

# Group Connectivity of Complementary Graphs

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**Abstract:** Let  $G$  be a 2-edge-connected undirected graph,  $A$  be an (additive) abelian group and  $A^* = A - \{0\}$ . A graph  $G$  is  $A$ -connected if  $G$  has an orientation  $D(G)$  such that for every function  $b: V(G) \mapsto A$  satisfying  $\sum_{v \in V(G)} b(v) = 0$ , there is a function  $f: E(G) \mapsto A^*$  such that for each vertex  $v \in V(G)$ , the total amount of  $f$  values on the edges directed out from  $v$  minus the total amount of  $f$  values on the edges directed into  $v$  equals  $b(v)$ . For a 2-edge-connected graph  $G$ , define  $\Lambda_G(G) = \min\{k: \text{for any abelian group } A \text{ with } |A| \geq k, G \text{ is } A\text{-connected}\}$ . In this article, we prove the following Ramsey type results on group connectivity:

- (i) Let  $G$  be a simple graph on  $n \geq 6$  vertices. If  $\min\{\delta(G), \delta(G^c)\} \geq 2$ , then either  $\Lambda_G(G) \leq 4$ , or  $\Lambda_G(G^c) \leq 4$ .

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- (ii) Let  $Z_3$  denote the cyclic group of order 3, and  $G$  be a simple graph on  $n \geq 44$  vertices. If  $\min\{\delta(G), \delta(G^c)\} \geq 4$ , then either  $G$  is  $Z_3$ -connected, or  $G^c$  is  $Z_3$ -connected. © 2011 Wiley Periodicals, Inc. J Graph Theory 69: 464–470, 2012

## 1. INTRODUCTION

We consider finite graphs which permit multiple edges but no loops, and refer to [1] for undefined terms and notations. In particular, the minimum degree, the maximum degree, and the edge-connectivity of a graph  $G$  are denoted by  $\delta(G)$ ,  $\Delta(G)$ , and  $\kappa'(G)$ , respectively. If  $G$  is a simple graph, then  $G^c$  denotes the *complement* of  $G$ . For a subset  $X \subseteq V(G)$  or  $X \subseteq E(G)$ ,  $G[X]$  denotes the subgraph of  $G$  induced by  $X$ . Unlike in [1], a 2-regular connected nontrivial graph is called a *circuit*, and a circuit on  $k$  vertices is also referred as a *k-circuit*. Throughout this article,  $A$  denotes an (additive) abelian group with identity 0. For an integer  $m \geq 1$ ,  $Z_m$  denotes the set of all integers modulo  $m$ , as well as the cyclic group of order  $m$ .

Let  $G$  be a graph with an orientation  $D = D(G)$ . For a vertex  $v \in V(G)$ , we use  $E^+(v)$  (or  $E^-(v)$ , respectively) to denote the set of edges with tails (or heads, respectively) at  $v$ . Following [8], define  $F(G, A) = \{f: E(G) \mapsto A\}$  and  $F^*(G, A) = \{f: E(G) \mapsto A - \{0\}\}$ . Given an  $f \in F(G, A)$ , the *boundary* of  $f$  is a map  $\partial f: V(G) \mapsto A$  defined by

$$\partial f(v) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e) \quad \forall v \in V(G),$$

where “ $\sum$ ” refers to the addition in  $A$ .

A map  $b: V(G) \mapsto A$  is called an *A-valued zero sum map* on  $G$  if  $\sum_{v \in V(G)} b(v) = 0$ . The set of all  $A$ -valued zero sum maps on  $G$  is denoted by  $Z(G, A)$ . A graph  $G$  is *A-connected* if  $G$  has an orientation  $D$  such that for every function  $b \in Z(G, A)$ , there is a function  $f \in F^*(G, A)$  such that  $b = \partial f$ . Define

$$\Lambda_g(G) = \min\{k: \text{for any abelian group } A \text{ with } |A| \geq k, G \text{ is } A\text{-connected}\}.$$

