the steady shock front is subsonic. But the velocity magnitude could actually be supersonic and this makes the governing system of partial differential equations to be hyperbolic, with the gas flow direction as the "time" direction. In this paper, we will show that this condition on the supersonic-ness, together with Lax' shock inequality, will guarantee the linear stability for oblique shock waves, see Theorem 1.1.

The linear stability of oblique shock waves studied in this paper is the stability with respect to small perturbation in the incoming supersonic flow and in the solid surface. The main work is to study a boundary value problem for hyperbolic system coupled with an unknown function defined on the boundary. We examine the uniform Kreiss condition for such coupled boundary value problem to determine the well-posedness of its linearization, and hence to derive the stability condition for the oblique shock front.

The uniform Kreiss condition is also called uniform Lopatinski condition in the study of L^2 well-posedness of linear initial-boundary value problem for hyperbolic systems. In [8], it was proved to be the necessary and sufficient condition for strictly hyperbolic systems. Later on, it was shown that the result also holds for symmetric hyperbolic systems with certain block structure so that a symmetrizer can be constructed. In particular, such block structure exists for linearized Euler system of gas dynamics, see [10, 15]. Indeed, Metivier proved the general result in [11] that all symmetric hyperbolic systems with eigenvalues of constant multiplicity has such block structure, including the linearized Euler system of gas dynamics as a special example. In this paper, we will apply the "uniform Kreiss condition" to the linearized Euler system in this sense.

We will limit ourselves in this paper to the simplified isentropic case. Even though actual entropy of the gas will increase across shock front, the model is justified for weak shock waves for the change of entropy across the shock wave is of the third order of shock strength. Based upon the result obtained in this paper, the well-posedness of nonlinear conical shock wave problem is discussed in [6]. And general non-isentropic case will be studied in later papers.

The main result of this paper is the following theorem.

Theorem 1.1 *For three-dimensional isentropic flow, a steady oblique shock wave is linearly stable with respect to the three dimensional perturbation in the incoming supersonic flow and in the sharp solid surface if*

1. *The usual entropy condition is satisfied across the shock front. For example, if shock is compressive, i.e., the density increases across the shock front:*

$$\rho_- > \rho_+. \tag{1.3}$$

Or equivalently, Lax' shock inequality is satisfied.

In (1.3), subscripts ρ_+ and ρ_- denote the status of upstream and downstream of the shock front, respectively.

2. The flow is supersonic behind the shock front

$$|\mathbf{v}| > a. \quad (1.4)$$

3. The shock strength $\frac{\rho_-}{\rho_+} - 1$ satisfies

$$\left(\frac{v_n}{|\mathbf{v}|}\right)^2 \left(\frac{\rho_-}{\rho_+} - 1\right) < 1. \quad (1.5)$$

Here v_n denotes the normal component of the downstream flow velocity \mathbf{v} .

The above conditions are also necessary for the linear stability of a plane oblique shock front.

Remark 1.1 The necessity part of the theorem follows from the fact that the uniform Kreiss condition is the necessary and sufficient condition for the well-posedness of the initial-boundary value problem for hyperbolic systems under consideration.

Remark 1.2 It is interesting to compare condition (1.5) with the following conditions in [10] (see (1.17) in [10]):

$$M^2 \left(\frac{\rho_-}{\rho_+} - 1\right) < 1, \quad M < 1. \quad (1.6)$$

(1.5) and (1.6) have very similar forms. The only difference is that the Mach number M in the first relation of (1.6) is replaced here by $v_n/|\mathbf{v}|$. Since the second relation in (1.6) requires that Mach number $M < 1$, and $|\mathbf{v}| > a$ from (1.4), we have

$$\frac{v_n}{|\mathbf{v}|} < M.$$

Hence condition (1.5) appears weaker than conditions (1.6) in [10].

However we emphasize that, despite apparent similarity, (1.5) and (1.6) deal with two different types of stability. (1.5) is about the stability with respect to the perturbation of incoming flow and solid surface, while (1.6) is with respect to the perturbation of initial data.

The paper is arranged as follows. For completeness, section 2 reviews the uniform Kreiss condition and derives the equivalent forms. Section 3 gives the formulation of linear stability of oblique shock front. The examination of Kreiss condition for linear stability is performed in detail in Section 4.

2 Kreiss condition for hyperbolic boundary value problems

In this section, we revisit the uniform Kreiss condition for hyperbolic boundary value problems. A generalization of such conditions can be found in [12]. For completeness, we give here a slightly generalized equivalent form which can be applied conveniently in section 3. For more details, also see [8,12,15].

Consider the boundary value problem of an $m \times m$ hyperbolic system:

$$\begin{cases} \partial_t u + \sum_{j=1}^n A_j(t, x) \partial_{x_j} u + C(t, x) u = f(t, x), & \text{in } x_1 > 0; \\ P(t, x') u = g(t, x') & \text{on } x_1 = 0. \end{cases} \quad (2.1)$$

In (2.1), $x = (x_1, x')$, $u(t, x)$ is an m -dimensional vector function, $A_j(t, x)$ ($j = 1, \dots, n$) are all $m \times m$ matrices, sufficiently smooth in (t, x) , and $P(t, x')$ is a $k \times m$ matrix, sufficiently smooth in (t, x) .

We assume that the system (2.1) is either strictly hyperbolic or symmetric hyperbolic. In the case of strictly hyperbolicity, the eigenvalues λ of the equation

$$\det(\lambda I - \sum \xi_j A_j) = 0$$

are distinct and real. In the case of symmetric hyperbolic system, the matrices A_j are all symmetric and the eigenvalues of $\sum \xi_j A_j$ have constant multiplicity for all $\xi \in R^n$ as in [11].

Also we assume that the boundary $x_1 = 0$ is non-characteristic with respect to the system (2.1), i.e., the matrix A_1 is nonsingular at $x_1 = 0$ and A_1 has k positive eigenvalues and $(m - k)$ negative eigenvalues.

Introduce the following norms in $R^1 \times R_+^n$ and $R^n \times R^{n-1}$:

$$\|u\|_\eta = \left(\int_{R^1} \int_{R^{n-1}} \int_0^\infty e^{-2\eta t} |u(t, x_1, x')|^2 dx_1 dx' dt \right)^{\frac{1}{2}}, \quad (2.2)$$

$$|u|_\eta = \left(\int_{R^1} \int_{R^{n-1}} e^{-2\eta t} |u(t, 0, x')|^2 dx' dt \right)^{\frac{1}{2}}. \quad (2.3)$$

The boundary value problem (2.1) is said to be *well-posed* if there are positive constants η_0 and C_0 such that

$$\eta \|u\|_\eta^2 + |u|_\eta^2 \leq C_0 \left(\frac{1}{\eta} \|f\|_\eta^2 + |g|_\eta^2 \right) \quad (2.4)$$

for all solutions $u \in C_0^\infty(R^1 \times R^n)$ of (2.1) and for all $\eta \geq \eta_0$.

At a fixed point on the boundary $x_1 = 0$, considers the matrix

$$M(s, i\omega) = -A_1^{-1} \left(sI + i \sum_{j=2}^n \omega_j A_j \right), \quad (2.5)$$

with $s = \eta + i\tau$ and $\omega \in R^{n-1}$.

It can be shown that for any $\eta > 0$, matrix $M(s, i\omega)$ has k eigenvalues with negative real parts, and $m - k$ eigenvalues with positive real parts, counting multiplicity. For the matrix $M(s, i\omega)$ at any fixed point $(t, 0, x')$, the bounded solution for the system of ordinary differential equations

$$\frac{du}{dx_1} = M(s, \omega)u \quad (2.6)$$

is a linear combination of k linearly independent solutions u_j ($j = 1, \dots, k$):

$$u = \sum_{j=1}^k \sigma_j u_j. \quad (2.7)$$

Substituting (2.7) into the boundary condition in (2.1), we obtain

$$Pu = \sum_{j=1}^k Pu_j \sigma_j \equiv \tilde{P}\sigma. \quad (2.8)$$

Here $\tilde{P}(t, x', s, \omega)$ is a $k \times k$ matrix and the vector $\sigma = (\sigma_1, \dots, \sigma_k)^T$. Then the uniform Kreiss condition can be stated as follows, see [8,15].

Theorem 2.1 (*Uniform Kreiss Condition*).

The boundary value problem (2.1) is well-posed in the sense of (2.4) if at every point (t, x') on the boundary $x_1 = 0$, the matrix $\tilde{P}(t, x', s, \omega)$ is uniformly nonsingular, i.e., there is a number $\delta > 0$ such that

$$|\det \tilde{P}| \geq \delta \quad (2.9)$$

uniformly for all $|s|^2 + |\omega|^2 = 1$ with $s = \eta + i\tau$ and $\eta > 0$.

Indeed, it can be shown [8] that the determinant in (2.9) is continuous in η up to $\eta = 0$. Therefore the condition (2.9) can also be re-stated in an equivalent form which is more convenient in application.

Theorem 2.2 (*Equivalent form of Theorem 2.1*).

