Chapter 1

Set Theory

This notes has two parts, theorems and definitions that are needed for the proves, the solution of all most all old entrance exams in WVU math dept. by 2009 and some homework questions in Rudin (v3) as well. Don't trust all of them as I made some mistakes when I was doing it. The comments are feedback from Dr. Lai. You are welcome to discuss with me. —- Ye

1.1 Notes

1.1.1 Ordered pairs: *$< x, y >$ is an ordered pair if we distinguish between first and second element. Thus $< x, y > \neq < y, x >$ if $x \neq y$.*

1.1.2 Cartersian product (direct product): *X and Y are two sets, the Cartersian product $X \times Y$ is the set x, y of all ordered pairs whose first element is from X and the second element is from Y.*

1.1.3 Finite {infinite} sequence: *a function whose domain is first n natural numbers {the set N of natural numbers}.*

1.1.4 Sequence: *finite or infinite sequence.*

1.1.5 Countable set: *a set is countable if it is the range of a sequence.*

Proposition 1.1.6 *The union of a countable collection of countable set is countable.*

Proposition 1.1.7 *The set of all rational numbers is countable.*

Proposition 1.1.8 *The set of all real numbers is not countable.*
Proof: 1 *It is suffices to prove that the subset $[0, 1]$ of real numbers is not countable.*
If $[0, 1]$ is countable, let f be an arbitrary function that $f : N \longrightarrow [0, 1]$. Then we can write down $[0, 1]$ as a sequence a_i such that $a_i = f(i)$, $i \in N$. Also, we can write down $a_i = 0.b_{i1}b_{i2}b_{i3}\dots$. Let $c_i = 1 - 0.b_{ii}$ for all $i \in N$, and $c = 0.c_1c_2c_3\dots$. Then $f(i) \neq c$ for $\forall i \in N$.
Need to add the explanation why $f(i) \neq c$, and need to complete the proof.
Let $c_i = 9 - b_{ii}$ for all $i \in N$, and $c = 0.c_1c_2c_3\dots$. Then $f(i) \neq c$ for $\forall i \in N$, else if there $\exists k$ such that

$f(k) = a_k = 0.b_{k1}b_{k2}\dots b_{kk}\cdots = c$, then $b_{kk} = c_k$. But $c_k = 9 - b_{kk}$, that leads to $b_{kk} = 4.5$, contradicts to the require that $b_{ij} \in Z^+$.

Proof: 2 *If R is countable, then it could write down as a sequence $\{a_1, a_2, a_3, \dots\}$. And we could find a covering of R like $\bigcup_n [a_n - \frac{1}{2^n}]$. And the length of this cover is at most 1, which is a contradiction.*

The notation of intervals are not right. It should be $B_n = [a_n, a_n + \frac{1}{2^n}]$ or $B_n = [a_n - \frac{1}{2^n}, a_n + \frac{1}{2^n}]$. To complete the proof, need to add the explanation of $R \subseteq \cup_n B_n$, and $|R| \leq \sum |B_n|$.

1.1.9 De Morgan's Laws:

$$\sim [\bigcup_{A \in C} A] = \bigcap_{A \in C} \widetilde{A}$$

$$\sim [\bigcap_{A \in C} A] = \bigcup_{A \in C} \widetilde{A}$$

1.1.10 Algebra of sets: *a collection $\mathscr{A}$ of subsets of X is called an algebra or Boolean algebra if (i) $A \cup B \in \mathscr{A}$, and (ii) $\sim A \in \mathscr{A}$ for any element A, B in $\mathscr{A}$.*

1.1.11 σ-algebra (Borel field): *an algebra $\mathscr{A}$ of sets is call a σ-algebra if every countable collection of sets in $\mathscr{A}$ is again in $\mathscr{A}$.*

1.1.12 Equivalence relation: *a relation that is transitive, reflexive, and symmetric on X is said to be an equivalence relation.*

1.1.13 Partial ordering: *a relation $\prec$ id said to be partial ordering of a set X if it is transitive and anti-symmetric on X. (eg. $\subset$ is a partial ordering on $\mathscr{P}(X)$.)*

1.1.14 Linear ordering: *a set X is said to be linear ordering if for any two elements x and y, we have either $x < y$ or $y < x$.*

1.1.15 First element: *an element $a \in E$ is the first element if, whenever $x \in E$ and $x \neq a$, we have $x < a$.*

1.1.16 Minimal element: *an element $a \in E$ is the minimal element of E if there is no $x \in E$ with $x \neq a$ and $x < a$.*

1.1.17 Well ordering: *a strict linear ordering $<$ on a set X is called a well ordering for X or is said to well order X if every nonempty subset of X contains a first element.*

Well-Ordering Principle: Every set X can be well ordered; that is, there is a relation $<$ well orders X.

A well-ordered set X is useful for constructing examples.

Chapter 2

The Real Number System

2.1 Notes

2.1.1 Upper bound: *If S is a set of real numbers, we say that b is an upper bound for S if for each $x \in S$, we have $x \leq b$.*

2.1.2 Least upper bound: *A number a is called a least upper bound for S if it is an upper bound for S, and for each upper bound b, we have $c \leq b$. We denote the least upper bound by $supS$.*

2.1.3 Completeness Axiom: *Every nonempty set S of real numbers which has an upper bound has a least upper bound.*

2.1.4 Axiom of Archimedes: *Given any real number x, there is an integer n such that $x < n$.*

Corollary 2.1.5 *Between any two real numbers is a rational; that is if $x < y$, then there is a rational r with $x < r < y$.*

Can you show also Corollary 2.1.5 by using the axioms and definitions?

Proof: By the Axiom of Archimedes, for the real number $1/(y - x)$, there is an integer b such that $1/(y - x) < b$. Now consider the set $\{n\}$ such that $\{n/b\}$ are rational numbers greater than y. By Axiom of Archimedes, the real number yb must have an integer n such that $yb < n$. That is equal to $y < n/b$, then the set $\{n\}$ is nonempty. By the Completeness Axiom, there must be a k such that k/b is the smallest one greater than y. Also, we have $(k-1)/b < y \leq k/b$ and $x = y-(y-x) < k/b-1/b = (k-1)/b < y$. Then $r = (k-1)/b$ is a rational such that $x < r < y$.

2.1.6 Extended real numbers: *Extension of real numbers by adding two elements ∞ and $-\infty$.*

2.1.7 Limit of a sequence: *Let $\{x_n\}$ be a sequence of real numbers. A real number l is the limit of $\{x_n\}$ if for any positive ϵ, there exist an N such that for any $n > N$ we have $x_n - l < \epsilon$.*

2.1.8 Cauchy sequence: *A sequence $\{x_n\}$ is a Cauchy sequence if for any $\epsilon > 0$, there exist an N such that for any n, m greater than N we have $|x_n - x_m| < \epsilon$.*

Theorem 2.1.9 Cauchy Criterion: *A sequence of real numbers converges if and only if it is a Cauchy sequence.*

Proof: "$\Longrightarrow$": If a sequence $\{x_n\}$ converges to l, then for any $\epsilon > 0$, there exist an N such that for any $n, m > N$, we have $|x_n - x_m| = |x_n - l + l - x_m| \leq |x_n - l| + |x_m - l| < 2\epsilon$. So $\{x_n\}$ is a Cauchy sequence.

"$\Longleftarrow$": If $\{x_n\}$ is a Cauchy sequence, then for any $\epsilon > 0$, there exists an N such that for any $n, m > N$ such that $|x_n - x_m| < \epsilon$. Then if we fix m, (**To indicate that m is fixed, it is much more clear to let $m = N + 1$.**) $x_m - \epsilon < x_n < x_m + \epsilon$ for any $n > N$. That means $\{x_n\}$ is bounded.

Suppose it is bounded by $[-M, M]$, then for $\epsilon_1 = 1$, at least one intervals $[x_i - \epsilon_1, x_i + \epsilon_1]$, $i = 1, 2, \ldots$ has infinitely many elements from $\{x_n\}$, let this interval be $[a_1, b_1]$. Let $\epsilon_2 = \frac{1}{2}$, then at least one interval $[x_i - \epsilon_2, x_i + \epsilon_2] \cap [a_1, b_1]$ contains infinitely many elements from $\{x_i\}$, let this interval $[x_i - \epsilon_2, x_i + \epsilon_2] \cap [a_1, b_1]$ be $[a_2, b_2]$. Continue such steps, let $\epsilon_n = \frac{1}{2^n}$, then we can get interval $[a_n, b_n]$ which contains infinitely many elements from $\{x_i\}$. Now we have $a_1 \leq a_2 \cdots \leq a_n \cdots \leq b_n \cdots \leq b_2 \leq b_1$, and $\lim b_n - a_n = 0$, by the nested intervals principle, there exist only one element $\xi \in [a_n, b_n]$ for all n, and that $\lim a_n = \lim b_n = \xi$.

Since for $\forall \epsilon$, there are infinitely many elements inside $[\xi - \epsilon, \xi + \epsilon]$, then there is a subsequence $\{x_{N_i}\}$ from $\{x_i\}$ converges to ξ. That means for $\forall \epsilon > 0$, $\exists N'$, such that for any $N_i > N'$, we have $|x_{N_i} - \xi| < \epsilon$. Since $\{x_i\}$ is a Cauchy sequence, then for $\forall \epsilon > 0$, $\exists N'' > 0$, such that for any $n, m > N''$, we have $|x_n - x_m| < \epsilon$. Let $N = max\{N', N''\}$, for any $n, N_i > N$,

$$|x_n - \xi| = |x_n - x_{N_i} + x_{N_i} - \xi| \leq |x_n - x_{N_i}| + |x_{N_i} - \xi| < 2\epsilon.$$

Then $x_n \longrightarrow \xi$.

Every claim must have a reason in the proof, and the reason is either a definition, or an axiom, or some established results. Here you need to add the explanation why there is such a subsequence, and why there are infinitely many points in the interval.

Suppose it is bounded by $[-M, M]$, $M \in Z^+$. Then for $\epsilon_1 = 1$, at least one interval $[d, d + 1]$, $d \in \{-M, -M + 1, \ldots M - 1\}$ has infinitely many elements from $\{x_n\}$. Else if all $[d, d + 1]$ have finitely many elements from $\{x_n\}$, since $[-M, M]$ has $2M$ such intervals which is also finite, then $[-M, M]$ contains only finite elements from $\{x_n\}$, which is a contradiction to $\{x_n\}$ is bounded by $[-M, M]$, let this interval be $[a_1, b_1]$ and choose one element inside this interval be x_{N_1}.

Next, let $\epsilon_2 = 1/2$, consider the elements in $[a_1, b_1]$. Then at least one of the intervals $[a_1, a_1 + 1/2]$ and $[a_1 + 1/2, b_1]$ must have infinitely elements from $\{x_n\}$ since $[a_1, b_1]$ contains infinitely elements from $\{x_n\}$. Let this interval be $[a_2, b_2]$ and choose one element which is not equal to x_{N_1} inside this interval be x_{N_2}.

