# Every $N_2$-locally connected claw-free graph with minimum degree at least 7 is $Z_3$-connected

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**Abstract**

Let $G$ be a 2-edge-connected undirected graph, $A$ be an (additive) abelian group and $A^* = A - \{0\}$. A graph $G$ is $A$-connected if $G$ has an orientation $D(G)$ such that for every function $b : V(G) \mapsto A$ satisfying $\sum_{v \in V(G)} b(v) = 0$, there is a function $f : E(G) \mapsto A^*$ such that for each vertex $v \in V(G)$, the total amount of $f$ values on the edges directed out from $v$ minus the total amount of $f$ values on the edges directed into $v$ equals $b(v)$. Let $Z_3$ denote the group of order 3. Jaeger et al conjectured that there exists an integer $k$ such that every $k$-edge-connected graph is $Z_3$-connected. In this paper, we prove that every $N_2$-locally connected claw-free graph $G$ with minimum degree $\delta(G) \geq 7$ is $Z_3$-connected.

## 1 Introduction

We consider finite graphs which permit multiple edges but no loops, and refer to [2] for undefined terminologies and notations. In particular, the minimum degree, the maximum degree of a graph $G$ are denoted by $\delta(G)$, $\Delta(G)$ respectively. If $G$ is a simple graph, then $G^c$ denotes the complement of $G$. For a subset $X \subseteq V(G)$ or $X \subseteq E(G)$, $G[X]$ denotes the subgraph of $G$ induced by $X$. Unlike in [2], a 2-regular connected nontrivial graph is called a **circuit**, and a circuit on $k$ vertices is also referred as a $k$**-circuit**. Throughout this paper, $A$ denotes an (additive) abelian group with identity 0. For an integer $m \geq 1$, $Z_m$ denotes the set of all integers modulo $m$, as well as the cyclic group of order $m$.

Let $G$ be a graph with an orientation $D = D(G)$. For a vertex $v \in V(G)$, we use $E^+(v)$ (or $E^-(v)$, respectively) to denote the set of edges with tails (or heads, respectively) at $v$. Following [9], define $F(G, A) = \{f : E(G) \mapsto A\}$ and $F^*(G, A) = \{f : E(G) \mapsto A - \{0\}\}$. Given an $f \in F(G, A)$, the **boundary** of $f$ is a map $\partial f : V(G) \mapsto A$ defined by

$$\partial f(v) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e), \ \ \forall v \in V(G),$$

where "$\sum$" refers to the addition in $A$.

A map $b : V(G) \mapsto A$ is called an $A$**-valued zero sum map** on $G$ if $\sum_{v \in V(G)} b(v) = 0$. The set of all $A$-valued zero sum maps on $G$ is denoted by $Z(G, A)$. A graph $G$ is $A$**-connected** if $G$ has an orientation $D$ such that for every function $b \in Z(G, A)$, there is a function $f \in F^*(G, A)$ such that $\partial f = b$. Define

$$\Lambda_g(G) = \min\{k : \text{ for any abelian group } A \text{ with } |A| \geq k, G \text{ is } A\text{-connected}\}.$$

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An $f \in F(G, A)$ is an $A$-**flow** of $G$ if $\partial f = 0$. If an $A$-flow $f \in F^*(G, A)$, then $f$ is an $A$-nowhere-zero flow (abbreviated as an $A$-**NZF**). When $A = \mathbf{Z}$ is the group of integers and $f$ is a $\mathbf{Z}$-NZF, if for $\forall e \in E(G)$, $|f(e)| < k$, then $f$ is a **nowhere-zero** $k$-**flow** (abbreviated as a $k$-**NZF**). It is noted in [9] that for a graph $G$, the property of being $A$-connected or having an $A$-NZF is independent of the choice of the orientation of $G$. Moreover, Tutte [25] showed that, for a finite abelian group $A$, a graph $G$ has an $A$-NZF if and only if $G$ has an $|A|$-NZF. The following conjectures on nowhere-zero flows, were first proposed by Tutte and supplemented by Jaeger.

**Conjecture 1.1** *(Tutte [25], [26], see also [8])*
*(i) Every graph $G$ with $\kappa'(G) \geq 4$ has a 3-NZF.*
*(ii) There exists an integer $k \geq 4$ such that every $k$-edge-connected graph has 3-NZF.*

As the nowhere-zero flow problem is the corresponding homogeneous case of the group connectivity problem, Jaeger, Linial, Payan and Tarsi proposed the following conjectures, which, as suggested by a result of Kochol [10], are stronger than the corresponding conjectures above.