The boundary value problem (2.1) is well-posed in the sense of (2.4) if at every point on the boundary $x_1 = 0$, the equation

$$|\det \tilde{P}| = 0 \quad (2.10)$$

has no solution (s, ω) on $|s|^2 + |\omega|^2 = 1$ with real part of s : $\Re s = \eta > 0$ or with $s = i\tau$ being admissible. Here, $s = i\tau$ is called admissible if for any positive sequence $\{\eta_n\}$, we have

$$\lim_{\eta_n \rightarrow 0} |\det \tilde{P}(\eta_n + i\tau, \omega)| = 0. \quad (2.11)$$

For a constant matrix $M(s, \omega)$ obtained by freezing the variables (x, t) and (s, ω) with $\eta > 0$, let λ_j be an eigenvalue with negative real part of multiplicity ℓ . The corresponding ℓ linearly independent solutions of (2.6) are,

$$e^{\lambda_j x_1} \xi_j, \quad e^{\lambda_j x_1} (x_1 \xi_j + \eta_1), \quad e^{\lambda_j x_1} \left(\frac{1}{2} x_1^2 \xi_j + x_1 \eta_1 + \eta_2 \right), \dots$$

Where ξ_j is an eigenvector of λ_j :

$$(A - \lambda_j I) \xi_j = 0$$

and η_p are generalized eigenvectors:

$$(A - \lambda_j I)^{p+1} \eta_p = 0$$

From this structure of the linearly independent solutions, the uniform Kreiss conditions (2.9) or (2.10) can be re-stated as the following equivalent theorem.

Theorem 2.3 *Let ξ_j ($j = 1, 2, \dots, k$) be k eigenvectors or generalized eigenvectors corresponding to eigenvalues with negative real parts of matrix $M(s, \omega)$. Let U be the $m \times k$ matrix with ξ_j as column vectors. The boundary value problem (2.1) is well-posed if at every point of the boundary $x_1 = 0$, the $k \times k$ matrix $PU(s, \omega)$ is nonsingular, i.e.,*

$$|\det(PU)(s, \omega)| \geq \delta_1 > 0 \tag{2.12}$$

for all $|s|^2 + |\omega|^2 = 1$ and $\eta > 0$.

Or equivalently, the equation

Theorem 2.4 *The boundary value problem (2.1) is well-posed if at every point of the boundary $x_1 = 0$, the equation*

$$|\det(PU)(s, \omega)| = 0 \tag{2.13}$$

has not solution (s, ω) on $|s|^2 + |\omega|^2 = 1$ with either $\eta > 0$ or $s = i\tau$ admissible.

For later application in sections 3 and 4, we state Kreiss condition for a slightly more general form of hyperbolic system. Consider the boundary value problem of general symmetric hyperbolic system

$$\begin{cases} A_0(t, x) \partial_t u + \sum_{j=1}^n A_j(t, x) \partial_{x_j} u + C(t, x) u = f(t, x), & \text{in } x_1 > 0; \\ P(t, x') u = g(t, x') & \text{on } x_1 = 0, \end{cases} \tag{2.14}$$

where matrices A_0, A_j are all symmetrical and A_0 is positively definite. We can rewrite it into the standard form (2.1) by a linear transformation of $u = Sv$ such that $S^T A_0 S = I$. The matrix S is invertible and can be written as $S = S_1 S_2$ with S_1 being an orthogonal matrix and S_2 is a positively definite diagonal matrix. The problem (2.14) can then be rewritten in v as

$$\begin{cases} \partial_t v + \sum_{j=1}^n S^T A_j S \partial_{x_j} v + C_1 v = S^T f, & \text{in } x_1 > 0; \\ PSv = g & \text{on } x_1 = 0. \end{cases} \quad (2.15)$$

The corresponding matrix $M(s, \omega)$ for (2.15) is

$$\begin{aligned} M(s, i\omega) &= -(S^T A_1 S)^{-1} \left(sI + i \sum_{j=2}^n \omega_j (S^T A_j S) \right) \\ &= S^{-1} \left(-A_1^{-1} (sA_0 + i \sum_{j=2}^n \omega_j A_j) \right) S \\ &\equiv S^{-1} M_0(s, i\omega) S. \end{aligned} \quad (2.16)$$

It is readily checked that matrices $M(s, i\omega)$ and $M_0(s, i\omega)$ have the same eigenvalues and ξ is an eigenvector (or generalized eigenvector) for M if and only if $\eta = S\xi$ is an eigenvector (or generalized eigenvector) for M_0 .

Let V be the $m \times k$ matrix with column vectors consisting of linearly independent eigenvectors and generalized eigenvectors for matrix $M(s, i\omega)$ corresponding to eigenvalues with negative real parts (as $\eta > 0$). The uniform Kreiss condition for the boundary value problem (2.15) is

$$|\det(PSV)(s, \omega)| \geq \delta_1 > 0, \quad \forall (s, \omega) \text{ on } |s|^2 + |\omega|^2 = 1, \quad \eta > 0 \quad (2.17)$$

which is obviously equivalent to

$$|\det(PU)(s, \omega)| \geq \delta_1 > 0, \quad \forall (s, \omega) \text{ on } |s|^2 + |\omega|^2 = 1, \quad \eta > 0, \quad (2.18)$$

where $U = SV$ is an $m \times k$ matrix. The column vectors of matrix $U = SV$ are linearly independent eigenvectors and generalized eigenvectors for matrix $M_0(s, i\omega)$ corresponding to eigenvalues with negative real parts (as $\eta > 0$). Similarly, condition (2.18) can be replaced by equivalent statement that the equation

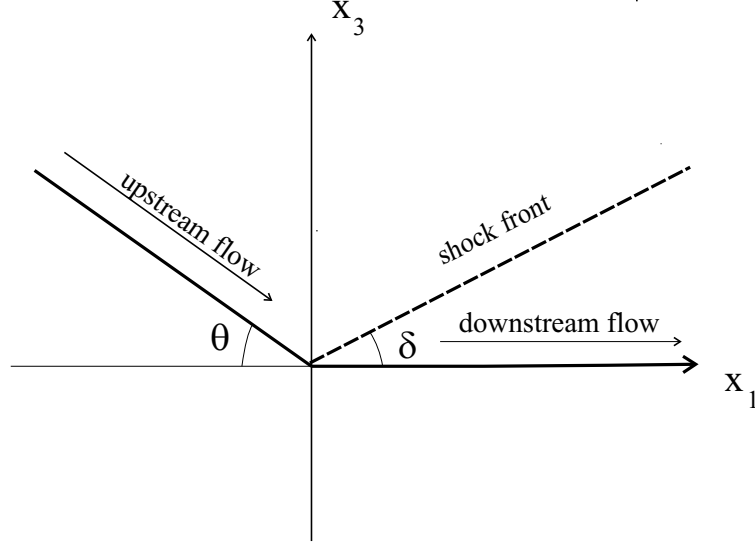
$$|\det(PU)(s, \omega)| = 0 \quad (2.19)$$

has no solution on $|s|^2 + |\omega|^2 = 1$ with either $\eta > 0$ or $s = i\tau$ admissible.

We conclude that the uniform Kreiss condition for general symmetric hyperbolic system (2.14) can be checked directly using matrix $M_0(s, i\omega)$ in (2.16) without transforming (2.14) into the standard form (2.15).

3 Linear stability of oblique shock waves

For simplicity, we choose the coordinate system (x_1, x_2, x_3) such that the solid wing surface is the plane $x_3 = 0$. In addition, we choose, as shown in the following figure,



- The downstream flow behind the oblique shock front is in the positive x_1 direction;
- The angle between the solid wing surface and oblique shock front is δ ;
- The angle between the incoming supersonic flow and the solid wing surface is θ .

We assume the incoming supersonic flow to be a small perturbation of the steady one and the downstream flow after shock front is close to the direction of positive x_1 -axis. Since the stability analysis is micro-local, the steady incoming flow needs not to be uniform. The solid surface of long wing is given by $x_3 = b(x_1, x_2)$ with $b(x_1, x_2) \sim 0$. The oblique shock front is described by $x_3 = s(x_1, x_2)$ such that $s_{x_1} \sim \lambda = \tan \delta > 0$. Obviously we have $b(x_1, x_2) < s(x_1, x_2)$ for all (x_1, x_2) . Without loss of generality, we assume that $b(0, 0) = b_{x_2}(0, 0) = 0$ and $s(0, 0) = s_{x_2}(0, 0) = 0$.

For steady isentropic flow in the region $b(x_1, x_2) < x_3 < s(x_1, x_2)$ the Euler system (1.1) becomes

$$\begin{cases} \sum_{j=1}^3 \partial_{x_j}(\rho v_j) = 0, \\ \sum_{j=1}^3 \partial_{x_j}(\rho v_i v_j + \delta_{ij} p) = 0, \quad i = 1, 2, 3. \end{cases} \quad (3.1)$$

On the shock front $x_3 = s(x_1, x_2)$, we have the Rankine-Hugoniot condition

$$s_{x_1} \begin{bmatrix} \rho v_1 \\ \rho v_1^2 + p \\ \rho v_1 v_2 \\ \rho v_1 v_3 \end{bmatrix} + s_{x_2} \begin{bmatrix} \rho v_2 \\ \rho v_1 v_2 \\ \rho v_2^2 + p \\ \rho v_2 v_3 \end{bmatrix} - \begin{bmatrix} \rho v_3 \\ \rho v_1 v_3 \\ \rho v_2 v_3 \\ \rho v_3^2 + p \end{bmatrix} = 0. \quad (3.2)$$

On the solid surface $x_3 = b(x_1, x_2)$ of the wing, the flow should be tangential to the surface and we have the boundary condition

$$v_1 \frac{\partial b}{\partial x_1} + v_2 \frac{\partial b}{\partial x_2} - v_3 = 0. \quad (3.3)$$

To study the steady oblique shock front $x_3 = s(x_1, x_2)$, we need to consider the system (3.1) with the boundary condition (3.2).