Continue such steps, on the k-th step, let $\epsilon_k = 1/2^k$, then at least one interval of $[a_{k-1}, a_{k-1} + 1/2^k]$ and $[a_{k-1} + 1/2^k, b_{k-1}]$ has infinitely many elements from $\{x_n\}$, let this interval be $[a_k, b_k]$ and also get an element x_{N_k}. Then we have $a_1 \leq a_2 \cdots \leq a_n \cdots \leq b_n \cdots \leq b_2 \leq b_1$, and $\lim b_n - a_n = 0$, by the nested intervals principle, there exists only one element $\xi \in [a_n, b_n]$ for all n, and that $\lim a_n = \lim b_n = \xi$.

Also for $\epsilon = 1/2^k$, let $N' = k + 1$, then for any $N_i > N'$, we have $|x_{N_i} - \xi| < \epsilon$. That means $x_{N_k} \to \xi$.

Since $\{x_i\}$ is a Cauchy sequence, then for $\forall \epsilon > 0$, $\exists N'' > 0$, such that for any $n, m > N''$, we have $|x_n - x_m| < \epsilon$. Let $N = max\{N', N''\}$, for any $n, N_i > N$,

$$|x_n - \xi| = |x_n - x_{N_i} + x_{N_i} - \xi| \leq |x_n - x_{N_i}| + |x_{N_i} - \xi| < 2\epsilon.$$

Then $x_n \longrightarrow \xi$.

2.1.10 Cluster point: *We say that l is a cluster point of $\{x_n\}$ if for any $\epsilon > 0$ and N, there exist an $n > N$ such that $|x_n - l| < \epsilon$.*

2.1.11 Limit superior, limit inferior:

$$\overline{\lim}\, x_n = \inf_n \sup_{k\geq n} x_k$$

$$\underline{\lim}\, x_n = \sup_n \inf_{k\geq n} x_k$$

Proposition 2.1.12 $\overline{\lim}\, x_n = l$ *iff i) for $\forall\epsilon > 0$, $\exists n$ such that $x_k < l + \epsilon$ for all $k > n$; ii) for $\forall\epsilon > 0$ and any n, $\exists k \geq n$ such that $x_k > l - \epsilon$.*

Proof: By definition, $\overline{\lim}\, x_n = \inf_n \sup_{k\geq n} x_k$. Let $s_n = \sup_{k\geq n} x_k$, then $\overline{\lim}\, x_n = \inf_n s_n = l \Longleftrightarrow$ i) For $\forall\epsilon > 0$, $\exists n$ such that $s_n < l + \epsilon$. ii) $\Longleftrightarrow$ For $\forall\epsilon > 0$, $\exists n$ such that $x_k < l + \epsilon$ for all $k > n$.

Proof of Prop 2.1.12: The second iff is not correct. Just apply definition of the upper limit.

By definition, $\overline{\lim}\, x_n = \inf_n \sup_{k\geq n} x_k$. Let $s_n = \sup_{k\geq n} x_k$, then $\overline{\lim}\, x_n = \inf_n s_n = l \Longleftrightarrow$ i) For $\forall\epsilon > 0$, $\exists n$ such that $s_n < l + \epsilon$. That means $x_k < l + \epsilon$ for $\forall k > n$. Also by the definition of limit superior, for $\forall\epsilon > 0$ and any n, $\exists k \geq n$ such that $x_k > l - \epsilon$. Since if it is not true, there $\exists\epsilon_0 > 0$ and $N \in Z^+$, and for any $k > N$, we have $x_k < 1 - \epsilon_0$, but this leads a contradiction that $\overline{\lim}\, x_n < l - \epsilon_0 < l$.

2.1.13 Open set: *A set O of real numbers is called open if for $\forall x \in O$, there $\exists\delta > 0$ such that each y with $|x - y| < \delta$ belongs to O.*

Proposition 2.1.14 *The intersection of any finite open set is open.*

Proof: *It suffices to prove that for any intersection of two open sets is open. Let O_1 and O_2 be any two open sets. By definition, for $\forall a_1 \in O_1$ and $\forall a_2 \in O_2$, there $\exists\delta_1, \delta_2 > 0$ such that $(a_1 - \delta_1, a_1 + \delta_1) \subset O_1$ and $(a_2 - \delta_2, a_2 + \delta_2) \subset O_2$. Then for $\forall a \in O_1 \cap O_2$, choose $\delta = \min\{\delta_1, \delta_2\}$, then we have $(a - \delta, a + \delta) \subset O_1 \cup O_2$.*

Proposition 2.1.15 *The union of any collection $\mathscr{C}$ of open sets is open.*

Proof: *For $\forall a \in \bigcup_{O\in\mathscr{C}} O$, a must be in at least one open set O. Then there $\exists\delta > 0$, such that $(a - \delta, a + \delta) \subset O \subset \bigcup_{O\in\mathscr{C}} O$. Thus $\bigcup_{O\in\mathscr{C}} O$ is open.*

Proposition 2.1.16 *Every open set of real numbers is the union of a countable collection of disjoint open intervals.*

Proposition 2.1.17 Lindelof: *Let $\mathscr{C}$ be a collection of open sets of real numbers. Then there is a countable subcollection $\{O_i\}$ of $\mathscr{C}$ such that*

$$\bigcup_{O\in\mathscr{C}} O = \bigcup_{i=1}^{\infty} O_i.$$

2.1.18 Point of closure: *A real number x is called a point of closure of a set E if for every $\delta > 0$ there is a $y \in E$ such that $|x - y| < \delta$. The closure of E is denoted $\overline{E}$.*

2.1.19 Closed set: *A set F is called closed if $F = \overline{F}$.*

Proposition 2.1.20 *The intersection of any collection $\mathscr{C}$ of closed set is closed.*

Theorem 2.1.21 Heine-Borel: *Let F be a closed and bounded set of real numbers. Then each open covering of F has a finite subcovering.*

2.1.22 continuous at one point: *We say that f is continuous at the point x in E if for $\forall\epsilon > 0$, there $\exists\delta > 0$ such that for any y in E with $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$.*

2.1.23 Continuous: *The function f is said to be continuous on a subset A of E if it is continuous at each point of A.*

Proposition 2.1.24 *Let f be a real valued function defined and continuous on a closed and bound set F. Then f is bounded on F and assumes its maximum and minimum on F.*
Proof: *By Heine-Borel open covering theorem.*

2.1.25 *(May, 2009)(Apr, 2005)*
Let $f(x)$ be monotone increasing on $[0, 1]$ with $f(0) = 0$ and $f(1) = 1$. If the set $\{f(x); x \in [0, 1]\}$ is dense in $[0, 1]$, show that f is a continuous function on $[0, 1]$. Is it absolutely continuous on $[0, 1]$? Prove your conclusion.

Proof: It is continuous on $[0, 1]$. Since $f(x)$ is monotone increasing on $[0, 1]$ with $f(0) = 0$ and $f(1) = 1$, the set $\{f(x); x \in [0, 1]\} \subset [0, 1]$. For an arbitrary $y \in (0, 1)$, since $\{f(x); x \in [0, 1]\}$ is dense in $[0, 1]$, then for $\epsilon = 1/n > 0$, there is a $f(x_n) \in (y, y + \epsilon)$ and $f(x'_n) \in (y - \epsilon, y)$. Since $f(x)$ is monotone increasing, then for $\forall f(x) = y \in (0, 1)$, let $\delta = min\{|x_n - x|, |x'_n - x|\}$, such that for any $|x' - x| < \delta$, we have $|f(x') - f(x)| < \epsilon$. Then $f(x)$ is continuous at $(0, 1)$.

Also for $x = f(x) = 0$ and $\forall\epsilon = 1/n > 0$, there is a $y_n = f(x_n) \in (0, \epsilon)$, let $\delta = |x_n - x|$, then for any $|x' - x| < \delta$, we have $|f(x') - f(x)| < \epsilon$. Then $f(x)$ is continuous at $x = 0$. The same for $x = 1$.

I want to use your proof for 2.1.25 as an example.

The current proof has the following problems:

(1) Confusing notations: x is used to mean a general variable in Line -10, but x also mean a special point in lines -7, -8, and -9.

(2) Lack of sufficient details: the monotone property must be used in the proof, and so the argument when $x < x'$ and when $x' < x$ are different, even though they are similar.

(3) When working on a proof, ask the question what do I need to prove? In this exercise, we need to prove, by definition of continuity, that $f(x)$ is continuous at every point $a \in [0, 1]$. (Use a instead of x to avoid notational confusion).

I am going to show you an example of this proof: (What are in parentheses are not part of the proof).

Proof of 2.1.25: It suffices to show that $f(x)$ is continuous at every point $a \in [0, 1]$. (This sentence tells the reader what you are going to do in the proof).

(To make you easy to understand the writing of the proof, I argue on the cases when $0 < a < 1$ and when $a \in \{0, 1\}$. These cases are not necessary and the proofs can be combined by using slightly more complicated notations).

First we assume that $a \in (0, 1)$. Let $b = f(a)$ (this is again a better notation than the confusing $y = f(x)$). Since $f(0) = 0$ and $f(1) = 1$, and since $f(x)$ is increasing (reason: assumption), we conclude that $0 \le b \le 1$ (conclusion).

For any $\epsilon > 0$, since $0 < b < 1$, and since $\{f(x) : x \in [0, 1]\}$ is dense in $[0, 1]$ (reason: assumption and established facts) , there must be an $x' \in [0, 1]$ such that $y' = f(x') \in (b, b + \epsilon)$, and an $x'' \in [0, 1]$ such that $y'' = f(x'') \in (b - \epsilon, b)$ (conclusion). Since f is increasing (reason: assumption), $x' > a > x''$ (conclusion).

Let $\delta = \min\{a - x'', x' - a\}$. Since $x' > a > x''$ (reason: established fact), $\delta > 0$ (conclusion). For any $x \in [0, 1]$ with $|x - a| < \delta$, either $x > a - \delta \ge a - (a - x'') = x''$, or $x < a + \delta \le a + (x' - a) = x'$ (establishing the fact by the definition of absolute values).

Since f is increasing, and by the definition of δ, (reason: assumption and established facts), when $x < x'$, we have $f(x) \le f(x') = y' < b + \epsilon$; and when $x > x''$, we have $f(x) \ge f(x'') = y'' > b - \epsilon$ (conclusion). It follows by the definition of absolute values that $|f(x) - b| < \epsilon$. Hence by the definition of continuity, $f(x)$ is continuous at a.

(The cases when $a = 0$ or $a = 1$ can be done by using help intervals. You can also combine the arguments, but separating the proofs makes the proof easier to be understood).

General Comments: Most of the other proofs in Chapter 1 (I am completing your Chapter 1 today) have the similar problems (confusing notations and lack of sufficient details). Need to train yourself to be a master of these.

2.1.26 *Let f be a function defined by setting*

$$f(x) = \begin{cases} x & \text{if } x \text{ irrational} \\ p \sin \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ in lowest terms.} \end{cases}$$

At what points is f continuous?

Proof: The function f is continuous on irrationals and discontinuous on rationals.