**Conjecture 1.2** *(Jaeger et. al., [9]) Let $G$ be a graph.*
*(i) If $\kappa'(G) \geq 5$, then $\Lambda_g(G) \leq 3$.*
*(ii) There exists an integer $k \geq 5$ such that if $\kappa'(G) \geq k$, then $\Lambda_g(G) \leq 3$.*

Many researchers have been studying these conjectures and a number of results towards these conjectures have been obtained. Steinberg and Younger [23], and independently Thomassen [24] proved that within the family of projective planar graphs, 4-edge-connectedness is sufficient for the existence of a 3-NZF. Lai and Li [14] proved that every 5-edge-connected planar graph $G$ satisfies $\Lambda_g(G) = 3$. Several researchers proved sufficient degree conditions for the existence of a 3-NZF or $Z_3$-connectedness. See [4], [5], [20], [28], and [29], among others. In [17] (see also [13]), it is shown that when the edge connectivity of a simple graph $G$ on $n$ vertices is at least $3\log_2(n)$, then $G$ is $Z_3$-connected. Recent studies also show that among certain triangulated graphs, high edge-connectivity will assure the existence of 3-NZF, or stronger, $Z_3$-connectedness. See [27], [6], [15], among others.

The **line graph** of a graph $G$, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in $G$ are adjacent. For a graph $G$, an induced subgraph $H$ isomorphic to $K_{1,3}$ is called a **claw** of $G$, and the only vertex of degree 3 of $H$ is called the **center** of the claw. A graph $G$ is **claw-free** if it does not have an induced subgraph isomorphic to $K_{1,3}$. Beineke ([1]) and Robertson ([21] and [7]) showed that every line graph is also a claw-free graph.

**Theorem 1.3** *Let $G$ be a graph and let $L(G)$ be the line graph of $G$.*
*(i) (Corollary 1.5 of [16]) Every line graph of a 4-edge-connected graph is $Z_3$-connected.*
*(ii) (Theorem 3.1 of [12]) Every 2-edge-connected, locally 3-edge-connected graph is $Z_3$-connected.*
*(iii) ([19]) Every 5-edge-connected graph is $Z_3$-connected if and only if every 5-edge-connected line graph is $Z_3$-connected.*

These recent researches motivate the current project. We are to investigate which families of claw-free graphs in which[1] certain connectivity property along with would imply $Z_3$-connectedness.

In [22], Ryjáček introduced the $N_2$-locally connected graphs. Let $G$ be a graph. Denoted by $N(v, G) = \{z \in V(G) : vz \in E(G)\}$ be the neighborhood of $v$ in $G$. For notational convenience, we shall also use $N(v, G)$ to denote the subgraph of $G$ induced by $N(v, G)$. When the context is clear, we can write $N(v)$ for abbreviation. Let $N_2(v, G)$ be the edge subset $\{e = xy \in E(G) : v \notin \{x, y\}$ and $\{x, y\} \cap N(v) \neq \emptyset\}$.

[1] I am not sure whether "in which" is needed here

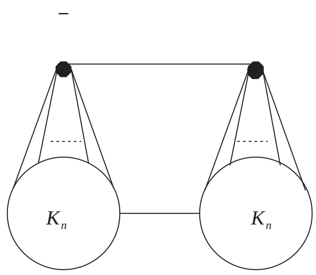


Figure 1: An $N_2$-locally connected claw-free graph with $\kappa'(G) = 2$.

A vertex $v$ is **$N_2$-locally connected** if the induced subgraph $G[N_2(v)]$ is connected; and $G$ is called **$N_2$-locally connected** if every vertex of $G$ is $N_2$-locally connected. It follows from the definitions that every locally connected graph is $N_2$-locally connected.

A result related to Hamilton connectivity of $N_2$-locally connected is as following:

**Theorem 1.4** *(Theorem 1.4 of [18]) Every 3-connected $N_2$-locally connected claw-free graph is Hamiltonian.*

The condition that a graph is $N_2$-locally connected does not imply high edge connectivity. Consider the graph $G$ shown in Figure 1, where each $K_n$ represents a complete graph on $n$ vertices. Then $G$ is an $N_2$-locally connected claw-free graph with $\kappa'(G) = 2$.