Using the first equation for conservation of mass in (3.1) to simplify the rest, we can rewrite the equations (3.1) as follows

$$\begin{cases} \partial_{x_1}(\rho v_1) + \partial_{x_2}(\rho v_2) + \partial_{x_3}(\rho v_3) = 0, \\ \frac{1}{\rho} \partial_{x_1} p + v_1 \partial_{x_1} v_1 + v_2 \partial_{x_2} v_1 + v_3 \partial_{x_3} v_1 = 0, \\ \frac{1}{\rho} \partial_{x_2} p + v_1 \partial_{x_1} v_2 + v_2 \partial_{x_2} v_2 + v_3 \partial_{x_3} v_2 = 0, \\ \frac{1}{\rho} \partial_{x_3} p + v_1 \partial_{x_1} v_3 + v_2 \partial_{x_2} v_3 + v_3 \partial_{x_3} v_3 = 0. \end{cases} \quad (3.4)$$

The study of multi-dimensional linear stability of the steady oblique shock front is to examine the well-posedness of the linearized problem consisting of system (3.4) under the boundary conditions (3.2).

System (3.4) can be written as a symmetric system for the unknown vector function $U = (\rho, v_1, v_2, v_3)^T$ in $b(x_1, x_2) < x_3 < s(x_1, x_2)$:

$$A_1 \partial_{x_1} U + A_2 \partial_{x_2} U + A_3 \partial_{x_3} U = 0 \quad (3.5)$$

where

$$A_1 = \begin{pmatrix} a^2 \rho^{-1} v_1 & a^2 & 0 & 0 \\ a^2 & \rho v_1 & 0 & 0 \\ 0 & 0 & \rho v_1 & 0 \\ 0 & 0 & 0 & \rho v_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} a^2 \rho^{-1} v_2 & 0 & a^2 & 0 \\ 0 & \rho v_2 & 0 & 0 \\ a^2 & 0 & \rho v_2 & 0 \\ 0 & 0 & 0 & \rho v_2 \end{pmatrix} \quad (3.6)$$

$$A_3 = \begin{pmatrix} a^2 \rho^{-1} v_3 & 0 & 0 & a^2 \\ 0 & \rho v_3 & 0 & 0 \\ 0 & 0 & \rho v_3 & 0 \\ a^2 & 0 & 0 & \rho v_3 \end{pmatrix}.$$

Under the assumption that downstream flow is supersonic, we have $v_1^2 > a^2$ and it is readily checked that matrix A_1 is positively definite. Therefore (3.5) is a hyperbolic symmetric system with x_1 being the time-like direction.

To study the three dimensional stability of the oblique steady shock front $x_3 = s(x_1, x_2)$, we perform the following coordinates transform to fix the shock front

$$x'_1 = x_1, \quad x'_2 = x_2, \quad x'_3 = x_3 - s(x_1, x_2). \quad (3.7)$$

In the new coordinates (x'_1, x'_2, x'_3) , the shock front is $x'_3 = 0$ and the shock front position $x_3 = s(x_1, x_2)$ becomes a new unknown function coupled with U . To simplify the notation, we will denote the new coordinates in the following again as (x_1, x_2, x_3) . The system (3.5) in the new coordinates becomes

$$A_1 \partial_{x_1} U + A_2 \partial_{x_2} U + \tilde{A}_3 \partial_{x_3} U = 0 \quad (3.8)$$

where $\tilde{A}_3 = A_3 - s_{x_1} A_1 - s_{x_2} A_2$. The Rankine-Hugoniot boundary condition (3.2) is now defined on $x_3 = 0$ and takes the same form:

$$s_{x_1} \begin{bmatrix} \rho v_1 \\ \rho v_1^2 + p \\ \rho v_1 v_2 \\ \rho v_1 v_3 \end{bmatrix} + s_{x_2} \begin{bmatrix} \rho v_2 \\ \rho v_1 v_2 \\ \rho v_2^2 + p \\ \rho v_2 v_3 \end{bmatrix} - \begin{bmatrix} \rho v_3 \\ \rho v_1 v_3 \\ \rho v_2 v_3 \\ \rho v_3^2 + p \end{bmatrix} = 0. \quad (3.9)$$

The system (3.8) with boundary condition (3.9) is a coupled boundary value problem for unknown variables (U, s) with U defined in $x_3 < 0$ and s being a function of (x_1, x_2) only. The study of the linear stability of steady oblique shock front is to study the well-posedness of the linearized problem of (3.8)(3.9). Since Kreiss condition is micro-local, we need only to study the linear stability of (3.8-3.9) at the uniform oblique shock front (U_0, s_0) :

$$U_0 = (\rho, v_1, 0, 0), \quad s_0 = \lambda x_1. \quad (3.10)$$

where $\lambda = \tan \delta$ with δ being the angle between solid surface and oblique shock front. Under the assumptions in Theorem 1.1, we have behind the shock front

$$v_1 > a, \quad v_n \equiv v_1 \sin \delta < a \quad (3.11)$$

where v_n is the flow velocity component normal to the shock front.

Let (V, σ) be the small perturbation of (U, s) with $V = (\dot{\rho}, \dot{v}_1, \dot{v}_2, \dot{v}_3)$. Consider the linearization of (3.8-3.9) at $(U, s) = (U_0, s_0)$.

The linearization of (3.8) is the following linear system

$$A_{10} \partial_{x_1} V + A_{20} \partial_{x_2} V + A_{30} \partial_{x_3} V + C_1 \sigma_{x_1} + C_2 \sigma_{x_2} + C_3 V = f. \quad (3.12)$$

Here $A_{10} = A_1$ and

$$A_{20} = \begin{pmatrix} 0 & 0 & a^2 & 0 \\ 0 & 0 & 0 & 0 \\ a^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_{30} = \begin{pmatrix} -a^2 \rho^{-1} \lambda v_1 & -\lambda a^2 & 0 & a^2 \\ -\lambda a^2 & -\rho \lambda v_1 & 0 & 0 \\ 0 & 0 & -\rho \lambda v_1 & 0 \\ a^2 & 0 & 0 & -\rho \lambda v_1 \end{pmatrix}. \quad (3.13)$$

For $v_1^2 > a^2$, matrix A_{10} is positively definite as in (3.5). Direct computation shows that A_{30} has a negative double eigenvalue $-\rho \lambda v_1$ and the other two eigenvalues satisfying the quadratic equation

$$y^2 + \lambda v_1 (\rho + a^2 \rho^{-1}) y - a^2 (a^2 + a^2 \lambda^2 - \lambda^2 v_1^2) = 0. \quad (3.14)$$

Lax' shock inequality implies that the normal velocity behind the shock front is subsonic, hence $a^2 - v_n^2 > 0$. The quantity $(a^2 + a^2 \lambda^2 - \lambda^2 v_1^2)$ in (3.14) will be used often later and will be denoted as

$$d^2 = (a^2 + a^2 \lambda^2 - \lambda^2 v_1^2) = (1 + \lambda^2)(a^2 - v_n^2) > 0. \quad (3.15)$$

Therefore (3.14) has one positive root and one negative root, and matrix A_{30} has three negative eigenvalues and one positive eigenvalue.

Denote U_+ the state ahead of shock front and $U_- = U_0$ the state behind shock front, i.e.

$$U_+ = (v_{1+}, 0, v_{3+}, \rho_+), \quad U_- = (v_{1-}, 0, 0, \rho_-) \equiv (v_1, 0, 0, \rho).$$

The linearization of boundary condition (3.9) has the form

$$\mathbf{a}_1 \partial_{x_1} \sigma + \mathbf{a}_2 \partial_{x_2} \sigma + BV = g. \quad (3.16)$$

Here \mathbf{a}_1 and \mathbf{a}_2 are vectors in R^4 :

$$\mathbf{a}_1 = \begin{pmatrix} \rho v_1 - \rho_+ v_{1+} \\ \rho v_1^2 + p_- - \rho_+ v_{1+}^2 - p_+ \\ 0 \\ -\rho_+ v_{1+} v_{3+} \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} 0 \\ 0 \\ p_- - p_+ \\ 0 \end{pmatrix}, \quad (3.17)$$

and B is a 4×4 matrix:

$$B = \begin{pmatrix} \lambda v_1 & \lambda \rho & 0 & -\rho \\ \lambda(v_1^2 + a^2) & 2\lambda \rho v_1 & 0 & -\rho v_1 \\ 0 & 0 & \lambda \rho v_1 & 0 \\ -a^2 & 0 & 0 & \lambda \rho v_1 \end{pmatrix}. \quad (3.18)$$

Similarly as in section 2, denote

$$\begin{aligned}\|u\|_\eta &= \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^{\infty} e^{-2\eta x_1} |u(x)|^2 dx_3 dx_2 dx_1 \right)^{\frac{1}{2}}, \\ |u|_\eta &= \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\eta x_1} |u(x_1, x_2, 0)|^2 dx_2 dx_1 \right)^{\frac{1}{2}}, \\ |u|_{1,\eta} &= \left(\sum_{t_0+t_1+t_2 \leq 1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \eta^{2t_0} e^{-2\eta x_1} |\partial_{x_1}^{t_1} \partial_{x_2}^{t_2} u(x_1, x_2, 0)|^2 dx_2 dx_1 \right)^{\frac{1}{2}}.\end{aligned}$$

The boundary value problem (3.12)(3.16) is said to be well-posed and the steady oblique shock front is linearly stable if there is an $\eta_0 > 0$ and a constant C_0 such that

$$\eta \|V\|_\eta^2 + |V|_\eta^2 + |\sigma|_{1,\eta}^2 \leq C_0 \left(\frac{1}{\eta} \|f\|_\eta^2 + |g|_\eta^2 \right) \quad (3.19)$$

for all solutions $(V, \sigma) \in C_0^\infty(R^1 \times R^2) \times C_1^\infty(R^2)$ of (2.1) and for all $\eta \geq \eta_0$.