First prove the case for irrationals. Since $\lim_{x \to 0} \frac{\sin a}{a} = \lim_{q \to \infty} \frac{\sin \frac{1}{q}}{\frac{1}{q}} = 1$. That means for $\forall \epsilon > 0$, there exists an sufficient large M, such that for any $q > M$, we have $\left| \frac{\sin \frac{1}{q}}{\frac{1}{q}} - 1 \right| < \epsilon$. Let x be an irrational number, for $\forall \epsilon > 0$, let $\delta_1 = \epsilon$, $\delta_2 = \min_{q \le M} |x - \frac{p}{q}|$, let $\delta = \min\{\delta_1, \delta_2\}$. Then for any rational $y = \frac{p}{q}$ with $|x - y| < \delta$, we have $|f(x) - f(y)| = |x - p \sin \frac{1}{q}| = |x - \frac{p}{q} + \frac{p}{q} - p \sin \frac{1}{q}| \le |x - \frac{p}{q}| + |\frac{p}{q} - p \sin \frac{1}{q}| \le 2\epsilon$.

Also we have for $\forall x \in [0, 1]$ and $\forall \epsilon > 0$, let $\delta = \epsilon > 0$ such that for any irrational y with $|x - y| < \delta$, we have that $|f(x) - f(y)| = |x - y| < \epsilon$.

Then we can get the conclusion that $f(x)$ is continuous on $[0, 1]$. It is easy to prove it is true for any $[N, N + 1]$, where N is an integer.

If $x = \frac{p}{q}$, let $\epsilon_0 = \frac{1}{2} p |\sin \frac{1}{q} - \frac{1}{q}|$, for any $0 < \delta < \epsilon_0$, there exists an irrational y with $|x - y| < \delta$, such that $2\epsilon_0 = |p \sin \frac{1}{q} - \frac{p}{q}| \le |p \sin \frac{1}{q} - y| + |y - \frac{p}{q}|$, implies $|f(x) - f(y)| = |p \sin \frac{1}{q} - y| \ge \epsilon_0$.

2.1.27 *(Apr, 2004)(May, 1996)*
Let f be a function in $(0,1)$ *defined by setting*

$$f(x) = \begin{cases} 0 & \text{if } x \text{ irrational} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ in lowest terms.} \end{cases}$$

Determine all the points where f *is continuous. Prove your conclusion.*
Proof: *The set where* f *is continuous is the irrationals in* $(0,1)$. *Since for* $\forall\epsilon = \frac{1}{n} > 0$, *the sequence* $\{x_i\}$*of rational number such that for each* $x_i = \frac{p}{q} \in (0,1)$ *and* $q \leq n$ *is finite, then let* $\delta = \min_i |x - x_i|$, *such that for any y with* $|x-y| < \delta$, *we have* $|f(x) - f(y)| < \epsilon$.
For any rational number $x = \frac{p}{q}$, *there* $\exists\epsilon_0 = \frac{1}{2q}$, *such that for any* $\delta > 0$, *there exists an irrational number y with* $|x-y| < \delta$, *such that* $|f(x) - f(y)| = |\frac{1}{q} - 0| > \epsilon_0$.

2.1.28 *(Aug, 2004)*
Let f be a function in $(0,1)$ *defined by setting*

$$f(x) = \begin{cases} 0 & \text{if } x \text{ irrational} \\ \sin\frac{1}{q} & \text{if } x = \frac{p}{q} \text{ in lowest terms.} \end{cases}$$

Find the set C of points where f *is continuous and the set D of points where* f *is discontinuous. Justify your conclusion.*
Is function f *Riemann integrable on* $(0,1)$*? Lebesgue integrable on* $(0,1)$*? Justify your conclusion.*
Proof: *The function* f *is continuous at irrationals. Since for any irrational x and* $\forall\epsilon = \frac{1}{n} > 0$, *let* $\delta = \min_{p \leq q \leq n} |x - \frac{p}{q}|$, *such that for any* $y = \frac{p}{q}$ *with* $|x-y| < \delta$, *we have that* $|f(x) - f(y)| = |\sin\frac{1}{q}| < |\frac{1}{q}| < \epsilon$.

2.1.29 *(Apr, 2004)(Apr, 2002)*
Let f *be an real function defined on R with* $f(0) = 1$. *If the set* $A = \{x : f(x) > 0\}$ *is both open and close, prove that* $f(x) > 0$ *everywhere on R.*
Proof: *Since* $f(0) = 1$, *we have* $0 \in A$. *If* $A \neq R$, *then there must be some real numbers not in A, then consider the smallest interval containing* 0. *At least one side of this interval must be a real number, if this side is open, then this will lead a contradiction to A is closed, for the same reason, it could not be closed. So A must be R.*

2.1.30 *(Aug, 2003)*
Let $f_n : (0,1] \to [0,\infty)$ *be a decreasing sequence of continuous function converging pointwise to a zero function* θ. *Must* f_n *converge uniformly?*

?

2.1.30: The answer is NO. When saying no, we need to construct a sequence of decreasing functions that converge point-wise to 0.

(How do we find such an example? Think this way: If the domain is a closed interval (or more generally, a compact set), then the answer will be a YES, as the point-wise convergence will turn into a finite number of subintervals, and the decreasing property assures that we only need to consider one function on each subinterval. Therefore, the problem must be the open interval.)

Define, for each $n > 2$,

$$f_n(x) = \begin{cases} 1 & \textbf{if } x < \frac{1}{n} \\ -nx+2 & \textbf{if } \frac{1}{n} \le x \le \frac{2}{n} \\ 0 & \textbf{if } 1 \ge x > \frac{2}{n} \end{cases}$$

(Then we need to verified that f_n is decreasing, point-wise convergent to 0, and not uniformly convergent.)

Proof: For $x = 1/m$ and any $\epsilon > 0$, there exists $N = 3m$ such that for any $n > N$, since f_n is a decreasing sequence, $f_n(x) = f_n(1/m) \le f_N(1/m) = f_{3m}(1/m) = \epsilon$. And for any $y \ge x$, we have $f_n(y) = f_n(x) = 0 < \epsilon$. Since x is arbitrary, $f_n(x) \nrightarrow 0$ point wise.

But $f_n(x) \nrightarrow 0$ uniformly. There exists $\epsilon_0 = 1/2$, for any N, there exists $x = 1/2N$ and $n = N$ such that $f_n(x) = f_N(1/2N) = 1 > \epsilon_0$. Therefore $f_n(x) \nrightarrow 0$ uniformly.

Additional Comment to this exercise: You should be able to prove, by using Theorem 2.1.21, the following:

2.1.30': Let $f_n : [0,1] \mapsto [0,\infty)$ be a sequence of decreasing and continuous functions point-wisely convergent to a zero function. Show that f_n must converge uniformly.

Hint: Use assumption to show $f_n(x) \ge 0$. Given $\epsilon > 0$, for each $a \in [0,1]$, there must be a N_a such that $n \ge N_a$, $|f_n(a)| < \epsilon$. Since f_n's are continuous, $\delta_a > 0$ such that when $x \in I_a = (a-\delta_a, a+\delta_a)$, $0 \le f_n(x) \le f_{N_a}(x)| < \epsilon$. Then use the compactness to show that there are only finitely many such I_a's that covers $[0,1]$, and choose the right δ.

Proof: Since $\{f_n\}$ is convergent pointwise to zero function, for $x_i \in [0,1]$, $\forall\epsilon$, there $\exists N_i \in Z^+$ such that for any $n \ge N_i$, we have $|f_n(x) - 0| = f_n(x) < \epsilon$. Since f_n is continuous, there $\exists\delta_i > 0$ such that for $\forall y \in (x_1-\delta_i, x_1+\delta_i)$, we have $|f_{N_i}(x_i) - f_{N_i}(y)| < \epsilon$. That is $f_{N_i}(y) < f_{N_i}(x_i)+\epsilon \le 2\epsilon$. Since $\{f_n\}$ is decreasing, we have for any $y \in (x_i-\delta_i, x_i+\delta_i)$ and any $n > N_i$ that $f_n(y) < f_{N_1}(y) < 2\epsilon$.

Since $[0,1]$ is compact, we could use finitely many $(x_i-\delta_i, x_i+\delta_i)$ to cover it. Then $[0,1] \subseteq \bigcup_{i=1}^m (x_i - \delta_i, x_i+\delta_i)$. Let $N = max\{N_1, N_2, \dots, N_m\}$. Then for any $x \in [0,1]$ and $n > N$, x must be in one of the $(x_i-\delta_i, x_i+\delta_i)$, therefore $|f_n(x) - f(x)| = f_n(x) < 2\epsilon$.

2.1.31 *(May, 1996)*

Let $\{f_n : R \to R\}$ be a sequence of continuous function and let f be a point wise limit of f_n. Show that for every $a < b$, $f^{-1}([a,b])$ is a G_δ set.

?

A G_δ set is a countable intersection of open sets. Here is a hint:

First show that

$$f^{-1}[a,b] = \cap_{m=1}^\infty \cap_{n=1}^\infty f_n^{-1}(a - \frac{1}{m}, b + \frac{1}{m}).$$

by showing each point on one side is also a point on the other side.

Then quoting the definition of continuous functions to show that each $f_n^{-1}(a-\frac{1}{m}, b+\frac{1}{m})$ is an open set.

Since $[a,b] = \cap_{m=1}^\infty (a-1/m, b+1/m)$, I can understand the part $\cap_{m=1}^\infty$. But for the part $\cap_{n=1}^\infty f_n^{-1}$, there is an example:

Let $f_n(x) = x^2 - 1/n$. Then $f_n(x) \to f(x) = x^2$ point wise. Consider the set $f_n^{-1}([0,1])$. We have $f_1^{-1}([0,1]) = [-2,-1] \cup [1,2]$, $f^{-1}([0,1]) = [0,1]$. Then $f^{-1}([0,1]) \nsubseteq f_1^{-1}([0,1])$

2.1.32 *(Apr, 2005)*
Let $f_n(x)$ be a sequence of continuous functions on $[0,1]$ and $f_n(x) \geq f_{n+1}(x)$ $n = 1,2,3\ldots.$ For every $x \in [0,1]$, $\lim_{n\to\infty} f_n(x) < 0$. Determine and prove if there is a $\delta > 0$ such that

$$\lim_{n\to\infty} f_n(x) \leq -\delta \quad \forall x \in [0,1].$$

Proof: *For $x_1 \in [0,1]$, let $y_1 = \lim_{n\to\infty} f_n(x_1)$. Then for $\epsilon_1 = -1/3y_1$, there $\exists N_1 > 0$, such that for any $n \geq N_1$, we have*

$$|f_n(x_1) - y_1| < \epsilon_1. \tag{2.1}$$

Also, $f_n(x)$ is a sequence of continuous functions, then there $\exists \delta_1 > 0$ such that for any $x \in (x_1 - \delta_1, x_1 + \delta_1)$, we have

$$|f_{N_1}(x) - f_{N_1}(x_1)| < \epsilon_1. \tag{2.2}$$

By 2.1, we have $f_{N_1}(x_1) < \epsilon_1 + y_1$, and by 2.2, $f_{N_1}(x) < \epsilon_1 + f_{N_1}(x_1)$. Since $f_n(x) \geq f_{n+1}(x)$, we have $f_n(x) \leq f_{N_1}(x) < \epsilon + f_{N_1}(x_1) < 1/3y_1$ for any $x \in (x_1 - \delta_1, x_1 + \delta_1)$ and $n \geq N_1$. Then we can find $\bigcup_{n_k=1}^{\infty} (x_i - \delta_i, x_i + \delta_i)$ to cover $[0,1]$, and since $[0,1]$ is closed and bounded, then we can find a finite sub-covering $\bigcup_{n_k=1}^{m} (x_{n_k} - \delta_{n_k}, x_{n_k} + \delta_{n_k})$ of $[0,1]$. Then let $\delta = \min\{-1/3y_{n_k}\}$, we have $\lim_{n\to\infty} f_n(x) \leq -\delta$ for $\forall x \in [0,1]$.