Our main result of this paper can be stated as follows.

**Theorem 1.5** *Every $N_2$-locally connected claw-free graph with $\delta(G) \geq 7$ is $Z_3$-connected.*

In Section 2, we present some of the preliminaries that will be needed in the proofs. The last section is devoted to the proof of the main theorem.

# 2 Preliminaries

Let $G$ be a graph and $X \subseteq E(G)$. The **contraction** $G/X$ is the graph obtained by identifying two ends of each edge in $X$ and then deleting the resulting loops. If $H$ is a subgraph of $G$, $G/H$ is the graph $G/E(H)$.

**Theorem 2.1** *(Proposition 3.2 of [11]) For any Abelian group $A$, $\langle A \rangle$ is a family of connected graphs satisfying each of the following:*
*(C1) $K_1 \in \langle A \rangle$,*
*(C2) if $e \in E(G)$ and if $G \in \langle A \rangle$, then $G/e \in \langle A \rangle$, and*
*(C3) if $H \in \langle A \rangle$ and if $G/H \in \langle A \rangle$, then $G \in \langle A \rangle$.*

Let $C_n$ denote the $n$-circuit, and $K_n$ denote the complete graph on $n$ vertices. We have the following result.

**Theorem 2.2** *([9], Proposition 3.2 of [12]) Let $G$ be a graph and $A$ be an Abelian group with $|A| \geq 3$. Then $\langle A \rangle$ satisfies each of the following:*
*(i) (Lemma 3.3 of [11]) $\Lambda_g(C_n) = n + 1$.*
*(ii) (Corollary 3.5 of [11]) Let $n \geq 5$ be an integer. Then $K_n \in \langle A \rangle$.*

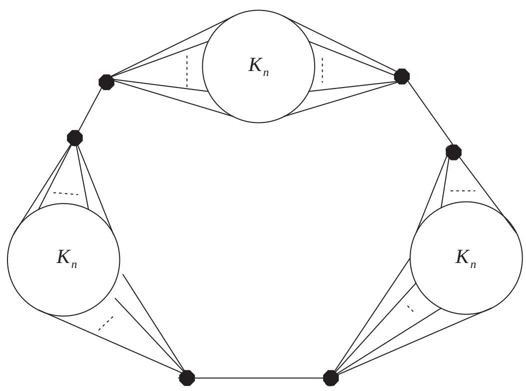


Figure 2: Figure for Example 1.

Next, we will give an example that shows the condition $N_2$-locally connected in Theorem 1.5 is necessary. We need another theorem. In [12], it is shown that for every Abelian group $A$, every graph $G$ has a unique subgraph $M_A(G)$ such that each component of $M_A(G)$ is a maximally $A$-connected subgraph of $G$. The contraction $G/M_A(G)$ is **the $A$-reduction** of $G$.

**Theorem 2.3** *(Corollary 2.3 of [12]) Let $G$ be a graph. Then each of the following holds.*
*(i) $G \in \langle A \rangle$ if and only if $G/M_A(G) \cong K_1$.*
*(ii) $G/M_A(G)$ does not have nontrivial subgraph that is $A$-connected.*

**Example 1** Let $G$ be the graph shown in Figure 2. Each $K_n$ in Figure 2 represents a complete graph with $n \geq 6$. Then $G$ is a claw-free graph with $\delta(G) \geq 7$, and $G$ is not $N_2$-locally connected. By Theorem 2.2, $K_n$ and $C_2$ is $Z_3$-connected. After we contract $K_n$ and $C_2$ successively, the resulting graph is a $C_3$. Since $C_3$ is not $Z_3$-connected, by Theorem 2.3(i), $G$ is not $Z_3$-connected.

# 3 Proof of the Main Theorem

**Lemma 3.1** *Let $G$ be a nonempty claw-free graph with $\delta(G) \geq 2$, and for any $v \in V(G)$, let $H = G[N(v)]$ denote the subgraph of $G$ induced by $N(v)$. Then $N(v)$ can be partitioned into $V_1$ and $V_2$ such that $G[V_1]$ and $G[V_2]$ are complete subgraphs.*