Denote

$$\tilde{\mathbf{a}}(s, i\omega) = s\mathbf{a}_1 + i\omega\mathbf{a}_2 \quad (3.20)$$

From (3.17),

$$\tilde{\mathbf{a}}(s, i\omega) \neq 0 \text{ on } |s|^2 + |\omega|^2 = 1. \quad (3.21)$$

Let Π be the projector in C^4 in the direction of vector $\tilde{\mathbf{a}}(s, i\omega)$, then

$$p(s, i\omega) = (I - \Pi)B \quad (3.22)$$

is a 4×4 matrix of rank 3, with elements being symbols in S^0 , i.e., functions of zero-degree homogeneous in $(s, i\omega)$, see [17]. The study of linear stability of oblique shock front under perturbation is reduced to the investigation of Kreiss condition for the following boundary value problem

$$\begin{cases} A_1 \partial_{x_1} V + A_{20} \partial_{x_2} V + A_{30} \partial_{x_3} V = f_1 \text{ in } x_3 < 0, \\ PV = g_1 \text{ on } x_3 = 0. \end{cases} \quad (3.23)$$

Here P is the zero-order pseudo-differential operator with symbol $p(s, i\omega)$ in (3.22).

The main result of the paper is the following theorem about the well-posedness of (3.23).

Theorem 3.1 *The linear boundary value problem (3.23), describing the linear stability of steady oblique plane shock front, is well-posed in the sense of Kreiss if*

1. $\rho_- > \rho_+$, i.e., the shock is compressive. This is the usual entropy condition.

2. The downstream flow is supersonic, i.e., $v_1 > a_-$. This guarantees the hyperbolicity of system in (3.23).
3. The following condition on the strength of shock front $\rho/\rho_+ - 1$ is satisfied

$$\frac{\rho}{\rho_+} - 1 < 1 + \frac{1}{\lambda^2}. \quad (3.24)$$

The above conditions are also necessary for the problem (3.23) with constant coefficients.

Remark 3.1 The condition (3.24) can also be written in a different form. Since $\lambda = \tan \delta$, (3.24) is equivalent to

$$\sin^2 \delta \left(\frac{\rho}{\rho_+} - 1 \right) < 1. \quad (3.25)$$

From (3.11), (3.25) can further be written as

$$\left(\frac{v_n}{|\mathbf{v}|} \right)^2 \left(\frac{\rho}{\rho_+} - 1 \right) < 1. \quad (3.26)$$

This is the condition (1.5) in Theorem 1.1.

About the condition (3.24) in Theorem 3.1 on the well-posedness of problem (3.23), we have the following

Theorem 3.2 For polytropic gas $p = A\rho^\gamma$ with $\gamma > 1$, (3.24) is always satisfied for oblique shock front satisfying the first two conditions in Theorem 1.1, i.e., $\rho_- > \rho_+$ and $v_1 > a_-$.

Proof: To show this, let q_+ and $q_- = v_1$ denote respectively the magnitude of upstream and downstream flow velocity and denote $r = \rho/\rho_+$. We write down the conservation of mass and momentum in the normal direction to the shock front to obtain

$$\begin{cases} \rho_- q_- \sin \delta = \rho_+ q_+ \sin \theta, \\ p_- + \rho_- q_-^2 \sin^2 \delta = p_+ + \rho_+ q_+^2 \sin^2 \theta. \end{cases} \quad (3.27)$$

Eliminating $q_+ \sin \theta$ from (3.27), we obtain

$$p_- - p_+ = (r - 1)\rho_- q_-^2 \sin^2 \delta. \quad (3.28)$$

From $\lambda = \tan \delta$, the condition (3.24) is equivalent to

$$p_- - p_+ < \rho_- q_-^2. \quad (3.29)$$

For polytropic gas $p = A\rho^\gamma$, (3.29) becomes

$$\left[1 - \frac{1}{r} \left(\frac{a_+}{a_-}\right)^2\right] < \gamma M_-^2. \quad (3.30)$$

(3.30) is always satisfied for supersonic downstream flow of shock front ($M_- > 1$), under entropy condition $\rho_- > \rho_+$ (and hence $a_+/a_- < 1$).

Indeed, it is easy to see that the conclusion in Theorem 3.2 remains to be valid for more general gas, as long as (3.29) is true.

By Theorem 3.2, condition (3.24) actually imposes no extra restriction for the linear stability of oblique shock, as long as the usual entropy condition and the downstream supersonic flow condition are satisfied. Therefore, the main theorem 1.1 follows directly from Theorem 3.1. The rest of the paper is devoted to the proof of Theorem 3.1.

4 Proof of Theorem 3.1

By the discussion in section 2, we construct the matrix $M(s, i\omega)$ as follows

$$M(s, i\omega) = -A_{30}^{-1}(sA_1 + i\omega A_{20}). \quad (4.1)$$

We have

$$sA_1 + i\omega A_{20} = \begin{pmatrix} sa^2\rho^{-1}v_1 & sa^2 & i\omega a^2 & 0 \\ sa^2 & s\rho v_1 & 0 & 0 \\ i\omega a^2 & 0 & s\rho v_1 & 0 \\ 0 & 0 & 0 & s\rho v_1 \end{pmatrix}$$

and

$$A_{30}^{-1} = \frac{\rho\lambda v_1}{|D|} \begin{pmatrix} -(\rho\lambda v_1)^2 & \lambda^2\rho v_1 a^2 & 0 & -\rho\lambda v_1 a^2 \\ \lambda^2\rho v_1 a^2 & -a^2(\lambda^2 v_1^2 - a^2) & 0 & \lambda a^4 \\ 0 & 0 & a^2 d^2 & 0 \\ -\rho\lambda v_1 a^2 & \lambda a^4 & 0 & a^2\lambda^2(a^2 - v_1^2) \end{pmatrix},$$

where $|D| = -(\rho\lambda v_1 a)^2 d^2 < 0$ is the determinant of A_{30} .

Obviously, we need only to consider the eigenvalue and eigenvectors of matrix $N(s, i\omega)$:

$$N(s, i\omega) \equiv -\frac{|D|}{\rho\lambda v_1 a^2} M(s, i\omega)$$

which has the following expression by straightforward computation:

$$N(s, i\omega) = \begin{pmatrix} s\lambda^2\rho v_1(a^2 - v_1^2) & 0 & -i\omega(\rho\lambda v_1)^2 & -s\lambda(\rho v_1)^2 \\ sa^4 & s\rho v_1 d^2 & i\omega\lambda^2\rho v_1 a^2 & s\rho\lambda v_1 a^2 \\ i\omega a^2 d^2 & 0 & s\rho v_1 d^2 & 0 \\ s\lambda a^2(a^2 - v_1^2) & 0 & -i\omega\rho\lambda v_1 a^2 & s\lambda^2\rho v_1(a^2 - v_1^2) \end{pmatrix}. \quad (4.2)$$

Beside one obvious eigenvalue $\xi_1 = s\rho v_1 d^2$, other eigenvalues are roots of

$$\det \begin{vmatrix} s\lambda^2 \rho v_1 (a^2 - v_1^2) - \xi & -i\omega(\rho\lambda v_1)^2 & -s\lambda(\rho v_1)^2 \\ i\omega a^2 d^2 & s\rho v_1 d^2 - \xi & 0 \\ s\lambda a^2 (a^2 - v_1^2) & -i\omega \rho \lambda v_1 a^2 & s\lambda^2 \rho v_1 (a^2 - v_1^2) - \xi \end{vmatrix} = 0.$$

It is easy to see that four eigenvalues for $N(s, i\omega)$ are

$$\begin{cases} \xi_1 = \xi_2 = s\rho v_1 d^2, \\ \xi_{3,4} = s\lambda^2 \rho v_1 (a^2 - v_1^2) \pm \rho \lambda v_1 a \sqrt{s^2(v_1^2 - a^2) + \omega^2 d^2}. \end{cases} \quad (4.3)$$

Since $d^2 = a^2 + \lambda^2 a^2 - \lambda^2 v_1^2 > 0$ by (3.15), we have

$$(\rho \lambda v_1 a)^2 (v_1^2 - a^2) > (\lambda^2 \rho v_1)^2 (v_1^2 - a^2)^2.$$

For $\eta = \Re s > 0$, one of the eigenvalues $\xi_{3,4}$ has positive real part and one has negative real part in (4.3). Consequently for $N(s, i\omega)$, there are three eigenvalues with positive real parts and one with negative real part when $\eta > 0$. This follows either directly from the general theorem in [8], or can be specifically derived from Lemma 5.1 in the Appendix.