2.1.33 *(Apr, 2002)*
Show that if a Cauchy sequence $\{f_n\}$ has a subsequence converges to l, then f_n converges to l.
Proof: *See the proof of Cauchy criteria.*

Chapter 3

Lebesgue Measure

3.1 Notes

3.1.1 Outer measure: *For each set A, define* $m^*(A) = \inf_{A \subset \cup I_n} \sum l(I_n)$, *where* $\{I_n\}$ *is the countable collection of open intervals that cover A.*

Proposition 3.1.2 *Let* $\{A_n\}$ *be countable collection of sets of real numbers. Then*

$$m^*(\cup A_n) \leq \sum m^*(A_n).$$

3.1.3 *A set E is measurable if for every set A, we have* $m^*A = M^*(A \cap E) + m^*(A \cap \widetilde{E})$.
Since $A = (A \cap E) \cup (A \cap \widetilde{E})$, *we have* $m^*A \leq m^*(A \cap E) + m^*(A \cap \widetilde{E})$. *So we only need to prove* $m^*A \geq m^*(A \cap E) + m^*(A \cap \widetilde{E})$.

Theorem 3.1.4 *The collection of measurable set is a* $\sigma -$ *algebra; that is, the complement of a measurable set is measurable, the union (intersection) of a countable collection of measurable set is measurable. Moreover, every set with outer measure 0 is measurable.*

Theorem 3.1.5 *Every Borel set is measurable. In particular each open set and each closed set is measurable.*

If E is a measurable set, define the Lebesgue measure mE to be the outer measure of E.

Proposition 3.1.6 *Let* $\{E_i\}$ *be a sequence of measurable set. Then*

$$m(\cup E_i) \leq \sum mE_i.$$

If the sets E_i *are pairwise disjoint, then*

$$m(\cup E_i) = \sum mE_i.$$

Proposition 3.1.7 *Let E be a given set. Then the following five statements are equivalent:*
(1) E is measurable.
(2) Given $\epsilon > 0$, *there is an open set* $O \supset E$ *with* $m^*(O \sim E) < \epsilon$.

(3) Given $\epsilon > 0$, there is a closed set $F \subset E$ with $m^*(E \sim F) < \epsilon$.
(4) There is a G in G_δ with $E \subset G$, $m^*(G \sim E) = 0$.
(5) There is an F in F_σ with $F \subset E$, $m^*(E \sim F) = 0$.
If m^*E is finite, the above statements are equivalent to:
(6) Given $\epsilon > 0$, there is a finite union U of open intervals such that $M^*(U \triangle E) < \epsilon$.

3.1.8 *An extended real-valued function f is said to be measurable if its domain is measurable, and for each real number r, the set $\{x : f(x) > r\}$ is measurable.*

3.2 Exercises

3.2.1 *(Apr, 2005)*
For a bounded set E, define

$$m_*(E) = b - a - m^*([a, b] \setminus E),$$

where $[a, b]$ is an interval containing E, and m^* denotes the usual outer measure. Prove the following statements.
(a)If E be the set of all irrational numbers in $[0, 1]$, then $m_*(E) = 1$.
(b)$m_*(E)$ is independent of the choice of $[a, b]$, as long as it contains E.
(c)$m_*(E) \leq m^*(E)$.

Proof: *(a)If E is the set of all irrational numbers in $[0, 1]$, then $[0, 1] \setminus E$ is all the rational numbers in $[0, 1]$, hence $m^*([0, 1] \setminus E) = 0$ and $m_*(E) = 1 - 0 - 0 = 1$.* **The conclusion "hence $m^*(0, 1] - E) = 0$" needs a reason. Exercise: If E is countable, then $m^*(E) = 0$.** *(See 3.2.3.)*

(b)Assume that $[a, b]$ is the smallest interval that contains E. Then any $[c, d]$ contains E, we have $c \leq a$ and $d \geq b$. Since the set $[a, b]$ is measurable, for the set $[c, d] \setminus E$ we have

$$\begin{aligned} m^*([c, d] \setminus E) &= m^*([c, d] \setminus E \cap [a, b]) + m^*([c, d] \setminus E \cap \widetilde{[a, b]}) \\ &= m^*([a, b] \setminus E) + m^*([c, a] \cup [b, d]). \end{aligned}$$

Then

$$\begin{aligned} m^*([c, d]) - m^*([c, d] \setminus E) &= m^*([c, d]) - m^*([a, d] \setminus E) - m^*([c, a] \cup [b, d]) \\ &= m^*([a, b]) - m^*([a, b] \setminus E) \\ &= m_*(E). \end{aligned}$$

We can get the conclusion that $m_*(E)$ is independent of the choice of $[a, b]$. **The equality needs the assumption $E \subseteq [a, b]$.** *(It is in the beginning that: assume that $[a, b]$ is the smallest interval that contains E.)*

(c)Since $[a, b] \setminus E \cup E = [a, b]$, we have

$$m^*([a, b]) \leq m^*([a, b] \setminus E) + m^*(E),$$

that is

$$m^*(E) \geq m^*([a, b]) - m^*([a, b] \setminus E) = b - a - m^*([a, b] \setminus E) = m_*(E).$$

3.2.2 *(Apr, 2005)(Aug, 2001)*
Let E be a measurable set in [0,1] with $mE = c(\frac{1}{2} < c < 1)$. *Let* $E_1 = E + E = x + y; x, y \in E$. *Show that there exists a measurable set* $E_2 \subset E_1$ *such that* $mE_2 = 1$.

Let a be the greatest element in E, then $E \cap E + a$ has at most one element, then $m(E \cup E + a) = m(E) + m(E + a) = 2m(E) > 1$.
Define $f(x) = m([0, x] \cap E_1)$, $0 \le x \le 2a$. Then $f(0) = 0$, $f(2a) > 1$. Now we want to prove f is a continuous function.

$$\begin{aligned} f(x + \Delta x) &= m([0, x + \Delta x] \cap E) \\ &= m(([0, x] \cap E) \cup [x, x + \Delta x] \cap E) \\ &\le f(x) + \Delta x \end{aligned}$$

That is also true for $\Delta x \le 0$. So $|f(x + \Delta x) - f(x)| \le |\Delta x|$. f is a continuous function. By mean value theorem, $\exists y' \in [0, 2a]$, such that $f(y) = m([0, y] \cap E) = 1$. And $[0, y] \cap E$ is the set E_2 we want.

Since E_2 is the intersection of two measurable sets, then it is also measurable.

Line 2: Since $E \cap E = E$, $E \cap E + a = E + a$**. Why it has at most one element? Also, why this would imply that** $m(E \cup E + a) > 1$**?**

Let a be the greatest element in E. Then for any $x \in E$, we have $0 \le x \le a$, that is $a \le x + a \le 2a$. So $E \cap E + a \subseteq \{a\}$. If $E \cap E + a =$, by the measurability of E and $E + a$, we have $m(E \cup E + a) = mE + m(E + a)$.

If $E \cap E + a = \{a\}$, then $E \cup E + a = E \setminus \{a\} \cup \{a\} \cup E + a \setminus \{a\}$. Since the set $\{a\}$ has measurable 0, it is measurable. Also $E \setminus \{a\} = E \cap \widetilde{\{a\}}$, then $E \setminus \{a\}$ is measurable, and $m(E) = m(E \setminus \{a\} \cup \{a\}) = m(E \setminus \{a\}) + m(\{a\}) = m(E \setminus \{a\}) + 0 = m(E \setminus \{a\})$. And for the same reason we have $m(E + a) = m(E + a \setminus \{a\})$ is measurable. Since the three sets $E \setminus a$, $\{a\}$ and $E + a \setminus \{a\}$ are disjoint,

$$m(E \cup E + a) = m(E \setminus \{a\}) + m(\{a\}) + m(E + a \setminus \{a\}) = m(E) + m(E + a) = 2m(E) > 1.$$

English: Sentences starting with "Let .." should end with a period ".", not a ",". Also, the next sentence should start with "Then", not "then".

3.2.3 *(Aug, 2004)*
Use the definition of Lebesgue measure show that the set of all rational numbers in $[0, 1]$ *is a Lebesgue measurable set.*

Because the rational numbers in $[0, 1]$ is a countable set, denoted as A, it can be written down as $x_1, x_2, \ldots$. Then for $\forall \epsilon, A \subset \bigcup_{i=1}^{\infty}(x_i - \frac{\epsilon}{2^{i+1}}, x_i + \frac{\epsilon}{2^{i+1}})$. So

$$\begin{aligned} m^*A &\le m^*(\bigcup_{i=1}^{\infty}(x_i - \frac{\epsilon}{2^{i+1}}, x_i + \frac{\epsilon}{2^{i+1}})) \\ &\le \sum_{i=1}^{\infty} 2^i \epsilon \\ &= \epsilon. \end{aligned}$$

This holds for any ϵ, so $m^*A = 0$.

Exercise 3.2.3: You need to carefully read the problem before you prove it. This exercise asks us to prove that A, the set of all rational numbers in [0,1], is a Lebesgue measurable set, and you prove that it has outer measure zero. This will be a BIG mistake. I was told that in the analysis exam, you also made similar errors and caused at least one problem with zero points. This is something we MUST correct.

Hint 1: To prove A is a measurable set, we need to verify Definition 3.1.3. (Remember what is the Purpose of doing exercise?)

Hint 2: You can also add to the current proof, a property that every subset A of the real numbers with $m^*(A) = 0$ is Lebesgue measurable., by verifying Definition 3.1.3.

By the definition of measurable set, we need to show that for any set E, $mE = m(E \cap A) + m(E \cap \tilde{A})$. That means we only need to show that $mE \geq m(E \cap A) + m(E \cap \tilde{A})$. Since $E \cap A \subseteq A$, we have $m(E \cap A) \leq mA$. Then $m(E \cap A) = 0$. And since $E \cap \tilde{A} \subseteq E$, $m(E \cap \tilde{A}) \leq mE$. Then we can get the conclusion that $mE \geq 0 + m(E \cap \tilde{A}) = m(E \cap A) + m(E \cap \tilde{A})$.

3.2.4 *(Aug, 2004)(Aug, 1997)*
Let $E \subset [0, 1]$ be closed and with no interior point. Is it true that the measure $m(E) = 0$? Justify your conclusion.