An  $f \in F(G, A)$  is an *A-flow* of  $G$  if  $\partial f = 0$ . If an  $A$ -flow  $f \in F^*(G, A)$ , then  $f$  is an *A-nowhere-zero-flow* (abbreviated as an *A-NZF*). When  $A = \mathbf{Z}$  is the group of integers and  $f$  is a  $\mathbf{Z}$ -NZF, if  $\forall e \in E(G)$ ,  $|f(e)| < k$ , then  $f$  is a *nowhere zero k-flow* (abbreviated as a *k-NZF*). It is noted in [8] that for a graph  $G$ , the property of being  $A$ -connected or having an  $A$ -NZF is independent of the choice of the orientation of  $G$ . Moreover, Tutte [18] showed that, for a finite abelian group  $A$ , a graph  $G$  has an  $A$ -NZF if and only if  $G$  has a  $|A|$ -NZF. Tutte ([18, 19], see also [7]) made several fascinating conjectures on the existence of nowhere zero flows. As the nowhere zero flow problem is the corresponding homogeneous case of the group connectivity problem, Jaeger, Linial, Payan, and Tarsi proposed in [8] several conjectures on group connectivity of graphs, which, as suggested by a result of Kochol [9], are stronger than the corresponding conjectures of Tutte.

One of the main problems in the theories of nowhere zero flows is that given a graph  $G$  and an abelian group  $A$ , determine if  $G$  has an  $A$ -NZF. Likewise, in the study of group connectivity of graphs, a natural corresponding problem will be to determine if  $G$  is  $A$ -connected. Jaeger [5, 6] was the first to show that every 2-edge-connected graph has a 8-NZF. He also showed that every 4-edge-connected graph has a 4-NZF. Seymour

in [17] proved the best result by far that every 2-edge-connected graph has a 6-NZF. For group connectivity of graphs, Jeager et al. in [8] improved the above-mentioned results and proved that every 4-edge-connected graph is  $A$ -connected for every abelian group  $A$  with  $|A| \geq 4$ , and that every 3-edge-connected graph is  $A$ -connected for every abelian group  $A$  with  $|A| \geq 6$ .

The famous Ramsey theorem [4] states that for any fixed integer  $k$ , when the order of a simple graph  $G$  is sufficiently large, either  $G$  or  $G^c$  contains  $K_k$  as a subgraph. There have been several Ramsey type results in the literature. In particular, Nebeský [15] proved that if  $n = |V(G)| \geq 5$ , then either  $L(G)$  or  $L(G^c)$  is hamiltonian; and that [16] if  $n = |V(G)| \geq 4$ , then either  $G$  or  $G^c$  has an Eulerian subgraph of order at least  $n-1$ , with an explicitly described class of exceptional graphs. Motivated by these results of Nebeský, Lai proved that (Theorem 1 and Corollary 3 of [10]) if  $n = |V(G)| \geq 61$ , and if  $\min\{\kappa'(G), \kappa'(G^c)\} \geq 2$ , then either  $G$  or  $G^c$  has a spanning Eulerian subgraph, (and so either  $G$  or  $G^c$  has a 4-NZF). Naturally, in this article, we shall investigate the group connectivity of complementary graphs. The main results are the following.

**Theorem 1.1.** *Let  $G$  be a simple graph of order  $n \geq 6$ . If  $\min\{\delta(G), \delta(G^c)\} \geq 2$ , then either  $\Lambda_g(G) \leq 4$  or  $\Lambda_g(G^c) \leq 4$ .*

If  $G = C_5$ , then  $G^c = C_5$ . By Lemma 2.1(ii), we know  $\Lambda_g(G) = \Lambda_g(G^c) = 6$ . Hence,  $n \geq 6$  cannot be relaxed. The condition  $\min\{\delta(G), \delta(G^c)\} \geq 2$  is also necessary in Theorem 1.1. To see that, let  $n \geq 6$  be an integer. Define  $K_{2,n-2}^-$  to be the graph obtained from the complete bipartite graph  $K_{2,n-2}$  by deleting any one edge. Note that  $G^c$  is a connected graph with a vertex of degree 1. Therefore, neither  $G$  nor  $G^c$  can be  $A$ -connected for any nontrivial abelian group  $A$ .

**Theorem 1.2.** *Let  $Z_3$  denote the cyclic group of order 3, and  $G$  be a simple graph on  $n \geq 44$  vertices. If  $\min\{\delta(G), \delta(G^c)\} \geq 4$ , then either  $G$  is  $Z_3$ -connected, or  $G^c$  is  $Z_3$ -connected.*

The next section will present some of the preliminaries on group connectivity of graphs, paving the way for the proofs. Sections 3 and 4 are devoted to the proofs for the two main results.