For the eigenvalues ξ_1, ξ_2, ξ_3 which have positive real parts when $\eta > 0$, we compute the corresponding eigenvectors or generalized eigenvectors for $N(s, i\omega)$.

The eigenvectors corresponding to the double eigenvalue $\xi_1 = \xi_2$ satisfy the system

$$\begin{pmatrix} -s\rho v_1 a^2 & 0 & -i\omega(\rho\lambda v_1)^2 & -s\lambda(\rho v_1)^2 \\ sa^4 & 0 & i\omega\lambda^2 \rho v_1 a^2 & s\rho\lambda v_1 a^2 \\ i\omega a^2 d^2 & 0 & 0 & 0 \\ s\lambda a^2 (a^2 - v_1^2) & 0 & -i\omega \rho \lambda v_1 a^2 & -s\rho v_1 a^2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = 0. \quad (4.4)$$

(4.4) has two linearly independent solutions α_1 and α_2 :

$$\begin{cases} \alpha_1 = (0, 1, 0, 0)^T, \\ \alpha_2 = (0, 0, s, -i\omega\lambda)^T. \end{cases} \quad (4.5)$$

The eigenvector α_3 corresponding to the eigenvalue ξ_3 satisfies the system

$$\begin{pmatrix} -\rho\lambda v_1 a \mu & 0 & -i\omega(\rho\lambda v_1)^2 & -s\lambda(\rho v_1)^2 \\ sa^4 & s\rho v_1 a^2 - \rho\lambda v_1 a \mu & i\omega\lambda^2 \rho v_1 a^2 & s\rho\lambda v_1 a^2 \\ i\omega a^2 d^2 & 0 & s\rho v_1 a^2 - \rho\lambda v_1 a \mu & 0 \\ s\lambda a^2 (a^2 - v_1^2) & 0 & -i\omega \rho \lambda v_1 a^2 & -\rho\lambda v_1 a \mu \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = 0. \quad (4.6)$$

where

$$\mu \equiv \sqrt{s^2(v_1^2 - a^2) + \omega^2 d^2}. \quad (4.7)$$

Therefore the eigenvector α_3 is

$$\alpha_3 = (\rho v_1(\lambda\mu - sa), (sa - \lambda\mu)a^2, i\omega ad^2, a[\mu a - s\lambda(v_1^2 - a^2)])^T. \quad (4.8)$$

It is obvious that three eigenvectors α_1, α_2 and α_3 are linearly independent at $sa \neq \lambda\mu$.

When $sa = \lambda\mu$, we have $s^2 = \lambda^2\omega^2$, and we have actually triple eigenvalue $\xi_1 = \xi_2 = \xi_3$. α_3 is now parallel to the vector $(0, 0, -i\omega\lambda, -s)^T$ which is parallel to α_2 at $sa = \lambda\mu$. At this point, we will need to find a generalized eigenvector α_3 , in addition to the eigenvectors α_1, α_2 to examine Kreiss condition.

The two cases $sa \neq \lambda\mu$ and $sa = \lambda\mu$ will be discussed one by one in the following.

4.1 Case I: $sa \neq \lambda\mu$

In the case $sa \neq \lambda\mu$, we need to consider the four vectors $(\zeta_1, \zeta_2, \zeta_3) = (B\alpha_1, B\alpha_2, B\alpha_3)$ and $\zeta_4 = s\mathbf{a}_1 + i\omega\mathbf{a}_2$, where B and \mathbf{a}_j are defined in (3.17) and (3.18).

- Vector $\zeta_1 = (\lambda\rho, 2\lambda\rho v_1, 0, 0)^T$ is parallel to, and hence can be replaced by

$$\zeta'_1 = (1, 2v_1, 0, 0)^T \quad (4.9)$$

- Vector $\zeta_2 = (i\omega\lambda\rho, i\omega\lambda\rho v_1, s\lambda\rho v_1, -i\omega\lambda^2\rho v_1)^T$ is parallel to, and hence can be replaced by

$$\zeta'_2 = (i\omega, i\omega v_1, sv_1, -i\omega\lambda v_1)^T. \quad (4.10)$$

- Vector $\zeta_3 = (-\rho\mu d^2, -\rho v_1\mu d^2, i\omega\lambda\rho v_1 ad^2, s\rho v_1 ad^2)^T$ is parallel to, and hence can be replaced by

$$\zeta'_3 = (-\mu, -v_1\mu, i\omega\lambda v_1 a, sv_1 a)^T, \quad (4.11)$$

- Vector ζ_4 is computed to be

$$\zeta_4 = (s(\rho v_1 - \rho_+ v_{1+}), s(\rho v_1^2 + p - \rho_+ v_{1+}^2 - p_+), i\omega(p - p_+), -s\rho_+ v_{1+} v_{3+})^T. \quad (4.12)$$

ζ_4 can be simplified from the Rankine-Hugoniot relations satisfied by the states U_+ and U_- :

$$\begin{cases} \lambda(\rho v_1 - \rho_+ v_{1+}) + \rho_+ v_{3+} = 0, \\ \lambda(\rho v_1^2 + p - \rho_+ v_{1+}^2 - p_+) + \rho_+ v_{1+} v_{3+} = 0, \\ \lambda\rho_+ v_{1+} v_{3+} + (p - \rho_+ v_{3+}^2 - p_+) = 0. \end{cases} \quad (4.13)$$

Solving $p - p_+$ from the third equation in (4.13)

$$p - p_+ = -\lambda\rho_+ v_{1+} v_{3+} + \rho_+ v_{3+}^2 = \rho_+ v_{3+}(v_{3+} - \lambda v_{1+})$$

and substituting it into the second equation in (4.13), we obtain

$$\lambda(\rho v_1^2 - \rho_+ v_{1+}^2 + \rho_+ v_{3+}(v_{3+} - \lambda v_{1+})) + \rho_+ v_{1+} v_{3+} = 0,$$

which simplifies to

$$\lambda \rho v_1^2 = \rho_+(v_{1+} + \lambda v_{3+})(\lambda v_{1+} - v_{3+}). \quad (4.14)$$

From the first equation in (4.13), we obtain

$$\lambda \rho v_1 = \rho_+(\lambda v_{1+} - v_{3+}). \quad (4.15)$$

Combining (4.14) and (4.15) we obtain

$$v_1 = v_{1+} + \lambda v_{3+}.$$

Therefore, we have

$$\rho_+ v_{1+} = \frac{\rho_+ + \lambda^2 \rho}{1 + \lambda^2} v_1, \quad \rho_+ v_{3+} = \frac{\lambda(\rho_+ - \rho)}{1 + \lambda^2} v_1.$$

Consequently we obtain

$$\left\{ \begin{array}{l} \rho v_1 - \rho_+ v_{1+} = \frac{\rho - \rho_+}{1 + \lambda^2} v_1 \\ \rho v_1^2 - \rho_+ v_{1+}^2 + p - p_+ = \frac{(\rho - \rho_+)(\rho_+ + \lambda^2 \rho)}{\rho_+(1 + \lambda^2)^2} v_1^2 \\ p - p_+ = \frac{\lambda^2 v_1^2}{1 + \lambda^2} \frac{\rho(\rho - \rho_+)}{\rho_+} = \frac{\rho(\rho - \rho_+)}{\rho_+} v_n^2 \\ -\rho_+ v_{1+} v_{3+} = \frac{\lambda(\rho - \rho_+)(\rho_+ + \lambda^2 \rho)}{\rho_+(1 + \lambda^2)^2} v_1^2. \end{array} \right. \quad (4.16)$$

By (4.16), we obtain that ζ_4 is parallel to

$$\zeta_4' = \left(s(1 + \lambda^2)\rho_+, s(\rho_+ + \lambda^2 \rho)v_1, i\omega(1 + \lambda^2)\lambda^2 \rho v_1, s\lambda(\rho_+ + \lambda^2 \rho)v_1 \right)^T. \quad (4.17)$$

Kreiss condition states that the oblique steady shock front is linearly stable if four vectors $\zeta_1', \zeta_2', \zeta_3', \zeta_4'$ are linearly independent, or the following matrix with these four vectors as column vectors is uniformly non-degenerate on $|s|^2 + |\omega|^2 = 1, \eta > 0$:

$$\left(\begin{array}{cccc} 1 & i\omega & -\mu & s(1 + \lambda^2)\rho_+ \\ 2v_1 & i\omega v_1 & -v_1 \mu & s(\rho_+ + \lambda^2 \rho)v_1 \\ 0 & s v_1 & i\omega \lambda v_1 a & i\omega(1 + \lambda^2)\lambda^2 \rho v_1 \\ 0 & -i\omega \lambda v_1 & s v_1 a & s\lambda(\rho_+ + \lambda^2 \rho)v_1 \end{array} \right) \quad (4.18)$$