Proof: *No. Consider the generalized Cantor set.*
Let α be an fixed number in $(0, 1)$, we construct the generalized Cantor set as follows: First delete the middle $\frac{1}{3}\alpha$ open interval. Second, for the remaining two intervals, delete each interval the middle $\frac{1}{9}\alpha$ open interval. In the n-th step, delete the middle $\alpha\frac{1}{3^n}$ open interval from each of the remaining 2^{n-1} intervals. Let $n \to \infty$, the measure of the remaining set E is $1 - \alpha \sum_{i=1}^{\infty} \frac{1}{3^i} = 1 - \alpha$.
In each step, we delete the open intervals, then E is a closed set. If E has an interior point x, then there exists a $\delta > 0$ with $(x - \delta, x + \delta) \subset E$. But each time we delete the middle $\frac{1}{3^n}$ open intervals, there must be an m, such that $\frac{1}{3^m} < 2\delta$. Then in the m-th step, we will delete part of $(x - \delta, x + \delta)$, contradicting to $(x - \delta, x + \delta) \subset E$.
The set $E \subset [0, 1]$ is closed and with no interior point and $mE = 1 - \alpha \neq 0$.

Cantor set is a nowhere dense set.

3.2.5 *(Aug, 2003)*
Let $E_1 \supset E_2 \supset \dots$ be an infinite sequence of measurable subest of $\mathbb{R}$ and assume that $\bigcap_{n=1}^{\infty} E_n = \emptyset$.
(a)Show that if $m(E_1) < \infty$, then $\lim_{n\to\infty} m(E_n) = 0$
(b)Give an example showing that the conclusion of (a) may be false when $m(E_1) = \infty$.

Proof: *Let $F_i = E_i \setminus E_{i+1}$, then we have $F_i \cap F_j = \emptyset$ if $i \neq j$, and that $E_1 = \bigcup_i F_i \cup \bigcap_i E_i$. Since the family of measurable sets is an algebra of sets, then $\bigcup_i F_i$ and $\bigcap_i E_i$ are measurable and that*

$$\begin{aligned} m^*(E_1) - m^*(\bigcap_i E_i) &= m^*(\bigcup_i F_i) \\ &= m^*(E_1) - m^*(E_2) + m^*(E_2) - m^*(E_3) \dots \\ &= m^*(E_1) - \lim_i E_i. \end{aligned}$$

Then $\lim_i E_i = m^*(\bigcap_i E_i) = m^*(\emptyset) = 0$.

(2)Let $E_i = [i, \infty]$, *then* $\bigcap_{n=1}^{\infty} E_n = \emptyset$, *but* $\lim E_i \neq 0$.

3.2.5: This is a well-written exercise.

English:

(1) Avoid ”Let ..., then ...”, which is incorrect English. Use ”Let.... Then ...” (This occurs very often, and almost everywhere in your writing.)

(2) ”... and that” should be ”that ... and that ...” This does not fit in both uses in your writing. You can delete the two ”that” in your writing here.

3.2.6 *(Apr, 2005)*
Let $\{f_n\}$ *be a sequence of measurable functions on* (a, b), *and*

$$E = \{x \in (a, b) : f_n(x) \textit{ is convergent}\}.$$

Show that E *is measurable.*

Since for a fixed x, if $\{f_n(x)\}$ convergent, then it is a Cauchy sequence. For $\epsilon_k = 1/k$, there $\exists N_k > 0$, such that for any $n, m > N_k$, we have $|f_n(x) - f_m(x)| < \epsilon_k$. Then the set E where $f_n(x)$ is convergent could write down as follows:

$$E = \bigcap_{k=1}^{\infty} F_k,$$

$$F_k = \bigcup_{n,m>N_k} \{x : |f_n(x) - f_m(x)| < \epsilon_k\}.$$

Since F_k is the union of countable collection of measurable set, then F_k is measurable, and E is measurable.

3.2.6:

Mathematical Logic: Need to quote a reason why F_k is measurable. (Hint: use definition of measurable functions).

A General Rule: You can always check your proof and see if all the conditions in the problems are applied. Reading your proof, you may ask yourself where did I use the assumption that the functions are measurable functions? If there is a condition not used in the proof, check the proof again.

English: Avoid ”Since ...., then ...” This is Chinese English. See Bondy and Murty’s book to see ”Since ..., (without then)...”

Now we want to show F_k is measurable. We can write down F_k like this

$$\begin{aligned} F_k &= \bigcup_{n,m>N_k}\{x : |f_n(x) - f_m(x)| < \epsilon_k\} \\ &= \bigcup_{n,m>N_k}\{x : -\epsilon_k + f_m(x) < f_n(x) < f_m(x) + \epsilon_k\} \\ &= \bigcup_{n>N_k} \bigcup_{m>N_k}\{x : f_n(x) > f_m(x) - \epsilon_k\} \cap \{x : f_n(x) < f_m(x) + \epsilon_k\} \end{aligned} .$$

Since each $f_n(x)$ is measurable, by the definition of measurable function,the set $\{x : f_n(x) > f_m(x) - \epsilon_k\}$ and $\{x : f_n(x) < f_m(x) + \epsilon_k\}$ are measurable. Since the collection of measurable set is a $\sigma-$algebra, F_k is measurable and hence E is measurable.

3.2.7 *(Apr, 2002)*
Prove that function $f(x)$ defined on R is measurable if for any rational number r, the set $E = \{x : f(x) < r\}$ is measurable.
Proof: *For each $x \in R$, there is a sequence of rational numbers $r_n \downarrow x$, then $\{x : f(x) < x\} = \bigcap_n \{x : f(x) < r_n\}$. Since the countable intersection of measurable set is measurable, then $\{x : f(x) < x\}$ is measurable, and therefore $f(x)$ is measurable.*

3.2.7: Good math. Same English problem as above.

3.2.8 *(Aug, 2001)*
Suppose $E_1, E_2, \ldots, E_n$ are n measurable sets in $[0,1]$, and every $x \in [0,1]$ belongs to at least q of these sets. Show that, there is at least an E_k such that $mE_k \geq q/n$.
Proof: *Since every $x \in [0,1]$ belongs to at least q sets from $\{E_1, E_2, \ldots, E_n\}$, then $\bigcup_k E_k$ covers $[0,1]$ q times, and that*

$$q = m(\bigcup_{k=1}^{n} E_k) \leq mE_1 + mE_2 + \cdots + mE_n.$$

Then there is at least an E_k such that $mE_k \geq q/n$. **3.2.8: Good.**

3.2.9 *(Apr, 2001)*
(a) Let $\{f_n\}$ be a sequence of measurable functions. Show that if $f_n \to f$ a.e., then f is measurable.
(b) Let f be a differentiable function. Show that f' is measurable.

(a)The function f is measurable if for each $r \in R$, the set $E_r = \{x : f(x) > r\}$ is measurable. Since $f_n \to f$ a.e., the set where $f_n \not\to f$ is measure 0 hence a measurable set, and any subset of it is also of measure 0 and hence measurable, then $\{x : f(x) > r, f_n(x) \not\to f(x)\}$ is measurable. And we only need to consider the part that $f_n(x) \to f(x)$. And for $f_n(x) \to f(x)$, we have $\overline{\lim} f_n(x) = \underline{\lim} f_n(x) = f(x)$ and:

$$\begin{aligned}
\{x : \lim f_n(x) = f(x) > r\} &= \{x : \overline{\lim} f_n(x) > r\} \\
&= \{x : \inf_k \sup_{n \geq k} f_n(x) > r\} \\
&= \{x : \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} f_n(x) > r\}.
\end{aligned}$$

Each $f_n(x)$ is measurable and the collection of measurable set is a σ-algebra, so $f(x)$ is measurable.

(b) Let $f_n(x) = \frac{f(x+\frac{1}{n})-f(x)}{\frac{1}{n}}$. Then $f'(x) = \lim_{n\to\infty} f_n(x)$. Since the set $\{x : f_n(x) > r\}$ is measurable, $\{x : f'(x) > r\} = \bigcap_n \{x : f_n(x) > r\}$ is also measurable.

Page 13:

Do you think you need to justify the equalities of the sets? I don't understand.

English:

(1) Avoid "Let ..., then ...", which is incorrect English. Use "Let.... Then ..." (This occurs very often, and almost everywhere in your writing.)

3.2.10 *(Apr, 2001)(Aug, 1996)*
Let $\{f_n\}$ be a sequence of measurable function on $[0,1]$ such that $\{f_n(x)\}$ is a bounded sequence for each x. Define the set $E = \{x \in [0,1] : \lim_{n\to\infty} f_n(x)$ exists.$\}$ Show that E is a measurable set.

See 3.2.6. **3.2.10: Why f_n's must be bounded?** The condition that f_n is bounded has nothing to do with the problem. Since the condition that $E = \{x \in [0,1] : \lim_{n\to\infty} f_n(x) \text{ exists.}\}$ only asks us to consider that set that f_n convergent to a real number smaller than ∞.

3.2.11 *(Aug, 2000)(Jan, 2001)(Apr, 2001)*
For each statement below, either prove it (if true) or give a counter example (if false).
(a) E is measurable if and only if $m^*(P \cup Q) = m^*(P) + m^*(Q)$ for any $P \subset E$ and any $Q \subset \widetilde{E}$.
(b) If E is a countable set, then E is measurable and $mE = 0$.
(c) If E is a measurable set with $mE = 0$, then E is a countable set.
(d) If f is measurable, then so is $|f|$.
(e) If $|f|$ is measurable, then so is f.
(f) If $f \geq 0$ is measurable on E and $\int_E f = 0$, then $f = 0$ a.e. on E.
(g) If f is continuous on $[a,b]$, then f is of bounded variation on $[a,b]$.
Proof: *(a) True. "$\Longrightarrow$" is obvious. "$\Longleftarrow$": For any set A, let $P = A \cap E$, $Q = A \cap \tilde{E}$, then $A = P \cup Q$, and $m^*(A) = m^*(P \cup Q) = m^*(P) + m^*(Q) = m^*(A \cap E) + m^*(A \cap \tilde{E})$.*
(b) True. If E is countable, the we could write down E as $\{x_1, x_2, \dots\}$, for $\forall \epsilon > 0$, the set $\bigcup_{n=1}^{\infty}(x_n - \frac{\epsilon}{2^n}, x_n + \frac{\epsilon}{2^n})$ covers E, and its length is ϵ, then $mE < \epsilon$. Since ϵ is arbitrary, $mE = 0$.
For any set A, $m(A \cap E) = 0$. Also $A \supset A \cap \tilde{E}$, and so $mA \leq m(A \cap \tilde{E}) = m(A \cap E) + m(A \cap \tilde{E})$, then therefore E is measurable.
(c) False. Cantor set is measurable but not countable. It has a one to one correspondence with $[0,1]$.
(d) True. If f is measurable, then both $\{x; f(x) > r\}$ and $\{x; f(x) < -r\}$ are measurable for $\forall r \in R$. Therefore $\{x; |f(x)| > r\} = \{x; f(x) > r\} \cup \{x; f(x) < -r\}$ is measurable.
(e) False. Define

$$f(x) = \begin{cases} 1 & x \notin S \\ -1 & x \in S, \end{cases}$$

where S is a nonmeasurable set. Then f is nonmeasurable but $|f|$ is measurable.
(f) Let $E_n = \{x : f(x) > 1/n\}$, then $\int_E f \geq mE_1 + 1/2m(E_2 \setminus E_1) + 1/3m(E_3 \setminus E_2) + \dots$, and $E = E_1 \cup (E_2 \setminus E_1) \cup (E_3 \setminus E_2) \cup \dots$. If one of the $E_n \setminus E_{n-1}$ has measure greater than 0, then we will have $\int_E f(x) > 0$ which is a contradiction. Therefore $mE = 0$.
(g) Consider the function $f(x) = \begin{cases} \sin\frac{1}{x} & x \in (0,1] \\ 0 & x = 0 \end{cases}$ on $[0,1]$. For any $M \in Z^+$, there is a subdivision $0 = x_1 < \frac{1}{2M\pi + \frac{3\pi}{2}} < \frac{1}{2M\pi + \frac{\pi}{2}} < \frac{1}{2(M-1)\pi + \frac{3\pi}{2}} < \frac{1}{2(M-1)\pi + \frac{\pi}{2}} < \cdots < x_{2M+2} = 1$ such that $\sum_{k=1}^{2M+2} |f(x_k) - f(x_{k-1})| > 4M$. Then $f(x)$ is not of bounded variation on $[0,1]$.