**Proof:** Let $H^c$ be the complement of $H = G[N(v)]$. And $|N(v)| \geq 2$ by $\delta(G) \geq 2$. If $E(H^c) = \emptyset$, then $G[N(v)]$ is a clique. Any partition $(V_1, V_2)$ of $N(v)$ has the property that $G[V_i]$ $(i = 1, 2)$ is a complete graph. If $E(H^c) \neq \emptyset$, since $G$ is a claw-free graph, every path in $H^c$ has length at most 1. Thus $H^c$ is the union of disjoint edges (and some isolated vertices). Let $V_1$ denote the vertex set that contains exactly one end of these disjoint edges, and let $V_2 = N(v) - V_1$. Then the subgraphs induced by $V_1$ and $V_2$ in $H^c$ are independent sets in $H^c$, and so $G[V_1]$ and $G[V_2]$ are both complete graphs. ■

Since $G$ is a claw-free graph with $\delta(G) \geq 2$, by Lemma 3.1, for any $v \in V(G)$, the subgraph $H = G[N(v)]$ induced by $N(v)$ contains two edge-disjoint cliques as subgraphs. Since $G$ is $N_2$-locally connected, we can classify $v$ into the following two types.

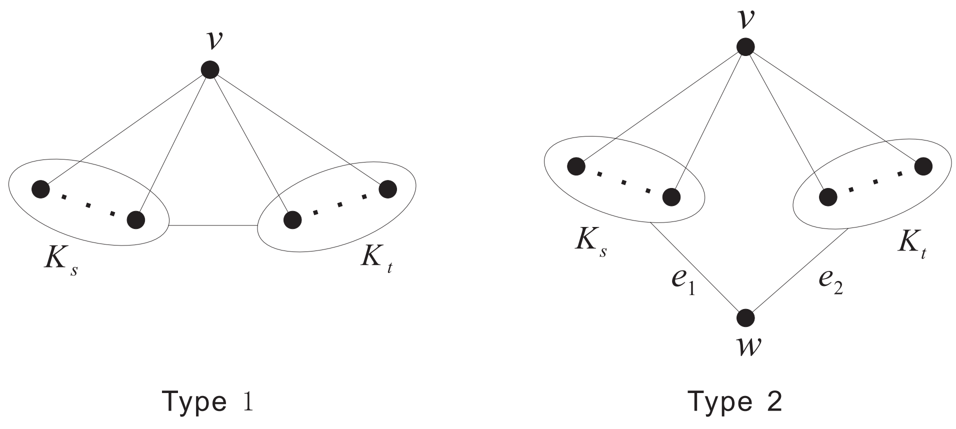


Figure 3: Two types of vertex $v$.

**Type 1**: Two cliques of $H$ are connected in the induced graph $G[N(v)]$ (see Figure 3).

**Type 2**: Two cliques of $H$ are disconnected in the induced graph $G[N(v)]$ (see Figure 3).

If $v$ is of Type 1, let $Q_v = G[N(v) \cup \{v\}]$ be the subgraph induced by $N(v) \cup \{v\}$ in $G$. If $v$ is of Type 2, let $Q_v$ be the subgraph induced by $N(v) \cup \{v, w\}$ in $G$ where $w \in V(G)$ is a vertex which is adjacent to both $K_s$ and $K_t$. Note that $w$ has neighbors in each of the two different cliques of $H$.

Let $G'$ be the $A$-reduction of $G$. By Theorem 2.3(ii), $G'$ does not have nontrivial subgraph that is $Z_3$-connected. By the definition of contraction, $E(G') \subseteq E(G)$. For any $v \in V(G')$, $G$ has a maximal $Z_3$-connected subgraph $H_v$ such that $v$ is the vertex in $G'$ onto which $H_v$ is contracted. We call $H_v$ **the preimage of** $v$.

**Lemma 3.2** *Let $G$ be an $N_2$-locally connected claw-free graph with $\delta(G) \geq 7$, and let $A = Z_3$. If $v$ is a vertex of Type 1, then $E(Q_v) \subset E(M_A(G))$, and so $E(Q_v) \cap E(G') = \emptyset$.*