## 2. PRELIMINARIES

Let  $K_n$  be a complete graph on  $n$  vertices and  $K_n^-$  denote the graph obtained from  $K_n$  by deleting one edge from  $K_n$ . We use  $C_n$  to denote a circuit on  $n$  vertices and  $K_{m,n}$  to denote the complete bipartite graph with two parts of orders  $m$  and  $n$ , respectively, and let  $K_{m,n}^-$  denote the graph obtained from  $K_{m,n}$  by removing an edge.

For any abelian group  $A$ , let  $\langle A \rangle$  be the family of all the  $A$ -connected graphs. The following displays some useful facts on group connectivity of graphs.

**Lemma 2.1.** *Let  $A$  be an abelian group with  $|A| \geq 3$ . Each of the following holds:*

- (i) (Corollary 3.5 of [12])  $K_n, K_n^- \in \langle A \rangle$ , if  $n \geq 5$ .
- (ii) ([8] and Lemma 3.3 of [12])  $\Lambda_g(C_n) = n + 1$ .
- (iii) (Chen et al., Theorem 4.6 of [3])  $\Lambda_g(K_{m,n}) = 3$  if  $m \geq n \geq 4$  and  $\Lambda_g(K_{m,3}) = 4$  for all  $m \geq 3$ .

**Lemma 2.2** (Lai [12]). *Let  $A$  be an abelian group with  $|A| \geq 3$ . Each of the following holds:*

- (C1)  $K_1 \in \langle A \rangle$ .
- (C2) if  $H \subseteq G$  and if  $H, G/H \in \langle A \rangle$ , then  $G \in \langle A \rangle$ .

**Lemma 2.3.** *Let  $A$  denote an abelian group with  $|A| \geq 3$ . Each of the following holds:*

- (i) Let  $v \in V(G)$  with  $d_G(v) \geq 2$ . If  $G - v \in \langle A \rangle$ , then  $G \in \langle A \rangle$ .
- (ii) (Lemma 2.1 of [13]) If  $G$  has a spanning  $A$ -connected subgraph, then  $G$  is also  $A$ -connected.

**Proof.** (i) Let  $G' = G/(G - v)$ . Then  $G'$  is spanned by a 2-circuit. By Lemma 2.1(ii), and since  $|A| \geq 3$ ,  $G' \in \langle A \rangle$ . Since  $G - v \in \langle A \rangle$ , it follows by Lemma 2.2(C2) that  $G \in \langle A \rangle$ . ■

**Lemma 2.4** (Yao [20]). *Let  $m \geq 2$  be an integer. If  $G$  is connected and if every edge of  $G$  lies in a circuit of length at most  $m$ , then  $\Lambda_g(G) \leq m + 1$ .*

A graph  $G$  is *collapsible* if for every vertex subset  $X \subseteq V(G)$  with  $|X| \equiv 0 \pmod{2}$ ,  $G$  has a spanning connected subgraph  $L_X$  such that  $X$  is the set of odd degree vertices in  $L_X$ .

**Lemma 2.5** (Catlin, Lemma 1 of [2]).  *$K_{3,3} - e$  is collapsible.*

**Lemma 2.6** (Theorem 1.5 of [11]). *Let  $G$  be a collapsible graph and let  $A$  be an abelian group with  $|A| = 4$ . Then  $G \in \langle A \rangle$ .*

### 3. PROPERTY OF BEING $A$ -CONNECTED WHEN $|A| \geq 4$

For disjoint vertex subsets  $X, Y \subseteq V(G)$ , let  $E_G[X, Y]$  denote the set of edges in  $G$  with one end in  $X$  and the other one in  $Y$ , and let  $e_G(X, Y) = |E_G[X, Y]|$ , and if the graph  $G$  is implicitly given by the context, we omit the subscript for convenience.

**Lemma 3.1.** *Let  $m \geq n \geq 3$  and  $A$  be an abelian group with  $|A| \geq 4$ . Then the following holds:*

- (i)  $K_n \in \langle A \rangle$ .
- (ii) Let  $m \geq n \geq 3$  be integers. Then, for any matching  $M$  in  $K_{m,n}$  with  $|M| \leq n - 2$ ,  $\Lambda_g(K_{m,n} - M) \leq 4$ .
- (iii) Let  $K_{m,3}$  be a bipartite graph with bipartition  $\{X, Y\}$  and with  $Y = \{y_1, y_2, y_3\}$ . Let  $H = K_{m,3} - \{y_1x_1, y_2x_2\} + y_1y_2$ , where  $x_1, x_2 \in X$ . If  $\delta(H) \geq 2$ , then  $H \in \langle A \rangle$ .