3.2.11:

(a) Never use the phrases like "obvious", "it is clear" ... Can you quote a definition, or a theorem to justify your conclusion?

English: (a), (f): Avoid "Let ..., then ...",

3.2.12 *(Jan, 2001)(Aug, 1997)*
(a) State the definition of a measurable function.
(b) Use the definition to deduce that, if f is measurable on a measurable set E, then for every $\alpha \in R$, the set

$E_\alpha = \{x \in E : f(x) = \alpha\}$ is measurable.
(c) Construct a function f on $E = (0,1)$ to show the converse of (b) is not true.

(a) The function f is measurable if for any $r \in R$, the set $\{x : f(x) > r\}$ is measurable.

(b) If $\{x : f(x) > r\}$ is measurable, then its complement $\{x : f(x) \le r\}$ is also measurable. Since the set $\{x : f(x) < r\} = \bigcup_{n=1}^{\infty}\{x : f(x) \le r - \frac{1}{n}\}$. The countable collection of measurable set is a σ-algebra, then $\{x : f(x) < r\} \cap \{x : f(x) > r\} = \{x : f(x) = r\}$ is measurable.

(c) Let S be a nonmeasurable set in $(0,1)$, define $f(x) = \begin{cases} 0 & x \in E \setminus S \\ x & x \in S \end{cases}$. Then the set $\{x : f(x) > 0\} = S$ which is also nonmeasurable.

3.2.12:

English: (a) "The function..." should be "A function..."

Math logic: (b) The proof is an example of a bad logic. Here is how you would think:

Want to prove $\{x : f(x) = r\}$ is measurable? It suffices to show that both $\{x : f(x) < r\}$ and $\{x : f(x) > r\}$ are measurable.

Definition says $\{x : f(x) > r\}$ is measurable. How do I prove $\{x : f(x) < r\}$ is measurable? As $\{x : f(x) < r\} = \{x : -f(x) > -r\}$, can I used property or definition to show that $-f(x)$ is also measurable?

(c) The example is correct.

English: Let.... Define... (Two sentences).

Do you need to explain why f satisfies the converse of (b)? It is better for you to write down clearly what the converse of (b) is before you display your example.

(b) If $\{x : f(x) > r\}$ is measurable, then its complement $\{x : f(x) \le r\}$ is also measurable. Since the set $\{x : f(x) < r\} = \bigcup_{n=1}^{\infty}\{x : f(x) \le r - \frac{1}{n}\} = \bigcup_{n=1}^{\infty}\{x : f(x) > r - \frac{1}{n}\}$. The countable collection of measurable set is a σ-algebra, then $\{x : f(x) < r\}$ is measurable. Also, $\{x : f(x) < r\} \cap \{x : f(x) > r\} = \{x : f(x) = r\}$ is measurable.

(c) The converse of (b) is: If for any $\alpha \in R$, the set $E_\alpha = \{x : f(x) = \alpha\}$ is measurable, then f is measurable. Let S be a nonmeasurable set in $(0,1)$. Define $f(x) = \begin{cases} 0 & x \in E \setminus S \\ x & x \in S \end{cases}$. Then the set $\{x : f(x) > 0\} = S$ which is also nonmeasurable. Therefore f is nonmeasurable.

3.2.13 *(May, 1996)*
Let the set $E \subset (0,2)$ be defined as follows:

$$E = \left\{\frac{1}{j} + \frac{\sin k}{k}; j, k = 1, 2, 3 \ldots\right\}.$$

Is there a finite collection of open intervals $\{I_k, k = 1, 2, 3, \ldots\}$ such that $E \subset \cup_{k=1}^{n} I_k$ and $\sum_{k=1}^{n} l(I_k) \le \frac{1}{2}$? Prove your conclusion.

We can order the set E as we order the rational numbers, then E is a countable set, hence $mE = 0$. By the definition, we have $mE = \inf_{E \subset I_n} \sum_n l(I_n)$. Then there must be an open covering $\{I_n\}$ of E with $\sum_n l(I_n) = 1/2$. Also the set E is bounded and closed, then any open covering has a finite sub-covering.

3.12.13

English: We ... Then ... (two sentences)

It is much better to explicitly show the open cover then saying ”there must be...” Saying ”there must be ...” in many cases is equivalent to say nothing. List elements in E by $a_1, a_2, ...$ and cover them with $I_n = (a_n - 1/2^{n+1}, a_n + 1/2^{n+1})$.

We can order the set E as we order the rational numbers. Then E is a countable set and $E = \{a_1, a_2, a_3 \ldots\}$, hence $mE = 0$. By the definition, we have $mE = \inf_{E \subset I_n} \sum_n l(I_n)$. There is an open covering $\{I_n\} = (a_n - 1/2^{n+1}, a_n + 1/2^{n+1})$ of E with $\sum_n l(I_n) = 1/2$. Also the set E is bounded and closed, then any open covering has a finite sub-covering $O \subset \bigcup_{n=1}^{\infty} I_n$, and $mO \leq l(I_n) = 1/2$.

How to prove E is closed?

3.2.14 *(may, 1996)*
Is there a monotone function on $[0, 1]$ *which is discontinuous at all rational points? Prove your conclusion.*
Proof: *No?* **3.2.14:**

I will think it over before answering it.

New begin from July 16

Proposition 3.2.15 *If $\{f_n\}$ is a sequence of measurable functions that converge to a real-valued function f a.e. on a measurable set E of finite measure, then give $\epsilon, \delta > 0$, there is a subset $A \subset E$ with $mA < \delta$ and an integer N such that for $\forall x \in E \setminus A$ and $\forall n \geq N$,*

$$|f_n(x) - f(x)| < \epsilon$$

Proof: Let

$$G_n = \{x : |f_n(x) - f(x)| \geq \epsilon\},$$

and

$$E_n = \{\bigcup_{k=n}^{\infty} G_k\} = \{x : |f_k(x) - f(x)| \geq \epsilon \text{ for some } k \geq n\}.$$

Then $E_1 \supseteq E_2 \supseteq E_3 \ldots$.

Since $f_n \to f(x)$ a.e., the set F that $f_n \nrightarrow f(x)$ has measure 0, and $\bigcap E_n = F$. By 3.2.5, we have $\lim mE_n = mF = 0$. Then for $\delta > 0$, there exists an N, such that for any $n \geq N$, we have $mE_n \leq \delta$. Let $A = E_N$. Then we have

$$\tilde{A} = E \setminus A = \{x : |f_n(x) - f(x)| < \epsilon \text{ for any } n \geq N\}.$$

□

Theorem 3.2.16 Egoroff's Theorem: *If $\{f_n\}$ is a sequence of measurable functions that converge to a real-valued function f a.e. on a measurable set E of finite measure, then give $\eta > 0$, there is a subset $A \subset E$ with $mA < \eta$ such that f_n converges to f uniformly on $E \setminus A$.*

Proof: By 3.2.15, we have for $\epsilon_k = 1/k$ and $\delta_k = 1/2^k\eta$, there exists a set A_k with $mA_k < \delta_k$ and N_k, such that for any $x \in \tilde{A_k}$ and $n \geq N_k$, we have $|f_n(x) - f(x)| < \epsilon_k$. Let $A = \bigcup A_k$. Then $mA \leq \sum A_k < \eta$. And for $\epsilon = 1/k$, there exists $N = k$, such that for any $n > N$ and any $x \in \tilde{A}$, $|f_n(x) - f(x)| < \epsilon$. □

Chapter 4

The Lebesgue Integral

4.1 Notes

The function χ_E defined by

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

is called the characteristic function of E. A linear combination

$$\varphi = \sum_{i=1}^{n} a_i \chi_{E_i}(x)$$

is called a simple function if all E_i are measurable. Note that we don't require E_i to be disjoint.

Proposition 4.1.1 *Let f be a bounded function defined on a measurable set with finite measure. In order that*

$$\inf_{\psi \geq f} \int_E \psi(x)dx = \sup_{f \geq \varphi} \int_E \varphi(x)dx$$

for all simple function ψ and φ, it is necessary and sufficient that f be measurable.

Proof: Long proof, omit.

□

4.1.2 *If f is a bounded measurable function defined on a measurable set E with mE finite, we define the (Lebesgue) integral of f over E by*

$$\int_E f(x)dx = \inf \int_E \psi(x)dx$$

for all simple functions $\psi \geq f$.

Proposition 4.1.3 *Let f be a bounded function defined on $[a, b]$. If f is Riemann integrable on $[a, b]$, then it is measurable and*

$$R\int_a^b f(x)dx = \int_a^b f(x)dx.$$

Proof: Since a step function is also a simple function,

$$R\underline{\int_a^b} f(x) \leq \sup_{f \geq \varphi} \int_E \varphi(x)dx \leq \inf_{\psi \geq f} \int_E \psi(x)dx \leq R\overline{\int_a^b} f(x).$$

Since f is Riemann integrable and by 4.1.1, we have f is measurable. By 4.1.2,

$$\int_a^b f(x)dx = \inf_{\psi \geq f} \int_E \psi(x)dx = R\int_a^b f(x)dx.$$

□

Theorem 4.1.4 Bounded Convergence Theorem: *Let $\{f_n\}$ be a sequence of measurable function defined on a set E of finite measure, and suppose that there is a real number M such that $|f_n(x)| < M$ for all n and all $x \in E$. If $f_n(x) \to f(x)$ pointwise for all $x \in E$, then*

$$\int_E f = \lim \int_E f_n.$$

Proof: By 3.2.16, we have for $\forall \epsilon > 0$, there exists a set A with $mA < \epsilon/2M$, such that for any $x \in \tilde{A}$, we have $f_n \to f$ uniformly. Then there exists an N, such that for any $n > N$, we have $|f_n(x) - f(x)| < \epsilon/mE$ for all $x \in \tilde{A}$. Thus for all $n > N$, we have

$$\begin{aligned} |\int_E f_n - \int_E f| &= |\int_E f_n - f| \\ &\leq \int_E |f_n - f| \\ &= \int_{E\setminus A} |f_n - f| + \int_A |f_n - f| \\ &\leq mE * \epsilon/mE + 2M * \epsilon/2M \\ &= 2\epsilon. \end{aligned}$$

Then $\int_E f = \lim \int_E f_n$.

□

Proposition 4.1.5 *A bounded function f on $[a, b]$ is Riemann integrable if and only if the set of points at which f discontinuous has measure zero.*

If f is a nonnegative measurable function defined on a measurable set E, we define

$$\int_E f = \sup_{h \leq f} \int_E h,$$

where h is a bounded measurable function such that $m(x : h(x) \neq 0)$ is finite.