**Proof:** Suppose that $v$ is of Type 1. Denote the two adjacent complete graphs in $G[N(v)]$ by $K_s$ and $K_t$ with $s \geq t$, and let $e = uu'$ be an edge joining $K_s$ and $K_t$, with $u \in V(K_s)$ and $u' \in V(K_t)$. As $\delta(G) \geq 7$, $s \geq 4$ (see Figure 3 for an illustration). Let $H' = H[V(K_s) \cup \{u', v\}]$ and let $H_1 = H[V(K_s) \cup \{v\}]$. Since $H_1 \cong K_{s+1}$ with $s + 1 \geq 5$, it follows by Theorem 2.2 (ii) that $H_1$ is $Z_3$-connected. Since $H'/H_1$ is a 2-circuit, by Theorem 2.2(i) that $H'/H_1$ is also $Z_3$-connected. Hence by Theorem 2.1(C3) that $H'$ is $Z_3$-connected. By the definition of Type 1 vertices, and since $v$ is of Type 1, every vertex of $Q_v/H'$ lies in a 2-circuit, and so by Theorem 2.2(i), $Q_v/H'$ must be $Z_3$-connected. Since $H'$ is $Z_3$-connected, it follows by Theorem 2.1(C3) that $Q_v$ is $Z_3$-connected. Thus $E(Q_v) \subset E(M_A(G))$, and so by the definition of contraction, $E(Q_v) \cap E(G') = \emptyset$. ■

Next, we shall prove the main theorem.

**Proof of Theorem 1.5:** Let $G$ be an $N_2$-locally connected claw-free graph with $\delta(G) \geq 7$. Let $G' = G/M_A(G)^2$ denote the $Z_3$-reduction of $G$. By Theorem2.3, if we can prove $G' \cong K_1$, then we have $G$ is $Z_3$-connected.

[2]in the last version, here is $M(G)$, and so on in the proof of this theorem. And I changed all of them to $M_A(G)$. Seven places together

We prove by way of contradiction. Suppose that $G'$ has an edge $e$. Then $e = uv \in E(Q_v)$ for some vertices $u, v \in V(G)$ as $E(G') \subseteq E(G)$. By Lemma 3.2, vertex $v$ cannot be of Type 1 in $G$, as in this case $E(Q_v) \cap E(G') = \emptyset$. Hence $v$ is of Type 2. Let the two nonadjacent complete graphs be $K_s$ and $K_t$ with $s \geq t$ (see Figure 3 for an illustration). Since $G$ is $N_2$-locally connected, there is a vertex $w$ connecting to both $K_s$ and $K_t$ via two edges $e_1 = wx_1$ and $e_2 = wx_2$, where $x_1 \in V(K_s)$ and $x_2 \in V(K_t)$. As $\delta(G) \geq 7$, $s \geq 4$. Then the subgraph $H_1$ induced by $V(K_s) \cup \{v\}$ is isomorphic to $K_{s+1}$. Since $s+1 \geq 5$, by Theorem 2.2 (ii), $H_1$ is $Z_3$-connected, and so by the definition of $G'$, $E(H_1) \cap E(G') = \emptyset$.

To find a contradiction, it suffices to show that $E(Q_v) \subseteq E(M_A(G))$, as this will imply that $E(Q_v) \cap E(G') = \emptyset$, contrary to the assumption that $e = uv \in E(Q_v) \cap E(G')$.

We first claim that $e_1, e_2 \notin E(M_A(G))$. If, to the contrary, that one of $e_i$ (say $e_1$) is in $E(M_A(G))$, then $M_A(G)$ has a maximal $Z_3$-connected subgraph $N$ which contains $E(H_1) \cup \{e_1\}$. Let $L = Q_v \cup N$ denote the subgraph of $G$ induced by $E(Q_v) \cup E(N)$, and let $L_1 = L[E(H_1) \cup \{e_1, e_2, vx_2\} \cup E(N)]$ be a subgraph of $L$. Since $E(H_1) \cup \{e_1\} \subseteq E(N)$, $L_1/N$ is a 2-circuit consisting of edges $\{e_2, vx_2\}$, and so by Theorem 2.2 (i), $L_1/N$ is $Z_3$-connected. As $N$ is $Z_3$-connected, by Theorem 2.1(C3), $L_1$ is also $Z_3$-connected. Since $N$ is a maximal $Z_3$-connected subgraph of $G$, we must have $E(L_1) \subseteq E(N)$.

It now follows by $E(L_1) \subseteq E(N)$, every vertex in $L/L_1$ lies in a 2-circuit, and so by Theorem 2.2 (i), $L/L_1$ is $Z_3$-connected. As $L_1$ is also $Z_3$-connected, it follows by Theorem 2.1(C3) that $L$ is also $Z_3$-connected. This proves that $E(L) \subseteq E(M_A(G))$. In particular, $E(Q_u) \subseteq E(L) \subseteq E(M_A(G))$, a contradiction obtains.

This contradiction implies the theorem. ■

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