**Proof.** (i) This can be obtained directly from Lemma 2.1(i), (ii) and Lemma 2.2(C2).  
 (ii) By Lemma 2.4 (with  $m = 4$ ),  $\Lambda_g(K_{m,n} - M) \leq 5$ . It suffices to show that for any abelian group  $A$  with  $|A| = 4$ ,  $K_{m,n} - M \in \langle A \rangle$ .

Let  $G = K_{m,n} - M$ . Since  $m \geq n \geq 3$  and  $|M| \leq n - 2$ ,  $G$  contains a subgraph  $H \cong K_{3,3} - e$ . By Lemmas 2.5 and 2.6,  $H \in \langle A \rangle$ . As every edge in  $G/H$  lies in a circuit of length at most 3, by Lemma 2.4,  $G/H \in \langle A \rangle$ . It then follows by Lemma 2.2(C2) that  $G \in \langle A \rangle$ .

(iii) Since  $e_H(y_1, X) = e_H(y_2, X) = m - 1$ ,  $|N_H(y_1) \cap N_H(y_2) \cap X| \geq |X| - 2$ . Since  $\delta(H) \geq 2$ ,  $|N_H(y_1) \cap N_H(y_2) \cap X| = |X| - 2$  (Otherwise,  $X$  contains a vertex of degree one.). Since  $y_1, y_2 \in E(H)$ ,  $H$  contains  $m - 2$  triangles. Then, contracting all of the triangles in  $H$ , we get a  $K_4^-$  (possibly with multiple edges), which is  $A$ -connected by Lemma 2.4. By Lemma 2.2(C2),  $H \in \langle A \rangle$ . ■

**Proof of Theorem 1.1.** Let  $A$  be an abelian group with  $|A| \geq 4$ . It is sufficient to show that either  $G \in \langle A \rangle$  or  $G^c \in \langle A \rangle$ . Suppose to the contrary that

$$\text{neither } G \text{ nor } G^c \text{ is } A\text{-connected.} \tag{1}$$

Note that the Ramsey number  $R(3, 3) = 6$ . Since  $n \geq 6$ ,  $G$  or  $G^c$  contains a subgraph  $K_3$ . Assume  $K_3 \subseteq G$ . By Lemma 3.1(i),  $K_3 \in \langle A \rangle$ . Let  $H$  be a maximal subgraph of  $G$  such that  $H \in \langle A \rangle$ . Let  $X = V(H)$  and  $Y = V(G) - V(H)$ . Then, by Lemma 2.3(i),  $e_G(y, X) \leq 1$  for each  $y \in Y$ . Let  $Y' = \{y \in Y | e_G(y, X) = 1\}$  and  $Y'' = Y - Y'$ .

**Claim A.**  $Y'$  is an independent set in  $G$ , that is  $E(G[Y']) = \emptyset$ .

If not, let  $uv \in E(G[Y'])$ , then  $e_G(\{u, v\}, X) = 2$ . Hence  $G[X \cup \{u, v\}] / H \cong K_3 \in \langle A \rangle$ . By Lemma 2.2(C2),  $G[X \cup \{u, v\}] \in \langle A \rangle$ , contrary to the maximality of  $H$ .

**Claim B.**  $|Y''| = 1$ .

By Claim A and  $\delta(G) \geq 2$ ,  $Y'' \neq \emptyset$ . If  $|Y''| \geq 2$ , choose  $y_1, y_2 \in Y''$  and  $y_3 \in Y$ . Then, in  $G^c$ , the subgraph  $I$  induced by  $X \cup \{y_1, y_2, y_3\}$  has a spanning subgraph isomorphic to  $K_{|X|, 3}^-$ . By Lemma 3.1(ii),  $K_{|X|, 3}^- \in \langle A \rangle$ . By Lemma 2.3(ii),  $I \in \langle A \rangle$ . Since, for any  $y \in Y - \{y_1, y_2, y_3\}$ ,  $e_G(y, X) \leq 1$ ,  $e_{G^c}(y, X) \geq |X| - 1 \geq 2$ . By Lemma 2.3(i),  $G^c \in \langle A \rangle$ , contrary to (1).