Theorem 4.1.6 Fatou's Lemma: *If $\{f_n(x)\}$ is a sequence of nonnegative measurable functions and $f_n(x) \to f(x)$ a.e. on a set E, then*

$$\int_E f \leq \underline{\lim} \int_E f_n.$$

Proof: Since $f_n(x) \to f(x)$ a.e., the integral on a measure 0 set is 0. Then we can assume that $f_n(x) \to f(x)$ everywhere. Let h be a bounded measurable function no greater than f which and vanishes outside a set E' with finite measure. Define $h_n(x) = \min\{h(x), f_n(x)\}$.

Now we want to show that $h_n(x) \to h(x)$ on E'. Since $f_n(x) \to f(x)$, for any $\epsilon > 0$, there exists an N, such that for any $n > N$, we have $|f_n(x) - f(x)| < \epsilon$. Also, for any $n > N$, if $h_n(x) = h(x)$, we have $|h_n(x) - h(x)| = 0 < \epsilon$. If $h_n(x) = f_n(x)$, that means $f_n(x) \leq h(x) \leq f(x)$, then $|h_n(x) - h(x)| = |f_n(x) - h(x)| \leq |f_n(x) - f(x)| < \epsilon$. Therefore $h_n(x) \to h(x)$. And by Bounded Convergence Theorem, we have

$$\int_E h = \int_{E'} h = \lim \int_{E'} h \leq \underline{\lim} \int_{E'} f_n.$$

Taking the supremum over h, we have $\int_E f \leq \underline{\lim} \int_E f_n$.

□

Theorem 4.1.7 Monotone Convergence Theorem: *Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions, and let $f_n(x) \to f(x)$ a.e. Then*

$$\int f = \lim \int f_n.$$

Proof: By Fatou's Lemma, we have

$$\int f \leq \underline{\lim} \int f_n.$$

Since $\{f_n\}$ is an increasing sequence of functions, we have $f_n(x) \leq f(x)$ a.e., and so $\int f_n(x) \leq \int f(x)$. Then

$$\overline{\lim} \int f_n \leq \int f.$$

Since $\overline{\lim} \int f_n \geq \underline{\lim} \int f_n$, we have

$$\int f = \lim \int f_n.$$

□

4.1.8 *A nonnegative measurable function f is called <u>integrable</u> over the measurable set E if*

$$\int_E f < \infty.$$

Proposition 4.1.9 *Let f be a nonnegative function which is integrable over a set E. Then given $\epsilon > 0$, there is a $\delta > 0$ such that for any subset $A \subset E$ with $mA < \delta$, we have*

$$\int_A f < \epsilon.$$

Proof: If f is bounded by M. Then given $\epsilon > 0$, for any subset $A \subset E$ with $mA < \epsilon/M$, we have $\int_A f < \epsilon$.

If f is unbounded, define

$$f_n = \begin{cases} f(x) & \text{if } f(x) \leq n \\ n & \text{if } f(x) > n \end{cases}.$$

Then each f_n is bounded, f_n is increasing and converges to f pint wise. Let A be any subset of E. By Monotone Convergence Theorem, we have for any $\epsilon > 0$, there exist an N such that $\int_A f_N - \int_A f = \int_A f_N - f \leq \epsilon$. Let $\delta = \epsilon/N$. Then for any $A \subseteq E$ with $mA < \delta$, we have

$$\begin{aligned} \int_A f &\leq \int_A [f(x) - f_N(x)] + \int_A f_N(x) \\ &\leq \epsilon + N * \epsilon/N \\ &= 2\epsilon. \end{aligned}$$

□

Let $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = \max\{-f(x), 0\}$, we have $f = f^+ - f^-$.

4.1.10 *A measurable function f is said to be integrable over E if f^+ and f^- are both integrable over E. In thus case we define*

$$\int_E f = \int_E f^+ + \int_E f^-.$$

Theorem 4.1.11 Lebesgue Convergence Theorem: *Let g be an integrable over E and let $\{f_n(x)\}$ be a sequence of measurable function such that $|f_n| \leq g$ on E and $f_n(x) \to f(x)$ a.e. Then*

$$\int_E f = \lim \int_E f_n.$$

Proof: Since $|f_n| \leq g$ and $f_n(x) \to f(x)$ a.e., $g - f_n$ is nonnegative and $g(x) - f_n(x) \to g(x) - f(x)$ a.e., by Fatou's Lemma,

$$\int_E g(x) - f(x) \leq \underline{\lim} \int_E g(x) - f_n(x).$$

Since $f_n(x) \to f(x)$ a.e., by 3.2.9, $f(x)$ is measurable. Since $|f_n| \leq g$, $\int f \leq \int g < \infty$, so f is integrable. Then we have

$$\int g - \int f \leq \int g - \overline{\lim} \int f_n,$$

that is equal to

$$\int f \geq \overline{\lim} \int f_n.$$

Again by Fatou's Lemma, $\int f \leq \underline{\lim} \int f_n$. So $\int_E f = \lim \int_E f_n$.

□

Chapter 4.

General Comment: Looks like you are good in understanding and applying the definition of integral, using simple functions. You are a lot better in using the epsilon argument this time.

Theorems 4.1.4, 4.1.6, and 4.1.7 are the most important, and commonly used tools. Need to remember them all. One way to remember them is to find more applications of them. When you work on an exercise related to convergence, ask yourself: can I apply one of them?

4.2 Exercises

4.2.1 *(Aug, 2005)*
Determine if each of the following is true or false:
(1) If f is continuous on $[0, 1]$, then it is uniformly continuous on $[0, 1]$.
(2) If $\{f_n\}$ is a sequence in $L^1(R)$, and $f_n(x) \to 0$ uniformly for all $x \in R$, then $\int_R f_n \to 0$.
(3) For measurable sets $E_n \subseteq [0, 1]$ with $\lim mE_n = 1$, there holds $\lim \int_{E_n} f = \int_0^1 f$ for every integrable function f on $[0, 1]$.

Proof: (1) True. For any $x \in [0, 1]$, and any $\epsilon > 0$, there exists a $\delta_x > 0$, such that for any $y \in (x - \delta_x, x + \delta_x)$, we have $|x_k - y| < \epsilon$. Let O_x be $(x - 1/2\delta_x, x + 1/2\delta)$. Now we can use $\{O_x : x \in [0, 1]\}$ to cover $[0, 1]$. Since $[0, 1]$ is a closed bounded interval, by Heine-Borel Open Covering Theorem, there is a finite sub-covering $\cup_{k=0}^n O_k \supset [0, 1]$. Let $\delta = 1/2 \min\{\delta_1, \delta_2, \ldots, \delta_n\}$, for any $|y - z| < \delta$, the element y must in $(x - 1/2\delta_{x_k}, x + 1/2\delta_{x_k})$ for some k. Since $|y - z| < \delta \leq 1/2\delta_{x_k}$, $|z - x| \leq |z - y| + |y - x| \leq 1/2\delta_{x_k} + 1/2\delta_{x_k} = \delta_{x_k}$. That means $z \in (x_k - \delta_{x_k}, x_k + \delta_{x_k})$. Therefore $|f(x_k) - f(y)| < \epsilon$ and $|f(x_k) - f(z)| < \epsilon$. Also

$$|f(y) - f(z)| \leq |f(x_k) - f(y)| + |f(x_k) - f(z)| < 2\epsilon.$$

Then f is uniformly continuous on $[0, 1]$.

(2) False. Define $f_n(x) = \frac{1}{n}\chi([0, n])$. Then for any $x \in R$ and $\epsilon = 1/k > 0$, there exists an $N = k$, such that for any $n > N$, we have $|f_n(x) - 0| \leq 1/n < 1/k = \epsilon$. Then $f_n(x) \to 0$ uniformly on $[0, 1]$. But $\int_R f_n = 1$ for all n.

(3) True. By the definition of Lebesgue integral, $\int_0^1 f = \sup_{\varphi \leq f} \int_0^1 \varphi$. That means for any $\epsilon > 0$, there exist a simple function $\varphi \leq f$, such that $\int_0^1 f - \int_0^1 \varphi \leq \epsilon$. Without generality, assume $\varphi = \sum_{i=1}^n a_i \chi(F_i)$ and the sets F_i are disjoint. Then $\int_0^1 \varphi = \sum_{i=1}^n a_i mF_i$. Let $a = \max\{a_1, a_2, \ldots, a_n\}$.

Since $\lim mE_n = 1$ and $E_n \subseteq [0, 1]$, then there exists an N, such that for any $n \geq N$, we have $1 - mE_n < \epsilon/a$. Let $F_i' = E_N \cap F_i$. Since F_i are disjoint measurable sets, we have the sets F_i' are disjoint and measurable. Also, $\int_0^1 \varphi \geq \int_{E_N} \varphi$. Then

$$\begin{aligned} |\int_0^1 \varphi - \int_{E_N} \varphi| &= \int_0^1 \varphi - \int_{E_N} \varphi \\ &= \sum_{i=1}^n a_i mF_i - \sum_{i=1}^n a_i mF_i' \\ &\leq \sum_{i=1}^n amF_i - \sum_{i=1}^n amF_i' \\ &= a(\sum_{i=1}^n mF_i - mF_i') \\ &= a(m([0, 1]) - mE_N) \\ &\leq a\frac{\epsilon}{a} \\ &= \epsilon \end{aligned}.$$

Now we have $\int_0^1 \varphi - \int_{E'} \varphi \leq \epsilon$, combine with $\int_0^1 f - \int_0^1 \varphi \leq \epsilon$, we have $\int_0^1 f - \int_{E_N} \varphi \leq 2\epsilon$. Since $\int_{E_N} \varphi \leq \int_{E_N} f \leq \int_0^1 f$, we have $|\int_0^1 f - \int_{E_N} f| \leq 2\epsilon$. Since for any $n > N$, $m([0, 1]) - mE_n \leq \epsilon$, we can also derive such inequalities. Therefore $\lim \int_{E_n} f = \int_0^1 f$.

2nd Proof for (3).

Since f is integrable over $[0,1]$, we have both f^+ and f^- are integrable. By 4.1.9, we have for any ϵ, there exists a $\delta > 0$, such that for any $A \subseteq E$ with $mA < \delta$, we have

$$\int_A f^+ < \epsilon \qquad \int_A f^- < \epsilon.$$

Since $\lim mE_n = 1$, then there exists an N, such that for any $n > N$, we have $1 - mE_n < \delta$. So

$$\int_{[0,1]\setminus E_n} f^+ < \epsilon \qquad \int_{[0,1]\setminus E_n} f^- < \epsilon.$$

Then

$$\int_{[0,1]} f - \int_{E_n} f = \int_{[0,1]\setminus E_n} f = \int_{[0,1]\setminus E_n} f^+ - \int_{[0,1]\setminus E_n} f^- < \epsilon.$$

Then $\lim \int_{E_n} f = \int_0^1 f$.

Ex 4.2.1(3) The first proof uses definition, which is OK. Can we have a better (here is simpler) proof?

The second (another proof) has a gap. The last line of Page 25 does not imply the first line on Page 26. Check the definition of a limit, and understand absolute values.