Obviously, it is non-degenerate if and only if the following matrix J is non-degenerate:

$$J = \begin{pmatrix} 1 & i\omega & -\mu & s(1 + \lambda^2)\rho_+ \\ 2 & i\omega & -\mu & s(\rho_+ + \lambda^2\rho) \\ 0 & s & i\omega\lambda a & i\omega(1 + \lambda^2)\lambda^2\rho \\ 0 & -i\omega\lambda & sa & s\lambda(\rho_+ + \lambda^2\rho) \end{pmatrix}. \quad (4.19)$$

The determinant of J can be computed as

$$\det J = \det J_{11} - 2 \det J_{21}$$

with

$$J_{11} = \begin{pmatrix} i\omega & -\mu & s(\rho_+ + \lambda^2\rho) \\ s & i\omega\lambda a & i\omega(1 + \lambda^2)\lambda^2\rho \\ -i\omega\lambda & sa & s\lambda(\rho_+ + \lambda^2\rho) \end{pmatrix}, \quad J_{21} = \begin{pmatrix} i\omega & -\mu & s(1 + \lambda^2)\rho_+ \\ s & i\omega\lambda a & i\omega(1 + \lambda^2)\lambda^2\rho \\ -i\omega\lambda & sa & s\lambda(\rho_+ + \lambda^2\rho) \end{pmatrix}.$$

Hence we have

$$\begin{aligned} \det J &= -i\omega \det \begin{pmatrix} i\omega\lambda a & i\omega(1 + \lambda^2)\lambda^2\rho \\ sa & s\lambda(\rho_+ + \lambda^2\rho) \end{pmatrix} - \mu \det \begin{pmatrix} s & i\omega(1 + \lambda^2)\lambda^2\rho \\ -i\omega\lambda & s\lambda(\rho_+ + \lambda^2\rho) \end{pmatrix} \\ &\quad + [s(\rho_+ + \lambda^2\rho) - 2s(1 + \lambda^2)\rho_+] \det \begin{pmatrix} s & i\omega\lambda a \\ -i\omega\lambda & sa \end{pmatrix} \\ &= \omega^2 sa \lambda^2 (\rho_+ - \rho) - \lambda \mu [s^2 (\rho_+ + \lambda^2 \rho) - \omega^2 (1 + \lambda^2) \lambda^2 \rho] \\ &\quad + sa (\lambda^2 \rho - \rho_+ - 2\lambda^2 \rho_+) (s^2 - \omega^2 \lambda^2). \end{aligned}$$

Consequently

$$\begin{aligned} \det J &= s^3 a (\lambda^2 \rho - \rho_+ - 2\lambda^2 \rho_+) - sa \omega^2 \lambda^2 (1 + \lambda^2) (\rho - 2\rho_+) \\ &\quad - \lambda \mu [s^2 (\rho_+ + \lambda^2 \rho) - \omega^2 (1 + \lambda^2) \lambda^2 \rho]. \end{aligned} \quad (4.20)$$

Using the density ratio parameter r :

$$r = \rho / \rho_+ > 1, \quad (4.21)$$

we conclude that $\det J \neq 0$ if and only if $J_1 \neq 0$ with:

$$\begin{aligned} J_1 &= s^3 a (\lambda^2 r - 1 - 2\lambda^2) - sa \omega^2 \lambda^2 (1 + \lambda^2) (r - 2) \\ &\quad - \lambda \mu [s^2 (1 + \lambda^2 r) - \omega^2 (1 + \lambda^2) \lambda^2 r]. \end{aligned} \quad (4.22)$$

Kreiss condition at $sa \neq \lambda \mu$ requires that (4.20) is uniformly bounded from zero for all $s = \eta + i\tau$ and real ω on $|s|^2 + |\omega|^2 = 1$ with $\eta > 0$. We study (4.20) in the following.

First consider the case $\omega = 0$. We have

$$J_1 = s^3[a(\lambda^2 r - 1 - 2\lambda^2) - \lambda\sqrt{v_1^2 - a^2}(1 + \lambda^2 r)].$$

By (3.24), we always have

$$[a(\lambda^2 r - 1 - 2\lambda^2) - \lambda\sqrt{v_1^2 - a^2}(1 + \lambda^2 r)] < 0,$$

and therefore Kreiss condition is satisfied at $\omega = 0$.

For the case $\omega \neq 0$, since ω appear in J_1 only in the form of ω^2 , we may assume $\omega > 0$. Let $s = y\lambda\omega$, and denote $m = v_1/a$ the Mach number behind the shock front. From the condition 2 in Theorem 1.1, $m > 1$. Then $J_1 \neq 0$ if and only if $J_2 \neq 0$ with $J_1 = \lambda^3\omega^3 a J_2$:

$$\begin{aligned} J_2 = y[y^2(\lambda^2 r - 1 - 2\lambda^2) - (1 + \lambda^2)(r - 2)] \\ - [y^2(1 + \lambda^2 r) - (1 + \lambda^2)r]\sqrt{y^2\lambda^2(m^2 - 1) + (1 + \lambda^2 - \lambda^2 m^2)}. \end{aligned} \quad (4.23)$$

For oblique shock wave satisfying entropy condition $r > 1$ and (3.24), we have

$$w_1 \equiv \frac{(1 + \lambda^2)(r - 2)}{\lambda^2(r - 2) - 1} < 1 < \frac{r(1 + \lambda^2)}{1 + \lambda^2 r} \equiv w_2. \quad (4.24)$$

Here $w_1 > 0$ for $r < 2$ and $w_1 \leq 0$ for $r \geq 2$. The equation $J_2(y) = 0$ can be written as

$$\begin{aligned} J_2(y) \equiv y[\lambda^2(r - 2) - 1](y^2 - w_1) \\ - (1 + \lambda^2 r)(y^2 - w_2)\sqrt{y^2\lambda^2(m^2 - 1) + (1 + \lambda^2 - \lambda^2 m^2)} = 0. \end{aligned} \quad (4.25)$$

The study of (4.25) is carried out in the following three lemmas.

1. For the positive real roots of (4.25), we have

Lemma 4.1 The equation (4.25) has only one positive real solution $y = 1$.

The only real solution $y = 1$ of (4.25) corresponds to the case $s = \lambda\omega$, or equivalently $sa = \lambda\mu$ when a generalized eigenvector needs to be introduced. We will consider this case later.

Proof: Consider the following equation $J_3(Y) = 0$ with $Y = y^2$:

$$\begin{aligned} J_3(Y) \equiv Y \{[\lambda^2(r - 2) - 1]Y - (1 + \lambda^2)(r - 2)\}^2 \\ - [(1 + \lambda^2 r)Y - r(1 + \lambda^2)]^2 [\lambda^2 Y(m^2 - 1) + (1 + \lambda^2 - \lambda^2 m^2)] \\ = Y[\lambda^2(r - 2) - 1]^2 (Y - w_1)^2 \\ - (1 + \lambda^2 r)^2 (Y - w_2)^2 [Y\lambda^2(m^2 - 1) + (1 + \lambda^2 - \lambda^2 m^2)] = 0. \end{aligned} \quad (4.26)$$

Actually $J_3(Y)$ is obtained by taking the difference of two squares of two terms in $J_2(y)$. Obviously, for every root y of $J_2(y)$, $Y = y^2$ is a root of $J_3(Y)$.

Denote $w_0 = \max(w_1, 0)$. For real $y = \eta > 0$, if $\eta^2 \geq w_2$, we have $J_2(y) < 0$. If $\eta^2 \leq w_1$ in the case $w_1 > 0$, we have $J_2(y) > 0$. Consequently, all possible positive real roots y of $J_2(y) = 0$ lie within the interval $(\sqrt{w_0}, \sqrt{w_2})$.

If we can show that $J_3(Y) = 0$ has no root within interval (w_0, w_2) except for $Y = 1$, then $y = 1$ is the only positive real root of $J_2(y) = 0$.

Since $J_3(w_1) < 0$ and $J_3(w_2) > 0$, we compute $J'_3(Y)$ in the interval (w_0, w_2) and obtain

$$J'_3(Y) = [\lambda^2(r-2) - 1]^2(Y - w_1)(3Y - w_1) - (1 + \lambda^2r)(Y - w_2)[(3Y - w_2)\lambda^2(m^2 - 1) + 2(1 + \lambda^2 - \lambda^2m^2)]. \quad (4.27)$$

- If $w_1 > 0$ and $w_1 \geq \frac{1}{3}w_2$, we have

$$J'_3(Y) \geq 0 \text{ in } (w_1, w_2). \quad (4.28)$$

Therefore, $J_3(Y) = 0$ has only one solution $Y = 1$ in (w_0, w_2) .

- If $w_1 > 0$ but

$$w_1 < \frac{1}{3}w_2, \quad (4.29)$$

Then $J'_3(Y) \geq 0$ in $(\frac{1}{3}w_2, w_2)$. If $J'_3(Y) \geq 0$ is not true in the interval $(w_1, \frac{1}{3}w_2)$, let Y_1 be the smallest number in $(w_1, \frac{1}{3}w_2)$ such that $J'_3(Y) \geq 0$. Then we must have $J'_3(Y) < 0$ in $Y_1 - \epsilon < Y < Y_1$.

Since $J'_3(w_1) < 0$ and $J'_3(Y)$ is quadratic in Y , we conclude that $J'_3(Y) < 0$ in (w_1, Y_1) . But $J_3(w_1) < 0$, so $Y = 1$ is the only solution in (w_1, w_2) .