**Claim C.**  $|Y'| = 2$ .

Since  $\delta(G) \geq 2$  and  $|Y''| = 1$ ,  $|Y'| \geq 2$ . If  $|Y'| \geq 3$ , then, by Claim A and Lemma 3.1(i),  $G^c[Y'] \cong K_{|Y'|} \in \langle A \rangle$ .

We claim that there exists at most one vertex  $x_0$  in  $X$  with  $e_{G^c}(x_0, Y') \leq 1$ . Suppose to the contrary that  $X$  has two such vertices. Then  $e_{G^c}(X, Y') \leq (|X| - 2)|Y'| + 2$ . This implies a contradiction:

$$(|X| - 2)|Y'| + 2 \geq e_{G^c}(X, Y') \geq |Y'|(|X| - 1) > (|X| - 2)|Y'| + 2.$$

By Lemma 2.3(i),  $G^c[(X - \{x_0\}) \cup Y'] \in \langle A \rangle$ . Since  $e_{G^c}(Y'', X - \{x_0\}) \geq |X| - 1 \geq 2$ , again by Lemma 2.3(i),  $G^c[(X - \{x_0\}) \cup Y] \in \langle A \rangle$ . Since  $\delta(G^c) \geq 2$ ,  $d_{G^c}(x_0) \geq 2$ . Applying Lemma 2.3(i) again, we have  $G^c \in \langle A \rangle$ , contrary to (1). Claim C must hold.

Now assume  $Y' = \{y_1, y_2\}$  and  $Y'' = \{y_3\}$ . Let  $y_1x_1, y_2x_2 \in E(G)$ . Then  $G^c$  has a spanning subgraph  $H$  isomorphic to  $K_{|X|, 3}^- - \{y_1x_1, y_2x_2\} + y_1y_2$ , and so by Lemma 3.1(iii),  $H \in \langle A \rangle$ . By Lemma 2.3(ii),  $G^c \in \langle A \rangle$ , contrary to (1). ■

#### 4. PROPERTY OF BEING A-CONNECTED WHEN $|A| \geq 3$

The following two lemmas will be used in the proof of Theorem 1.2.

**Lemma 4.1.** Let  $B = B_{m,n}$  be a bridgeless bipartite graph with two parts of orders  $m$  and  $n$ , respectively. If  $m \geq n + 4 \geq 8$  and  $|E(B_{m,n})| \geq n(m - 1)$ , then  $B_{m,n} \in \langle Z_3 \rangle$ .

**Proof.** Let  $\{X, Y\}$  be the bipartition of  $V(B_{m,n})$  with  $|X| = m$  and  $|Y| = n$ , and  $X'$  be the set of vertices in  $X$  with degree  $n$ . If  $|X'| < m - n$ , then  $|E(B_{m,n})| \leq (m - n - 1)$

$n+(n+1)(n-1)=n(m-1)-1$ , contrary to the assumption that  $|E(B_{m,n})| \geq n(m-1)$ . Hence  $|X'| \geq m-n \geq 4$ . Thus  $B[X' \cup Y] \cong K_{|X'|,|Y|} \in \langle Z_3 \rangle$ , by Lemma 2.1(iii). Since  $B_{m,n}$  is bridgeless,  $e(x, Y) \geq 2$  for each  $x \in X - X'$ . It follows by Lemma 2.3(i) that  $B_{m,n} \in \langle Z_3 \rangle$ . ■

**Lemma 4.2** (Theorem 3.6 of [14]). *Let  $G$  be a graph with  $n$  vertices. If  $|E(G)| \geq 3n \lceil \log_2 n \rceil / 2$ , then  $G$  has a nontrivial subgraph  $H$  with  $H \in \langle Z_3 \rangle$ .*

**Proof of Theorem 1.2.** Argue by contradiction, assume that

$$G \notin \langle Z_3 \rangle \quad \text{and} \quad G^c \notin \langle Z_3 \rangle. \tag{2}$$

Let  $f(n) = 3n \lceil \log_2 n \rceil / 2$ . Then

$$|E(K_n)| \geq 2f(n) \quad \text{if } n \geq 37. \tag{3}$$

Since  $n \geq 44 > 37$ ,  $|E(G)| \geq f(n)$  or  $|E(G^c)| \geq f(n)$ . By Lemma 4.2,  $G$  or  $G^c$  has a nontrivial subgraph  $H$  such that  $H \in \langle Z_3 \rangle$ . Suppose  $H \subseteq G$  and  $H$  is maximal. Let  $X = V(H)$  and  $Y = V(G) - V(H)$ . Then  $e_G(y, X) \leq 1$  for each  $y \in Y$ . Since  $\delta(G) \geq 4$ ,  $|Y| \geq 4$ .