- If $w_1 \leq 0$, then $J_3(w_2) > 0$ and $J_3(0) < 0$. We will show that the equation $J_3(Y) = 0$ has only solution $Y = 1$ in $(0, w_2)$.

Obviously $J'_3(w_2) > 0$. Let Y_2 be the smallest number in $[0, w_2]$ such that $J'_3(Y) \geq 0$ in $[Y_2, w_2]$. If $Y_2 > 0$, $J'_3(Y)$ in $(0, Y_2)$ is monotone increasing of Y . So

$$J'_3(Y) < 0 \text{ in } 0 < Y < Y_2.$$

Since $J_3(0) < 0$, so $J_3(Y) < 0$ in $[0, Y_2]$, and there is no solution $Y \in [0, w_2]$ except $Y = 1$.

This finishes the proof of Lemma 4.1.

2. For the purely imaginary roots of (4.25), we have

Lemma 4.2 The equation (4.25) has no admissible purely imaginary root, i.e., there is no root $y = i\tau$ such that $y_n = \eta_n + i\tau$ with $\eta_n > 0$ satisfies $J_2(y_n) \rightarrow 0$.

Proof: Without loss of generality, assume $\tau > 0$. Let $y_n = \eta_n + i\tau$ with $\eta_n \ll 1$. If $y = i\tau$ is a solution for $J_2(y) = 0$, then we have

$$-\tau^2\lambda^2(m^2 - 1) + (1 + \lambda^2 - \lambda^2m^2) < 0. \quad (4.30)$$

Since $\eta_n \ll 1$, for the imaginary part of the first term in $J_2(y_n)$ in (4.25) we have for some $\epsilon > 0$:

$$\text{Im} \left\{ y_n [\lambda^2(r - 2) - 1] (y_n^2 - w_1) \right\} \geq \epsilon, \quad \forall n. \quad (4.31)$$

On the other hand, since

$$(\eta_n + i\tau)^2 \lambda^2 (m^2 - 1) + (1 + \lambda^2 - \lambda^2 m^2) \quad (4.32)$$

has negative real part and small positive imaginary part, its square root with positive real part must have uniformly positive imaginary part for all n . Consequently, for the imaginary part of the second term of $J_2(y_n)$ in (4.25) we have

$$\text{Im} \left[-(1 + \lambda^2 r) (y_n^2 - w_2) \sqrt{(\eta_n + i\tau)^2 \lambda^2 (m^2 - 1) + (1 + \lambda^2 - \lambda^2 m^2)} \right] \geq \epsilon, \quad \forall n. \quad (4.33)$$

Combining (4.31) and (4.33), we obtain $\text{Im} J_2(y_n) > 2\epsilon, \forall n$. This contradicts the fact that $J_2(y_n) \rightarrow 0$.

The case of $\tau < 0$ can be discussed similarly.

Hence we conclude that $J_2(y) = 0$ has no admissible purely imaginary solution.

3. For the complex roots of (4.25), we have

Lemma 4.3 Equation (4.25) has no complex solution $y = \eta + i\tau$ with real part $\eta > 0$.

Proof: First we assume $\tau > 0$. Denote $y^2 = (\eta + i\tau)^2 = \alpha + i\beta$, then $\beta > 0$.

Since the following discussion concerns only with the sign of real and imaginary parts, we replace Equation (4.25) with the following simplified equation

$$-y(y^2 - w_1) = (y^2 - w_2) \sqrt{y^2 + c^2}. \quad (4.34)$$

By the convention of square root, we have $\sqrt{y^2 + c^2} = a + ib$ with $a, b > 0$. (4.34) is equivalent to two equations

$$\begin{cases} -\eta(\alpha - w_1) + \tau\beta = a(\alpha - w_2) - \beta b, \\ -\eta\beta - \tau(\alpha - w_1) = \beta a + b(\alpha - w_2). \end{cases} \quad (4.35)$$

- Case $\alpha < w_1$: we have

$$-\eta(\alpha - w_1) + \tau\beta > 0, \quad a(\alpha - w_2) - \beta b < 0. \quad (4.36)$$

Hence (4.34) has no solution.

- Case $\alpha > w_2$: we have

$$-\eta\beta - \tau(\alpha - w_1) < 0, \quad \beta a + b(\alpha - w_2) > 0. \quad (4.37)$$

Hence (4.34) has no solution.

- Case $w_1 < \alpha < w_2$: Since the argument of y^2 is always larger than the argument of $y^2 + c^2$, we have

$$\tau/\eta > b/a. \quad (4.38)$$

Eliminating the term $(\alpha - w_1)$ on the left side of (4.35), we obtain

$$\beta(\tau^2 + \eta^2) = (\tau a - b\eta)(\alpha - w_2) - \beta(\tau b + a\eta). \quad (4.39)$$

(4.39) implies $\tau a - b\eta < 0$ which contradicts (4.38). Hence (4.34) has no solution for $\tau > 0$.

If $\tau < 0$, it is easy to see that we have $\beta < 0, b < 0$ in the above discussion. Therefore, in the case of $\alpha < w_1$, (4.36) remains true. In the case of $\alpha > w_2$, both two terms in (4.37) change signs. In the case of $w_1 < \alpha < w_2$, we have

$$\tau/\eta < b/a. \quad (4.38')$$

But (4.39) implies $\tau a - b\eta > 0$ since τ, β, b are all negative. This contradicts (4.38').

This concludes the proof for Lemma 4.3.

4.2 Case II: $sa = \lambda\mu$

In the case $sa = \lambda\mu$, we have $s = \lambda\omega > 0$. Matrix $N(s, i\omega)$ in (4.2) has triple eigenvalue $\xi_1 = \xi_2 = \xi_3 = s\rho v_1 d^2 = \lambda\omega\rho v_1 d^2$. The system (4.4) at this point can be written as $\omega P\mathbf{u} = 0$ with

$$P\mathbf{u} \equiv \begin{pmatrix} -\lambda\rho v_1 a^2 & 0 & -i(\rho\lambda v_1)^2 & -\lambda^2(\rho v_1)^2 \\ \lambda a^4 & 0 & i\lambda^2\rho v_1 a^2 & \lambda^2\rho v_1 a^2 \\ ia^2 d^2 & 0 & 0 & 0 \\ \lambda^2 a^2(a^2 - v_1^2) & 0 & -i\rho\lambda v_1 a^2 & -\lambda\rho v_1 a^2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = 0. \quad (4.40)$$

(4.40) has only two linearly independent solutions α_1 and α_2 :

$$\begin{cases} \alpha_1 = (0, 1, 0, 0)^T, \\ \alpha_2 = (0, 0, 1, -i)^T. \end{cases} \quad (4.41)$$

A generalized eigenvector α'_3 can be found by solving the equation $P\alpha'_3 = \alpha_2$, i.e.,

$$\begin{cases} a^2u_1 + \lambda\rho v_1(iu_3 + u_4) = 0, \\ a^2u_1 + \lambda\rho v_1(iu_3 + u_4) = 0, \\ id^2u_1 = 1, \\ \lambda^2(a^2 - v_1^2)u_1 - \lambda\rho v_1(iu_3 + u_4) = -i. \end{cases} \quad (4.42)$$

System (4.42) is solvable and has a solution of generalized eigenvector

$$(-id^{-2}, 0, a^2(\lambda\rho v_1)^{-1}d^{-2}, 0)^T,$$

which is parallel to

$$\alpha'_3 = (\lambda\rho v_1, 0, ia^2, 0)^T. \quad (4.43)$$

Using (3.18) to compute $\zeta_3 = B\alpha'_3$, we obtain that ζ_3 is parallel to

$$\zeta'_3 = v_1(\lambda v_1, \lambda(v_1^2 + a^2), ia^2, -a^2)^T. \quad (4.44)$$

Noticing $s = \lambda\omega$, we can write the matrix corresponding to (4.18) as follows

$$\begin{pmatrix} 1 & i\omega & \lambda v_1^2 & \lambda\omega(1 + \lambda^2)\rho_+ \\ 2v_1 & i\omega v_1 & \lambda(v_1^2 + a^2)v_1 & \lambda\omega(\rho_+ + \lambda^2\rho)v_1 \\ 0 & \lambda\omega v_1 & ia^2v_1 & i\omega(1 + \lambda^2)\lambda^2\rho v_1 \\ 0 & -i\lambda\omega v_1 & -a^2v_1 & \lambda^2\omega(\rho_+ + \lambda^2\rho)v_1 \end{pmatrix}. \quad (4.45)$$

Eliminating the non-zero factors in (4.45), we see that (4.45) is non-degenerate if and only if

$$\det J' = \det \begin{pmatrix} 1 & 1 & \lambda v_1^2 & (1 + \lambda^2)\rho_+ \\ 2 & 1 & \lambda(v_1^2 + a^2) & (\rho_+ + \lambda^2\rho) \\ 0 & -\lambda & a^2 & \rho\lambda(1 + \lambda^2) \\ 0 & -\lambda & -a^2 & \lambda(\rho_+ + \lambda^2\rho) \end{pmatrix} \neq 0. \quad (4.46)$$

The determinant of J' can be computed as

$$\det J' = \det J'_{11} - 2 \det J'_{21}$$

with

$$J'_{11} = \begin{pmatrix} 1 & \lambda(v_1^2 + a^2) & (\rho_+ + \lambda^2\rho) \\ -\lambda & a^2 & \rho\lambda(1 + \lambda^2) \\ -\lambda & -a^2 & \lambda(\rho_+ + \lambda^2\rho) \end{pmatrix}, \quad J'_{21} = \begin{pmatrix} 1 & \lambda v_1^2 & (1 + \lambda^2)\rho_+ \\ -\lambda & a^2 & \rho\lambda(1 + \lambda^2) \\ -\lambda & -a^2 & \lambda(\rho_+ + \lambda^2\rho) \end{pmatrix}.$$

Therefore we have

$$\begin{aligned}
\det J' &= -\det \begin{pmatrix} a^2 & \rho\lambda(1+\lambda^2) \\ -a^2 & \lambda(\rho_+ + \lambda^2\rho) \end{pmatrix} + \lambda(v_1^2 - a^2) \det \begin{pmatrix} -\lambda & \rho\lambda(1+\lambda^2) \\ -\lambda & \lambda(\rho_+ + \lambda^2\rho) \end{pmatrix} \\
&\quad + [\lambda^2(\rho - \rho_+) - (1 + \lambda^2)\rho_+] \det \begin{pmatrix} -\lambda & a^2 \\ -\lambda & -a^2 \end{pmatrix} \\
&= -\lambda a^2(\rho_+ + 2\rho\lambda^2 + \rho) + \lambda^3(v_1^2 - a^2)(\rho - \rho_+) + 2\lambda a^2[\lambda^2(\rho - \rho_+) - (1 + \lambda^2)\rho_+] \\
&= -\lambda(\rho - \rho_+)[a^2 - \lambda^2(v_1^2 - a^2)] - 4\rho_+ a^2(1 + \lambda^2).
\end{aligned}$$

Since $a^2 - \lambda^2(v_1^2 - a^2) = d^2 > 0$ by (3.15), we obtain

$$\det J' < 0. \tag{4.47}$$

This concludes the proof for the case $sa = \lambda\mu$.

4.3 (3.24) is a necessary condition

It remains to show that the condition (3.24) is necessary for the linear stability of plane oblique shock front.

Since Kreiss condition is stable under perturbation of the coefficients in the problem, the coefficients set which guarantees the energy estimate (3.19) is an open set. To prove the necessity of (3.24), it suffices to show that for

$$r > 2 + \frac{1}{\lambda^2}, \tag{4.48}$$

Kreiss condition is not satisfied. We have the following

Lemma 4.4 For r satisfying (4.48), Equation (4.25) has either a positive real root other than $y = 1$, or an admissible imaginary root.

First of all it is readily checked that (4.48) implies

$$w_1 > w_2 > 1, \tag{4.49}$$

and

$$J_2(0) > 0, \quad J_2(1) = 0, \quad J_2(\sqrt{w_1}) < 0, \quad J_2(\sqrt{w_2}) < 0. \tag{4.50}$$

Rewrite the function $J_3(Y)$ in (4.26) as

$$J_3(Y) = a_2 Y^2 + a_1 Y + a_0. \tag{4.51}$$

We can express the coefficient a_2 as a function of parameter r :

$$\begin{aligned} a_2 = a_2(r) &= [\lambda^2(r-2) - 1]^2 - (1 + \lambda^2 r)^2 \lambda^2 (m^2 - 1) \\ &= b_2 r^2 + b_1 r + b_0. \end{aligned} \quad (4.52)$$

In (4.52), the coefficient b_2 , by (3.15), has the form

$$b_2 = \lambda^4 [1 - \lambda^2 (m^2 - 1)] > 0. \quad (4.53)$$

Therefore for r satisfying (4.48), we have

$$a_2(r) < 0, \quad \text{for } r \sim 2 + \frac{1}{\lambda^2} \quad (4.54)$$

and

$$a_2(r) > 0, \quad \text{for } r \gg 1. \quad (4.55)$$

Consequently, there exists a unique $r_0 \in (2 + 1/\lambda^2, +\infty)$ such that

$$a_2(r_0) = 0. \quad (4.56)$$

1. Case I: $r > r_0$.

In this case, we have $a_2(r) > 0$. Hence for real $y = \eta$

$$\lim_{\eta \rightarrow +\infty} J_2(\eta) = +\infty. \quad (4.57)$$

This means that there exists $y_1 \in (\sqrt{w_1}, +\infty)$ such that $J_2(y_1) = 0$. Therefore Kreiss condition is not satisfied because Equation (4.25) has positive real root other than $y = 1$.

2. Case II: $2 + \frac{1}{\lambda^2} < r < r_0$.

In this case, we claim that Equation (4.25) has an admissible imaginary root $y = i\tau_0$ with $\tau_0 > 0$.

Let $\tau_1 > 0$ be defined such that

$$\tau_1^2 \lambda^2 (m^2 - 1) - (1 + \lambda^2 - \lambda^2 m^2) = 0. \quad (4.58)$$

For $\tau > \tau_1$, we have

$$\begin{aligned} J_2(i\tau) &= -i\tau[\lambda^2(r-2) - 1](\tau^2 + w_1) + (1 + \lambda^2 r)(\tau^2 + w_2) \\ &= -i\tau[\lambda^2(r-2) - 1](\tau^2 + w_1) + iQ(\tau)(1 + \lambda^2 r)(\tau^2 + w_2), \end{aligned} \quad (4.59)$$

where

$$Q(\tau) = \sqrt{\tau^2 \lambda^2 (m^2 - 1) - (1 + \lambda^2 - \lambda^2 m^2)} > 0. \quad (4.60)$$

Since $a_2(r) < 0$ for $2 + \frac{1}{\lambda^2} < r < r_0$, we have

$$\lim_{\tau \rightarrow +\infty} \operatorname{Im} J_2(i\tau) = +\infty. \quad (4.61)$$

On the other hand, we have

$$\operatorname{Im} J_2(i\tau_1) < 0. \quad (4.62)$$

Therefore, there exists $\tau_0 > \tau_1$ such that

$$J_2(i\tau_0) = -i\tau_0[\lambda^2(r-2) - 1](\tau_0^2 + w_1) + i(1 + \lambda^2 r)(\tau_0^2 + w_2)Q(\tau_0) = 0. \quad (4.63)$$

We prove that the imaginary root $i\tau_0$ is admissible, i.e., for $\eta_n > 0$ and $\eta_n \rightarrow 0$,

$$J_2(\eta_n + i\tau_0) \rightarrow 0. \quad (4.64)$$

Indeed we have

$$\begin{aligned} J_2(\eta_n + i\tau_0) &= -i\tau[\lambda^2(r-2) - 1](\tau^2 + w_1) + O(\eta_n) \\ &\quad + (1 + \lambda^2 r)(\tau_0^2 + w_2)\sqrt{(\eta_n + i\tau_0)^2 \lambda^2 (m^2 - 1) + (1 + \lambda^2 - \lambda^2 m^2)}. \end{aligned} \quad (4.65)$$

For small $\eta_n > 0$, the following complex number

$$\begin{aligned} &(\eta_n + i\tau_0)^2 \lambda^2 (m^2 - 1) + (1 + \lambda^2 - \lambda^2 m^2) \\ &= -[(\tau_0^2 - \eta_n^2) \lambda^2 (m^2 - 1) - (1 + \lambda^2 - \lambda^2 m^2)] + 2i\eta_n \tau_0 \lambda^2 (m^2 - 1) \end{aligned} \quad (4.66)$$

has negative real part and positive imaginary part. Hence its square root with positive real part must have positive imaginary part. Therefore

$$\sqrt{(\eta_n + i\tau_0)^2 \lambda^2 (m^2 - 1) + (1 + \lambda^2 - \lambda^2 m^2)} = iQ(\tau_0) + O(\eta_n). \quad (4.67)$$

Substituting (4.67) into (4.65) and letting $\eta_n \rightarrow 0$, we get $J_2(\eta_n + i\tau_0) \rightarrow 0$ by noticing (4.63).

This concludes the proof of Lemma 4.4, and also Theorem 3.1.

5 Appendix

Lemma 5.1 Let $a > 0$ and $c \geq 0$, then for any real numbers b , we have

$$\operatorname{Re}\sqrt{(a+ib)^2+c} \geq a. \quad (\text{A1})$$

Proof: Without loss of generality, we can assume $a = 1$. Writing $(1+ib)^2+c = re^{i2\theta}$, we have

$$r = \sqrt{(1-b^2+c)^2+4b^2}, \quad \cos(2\theta) = \frac{1-b^2+c}{r}. \quad (\text{A2})$$

It is easy to see that (A1) is equivalent to

$$r \cos^2 \theta \geq 1. \quad (\text{A3})$$

Since

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2} = \frac{r + 1 - b^2 + c}{2r},$$

then (A3) is equivalent to

$$r + 1 - b^2 + c \geq 2. \quad (\text{A4})$$

(A4) is true if $r \geq 1 + (b^2 - c)$, or

$$r^2 \geq 1 + (b^2 - c)^2 + 2(b^2 - c). \quad (\text{A5})$$

From (A2), we see (A5) is

$$(1 - b^2 + c)^2 + 4b^2 \geq 1 + (b^2 - c)^2 + 2(b^2 - c). \quad (\text{A6})$$

(A6) is readily checked to be true, since $c \geq 0$.

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