If  $|Y| < 37$ , then  $|X| = n - |Y| \geq 44 - 36 \geq 8$ . Choose  $Y_1 \subseteq Y$  with  $|Y_1| = 4$ . Then  $I = G^c[X \cup Y_1]$  is spanned by  $J = B_{|X|,|Y_1|}$  such that  $d_J(y) = e_{G^c}(y, X) \geq |X| - 1$ ,  $\forall y \in Y_1$ . Hence  $|E(J)| \geq |Y_1|(|X| - 1)$ . Since  $|X| \geq 8 = |Y_1| + 4$ , by Lemma 4.1,  $J \in \langle Z_3 \rangle$ . By Lemma 2.3(ii),  $I \in \langle Z_3 \rangle$ . As  $\forall y \in Y - Y_1, e_{G^c}(y, X) \geq |X| - 1 \geq 2$ , it follows from Lemma 2.3(i) that  $G^c \in \langle Z_3 \rangle$ , contrary to (2).

Hence we assume that  $|Y| \geq 37$ . By (3),  $G[Y]$  or  $G^c[Y]$  has a subgraph  $H'$  with  $H' \in \langle Z_3 \rangle$ .

Suppose first that  $H' \in G[Y]$ . Let  $Y' = V(H')$ . We claim that  $e_G(X, Y') \leq 1$ . If  $e_G(X, Y') \geq 2$ , then  $G[X \cup Y'] \in \langle Z_3 \rangle$  by Lemmas 2.2(C2) and 2.1(ii), contrary to the maximality of  $H$ . Thus, we must have  $e_G(X, Y') \leq 1$ . Note that  $I' = G^c[X \cup Y']$  is spanned by  $J' \cong K_{|X|,|Y'|}$ . Since  $\min\{|X|, |Y'|\} \geq 5$ ,  $J' \in \langle Z_3 \rangle$  by Lemma 2.1(iii), and so  $I' \in \langle Z_3 \rangle$  by Lemma 2.3(ii). As  $\forall y \in Y - Y', e_{G^c}(y, X) \geq |X| - 1 > 2$ , it follows by Lemma 2.3(i) that  $G^c \in \langle Z_3 \rangle$ , contrary to (2).

Thus, we must assume that  $H' \in G^c[Y]$ . Note that  $\forall y \in Y'$ , as  $e_G(y, X) \leq 1$ ,  $e_{G^c}(y, X) \geq |X| - 1$ . Hence  $e_{G^c}(X, Y') \geq |Y'|(|X| - 1)$ .

We claim that there exists at most one vertex  $x_0$  of  $X$  such that  $e_{G^c}(x_0, Y') \leq 1$ . Suppose that there are two such vertices in  $X$ . Then  $e_{G^c}(X, Y') \leq (|X| - 2)|Y'| + 2$ , leading to a contradiction:

$$(|X| - 2)|Y'| + 2 \geq e_{G^c}(X, Y') \geq |Y'|(|X| - 1) > (|X| - 2)|Y'| + 2,$$

and the claim follows. By the claim,  $\forall x \in X - \{x_0\}, e_{G^c}(x, Y') \geq 2$ , and by Lemma 2.3(i),  $I'' = G^c[(X - \{x_0\}) \cup Y'] \in \langle Z_3 \rangle$ . As  $\forall y \in Y - Y', e_G(y, X) \leq 1$ , we have  $e_{G^c}(y, X - \{x_0\}) \geq |X| - 2 \geq 2$ . Again by Lemma 2.3(i),  $J'' = G^c[(X - \{x_0\}) \cup Y] \in \langle Z_3 \rangle$ . Since  $\delta(G^c) \geq 4$ ,  $e_{G^c}(x_0, V(G) - \{x_0\}) \geq 4$ . It follows once more by Lemma 2.3(i) that  $G^c \in \langle Z_3 \rangle$ , contrary to (2). This proves the theorem. ■

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