The might be a third proof, which will train you how to apply a Theorem. Suppose that $f \geq 0$ (or $f \geq 0$ almost every where). Define $f_n = f$ when restricted to E_n. Then $f_n \leq f$, and f_n is increasing. Therefore, Theorem 4.1.7 can give you the answer.

How about if we do not have $f \geq 0$ almost everywhere? Then f^+ and f^- will. You may need to use the "Twin Sister" theorem of Theorem 4.1.7 when the sequence f_n is decreasing.

3rd Proof for (3).

Since f is integrable over $[0,1]$, we have both f^+ and f^- are integrable. Let f_n^+ and f_n^- be restricted on E_n. Then both f_n^+ and f_n^- are increasing sequences. Since $E_n \subseteq [0,1]$ with $\lim mE_n = 1$, we have f_n^+ converges to f^+ a.e. and f_n^- converges to f^- a.e.. By Monotone Convergence Theorem, we have

$$\int_E f^+ = \lim \int_{E_n} f_n^+ = \lim \int_{E_n} f^+ \quad \textit{and} \quad \int_E f^- = \lim \int_{E_n} f_n^- = \lim \int_{E_n} f^-.$$

And so

$$\int_E f = \int_E (f^+ - f^-) = \int_E f^+ - \int_E f^- = \lim \int_{E_n} f^+ - \lim \int_E f^- = \lim \int_{E_n} (f^+ - f^-) = \lim \int_{E_n} f.$$

□

4.2.2 *(Aug, 2005)*

Let E be a measurable set in $[0,1]$ with $mE > 0$. Then there are two numbers x and y in $E (x \neq y)$ such that $x - y$ is a rational number.

Proof: By way of contradiction. Assume that any two elements x, y in E, $x - y$ is not a rational number. Since the rational number is a countable set, let the rational number in $[0,1]$ be $\{r_1, r_2, r_3, \ldots\}$. Then for any two different rational numbers r_i and r_j, if the set $E + r_i \cap E + r_j$ is not empty, then there exists an element z inside, also $z - r_i \neq z - r_j$ and $z - r_i, z - r_j \in E$, but $z - r_i - (z - r_j)$ is a rational number, contradicts to our assumption. Then we assume that $E + r_i \cap E + r_j$ for all $i \neq j$ is empty.

Since E is a subset of $[0,1]$, we have $E + r_k$ is in $[0,2]$ for $k = 0,1,2,\ldots$. And so $\bigcup_{k=1}^{\infty} E_k \subseteq [0,2]$. We have $\sum mE_k \leq m([0,2]) = 2$. But $mE = mE_k > 0$, the summation $\sum mE_k$ should be ∞, which is a contradiction to $\sum mE_k \leq m([0,2]) = 2$. So there must be two numbers x and y in $E(x \neq y)$ such that $x - y$ is a rational number.

□

4.2.3 *(Aug, 2005)*
let $\{f_n\}$ be a sequence of integrable functions on $[0,1]$ such that $f_n \to f$ a.e. in $[0,1]$ with f integrable. Then

$$\lim \int_0^1 |f_n - f| = 0 \qquad \Longleftrightarrow \qquad \lim \int_0^1 |f_n| = \int_0^1 |f|.$$

Proof:" $\Longrightarrow$ ":

If $\lim \int_0^1 |f_n - f| = 0$, then for any $\epsilon > 0$, there exists an N, such that for any $n > N$, we have $\int_0^1 |f_n - f| < \epsilon$. Since $\int_0^1 (|f_n| - |f|) \leq \int_0^1 |f_n - f|$, we also have $\int_0^1 (|f_n| - |f|) = \int_0^1 |f_n| - \int_0^1 |f| < \epsilon$. Then $\lim \int_0^1 |f_n| = \int_0^1 |f|$.

4.2.3 " $\Longrightarrow$ ": same gap in absolute values. You need to show $|\int_0^1 (|f_n| - |f|)| < \epsilon$, and so your arguments need to be used twice.

If $\lim \int_0^1 |f_n - f| = 0$, then for any $\epsilon > 0$, there exists an N, such that for any $n > N$, we have $\int_0^1 |f_n - f| < \epsilon$. Since $\int_0^1 (|f_n| - |f|) \leq \int_0^1 |f_n - f|$ and $\int_0^1 (|f| - |f_n|) \leq \int_0^1 |f_n - f|$, we have $|\int_0^1 (|f_n| - |f|)| \leq \int_0^1 |f_n - f| < \epsilon$. Then $\lim \int_0^1 |f_n| = \int_0^1 |f|$.

" $\Longleftarrow$ ":

Since f is integrable in $[0,1]$, by 4.1.9, for any $\epsilon > 0$, there exists a $\delta > 0$, and any $E \in [0,1]$ with $mE < \delta$, we have $\int_E f < \epsilon$.

By Egoroff's Theorem, if $f_n \to f$ a.e. on a bounded measurable set $[0,1]$, for such $\delta > 0$, there exists a set $A \subseteq [0,1]$ with $mA < \delta$, such that $f_n \to f$ uniformly on $\tilde{A}$. Then there exists an N_1, such that for any $n > N_1$ and $x \in \tilde{A}$, $|f_n(x) - f(x)| < \epsilon$.

Since $\lim \int_0^1 |f_n| = \int_0^1 |f|$, there exists an N_2, such that for any $n > N_2$, $|\int_0^1 |f_n| - \int_0^1 |f|| < \epsilon$. Then we have that $\int_0^1 |f_n| < \int_0^1 |f| + \epsilon$. And so for the set $A \subset [0,1]$ with $mA < \delta$ and any $n > N = \max\{N_1, N_2\}$, we have

$$\begin{array}{rcl} \int_0^1 |f_n| & = & \int_A |f_n| + \int_{\tilde{A}} |f_n| \\ & < & \int_A |f| + \int_{\tilde{A}} |f| + \epsilon \end{array}.$$

Then we have that

$$\begin{array}{rcl} \int_A |f_n| & < & \int_A |f| + \int_{\tilde{A}} |f| - \int_{\tilde{A}} |f_n| + \epsilon \\ & \leq & \int_A |f| + \int_{\tilde{A}} |f - f_n| + \epsilon \\ & < & \epsilon + \epsilon + \epsilon \\ & = & 3\epsilon \end{array}$$

Now we have that for any any $A \subset [0,1]$ with $mA < \delta$ and any $n > N = \max\{N_1, N_2\}$,

$$\begin{array}{rcl} \int_0^1 |f_n - f| & \leq & \int_A |f_n - f| + \int_{\tilde{A}} |f_n - f| \\ & < & \epsilon + \int_{\tilde{A}} |f_n| + \int_{\tilde{A}} |f| \\ & < & 5\epsilon \end{array}.$$

Therefore $\lim \int_0^1 |f_n - f| = 0$.

4.2.4 *(Real Analysis, Royden, Page 94, Problem 17)(Apr,2004)(May, 1996)*
(a) Let f be integrable over R. Then

$$\int f(x)dx = \int f(x+t)dx.$$

(b) Let g be a bouned measurable function. Then

$$\lim_{t\to 0}\int_{-\infty}^{\infty}|g(x)[f(x)-f(x+t)]| = 0.$$

Proof: (a)By the definition of Lebesgue integral, we have

$$\int f(x)dx = \inf_{\varphi\geq f}\int \varphi(x)dx.$$

So for any $\epsilon > 0$, there exists a simple function $\varphi(x)$ such that $\left|\int f(x)dx - \int \varphi(x)dx\right| < \epsilon$, and therefore $\left|\int_R f(x+t)dx - \int_R \varphi(x+t)dx\right| = \left|\int_R f(x+t)d(x+t) - \int_R \varphi(x+t)d(x+t)\right| = \left|\int_R f(y)dy - \int_R \varphi(y)dy\right| < \epsilon$, for $y = x+t$.

For such simple function φ, we have

$$\begin{aligned}\int \varphi(x)dx &= \textstyle\sum_{k=1}^{n} a_i mE_i\\ &= \textstyle\sum_{k=1}^{n} a_i m\{E_i + t\}\\ &= \int \varphi(x+t)dx.\end{aligned}$$

Then

$$\begin{aligned}\left|\int f(x)dx - \int f(x+t)dx\right| &\leq \left|\int f(x)dx - \int \varphi(x)dx\right| + \left|\int \varphi(x)dx - \int \varphi(x+t)dx\right|\\ &\quad + \left|\int \varphi(x+t)dx - \int f(x+t)dx\right|\\ &< 2\epsilon.\end{aligned}$$

Since ϵ is arbitrary, we have $\int f(x)dx = \int f(x+t)dx$.

(b) Assume that $|g|$ is bounded by M. Since $f(x)$ is integrable over R, given $\epsilon > 0$, there is a continuous function h defined on a closed interval $[-N, N]$ and vanishes outside, such that

$$\int_{-\infty}^{\infty}|f-h| < \epsilon/M.$$

Then h is a continuous function defined on $[-N, N]$, and so it is uniformly continuous. That means for any $x \in R$ there exists a $\delta > 0$, such that for any $|t| < \delta$, we have $|h(x) - h(x+t)| < \frac{\epsilon}{2MN}$. And so $\int_{-N}^{N}|h(x) - h(x+t)| < \frac{\epsilon}{M}$. Then we have for any $\epsilon > 0$, there exists $\delta > 0$, such that for any $|t| < \delta$, we have

$$\begin{aligned}\int_{-\infty}^{\infty}|g(x)[f(x)-f(x+t)]| &= \int_{-\infty}^{\infty}|g(x)[f(x)-f(x+t)]|\,dx\\ &= \int_{-\infty}^{\infty}|g(x)|\,|[f(x)-f(x+t)]|\,dx\\ &\leq M\int_{-\infty}^{\infty}|[f(x)-f(x+t)]|\,dx\\ &\leq M\int_{-N}^{N}|[f(x)-h(x)| + |h(x)-h(x+t)| + |h(x+t)-f(x+t)]|\,dx\\ &< 3\epsilon.\end{aligned}$$

Therefore $\lim_{t\to 0}\int_{-\infty}^{\infty}|g(x)[f(x)-f(x+t)]| = 0$.

□

4.2.4 (a) Over what set the integration takes place? Without it, the equality might be false. I think it should be over R, the real number line.

Why do you think you do not need a reason for this "therefore" conclusion:
such that $\left|\int f(x)dx - \int \varphi(x)dx\right| < \epsilon$, and therefore $\left|\int f(x+t)dx - \int \varphi(x+t)dx\right| < \epsilon$.
(You presented the reason after the conclusion, without mentioning the relation between the two).

4.2.5 *(Aug, 2005)*
For $f \in L^1(R)$ and $g \in L^\infty(R)$, define

$$F(t) = \int_R g(x)f(t-x)dx \qquad \text{for } t \in R.$$

Then F is uniformly continuous in R.

Proof: The condition that $g \in L^\infty(R)$ means g is bounded except for a measure 0 set, so it is the same with 4.2.4.

4.2.6 *(May, 2009)(Apr, 2005)(Jan, 2001)*
Let $f \in L^1(R^1)$ and define

$$F(t) = \int_{-\infty}^{\infty} f(x) sin(xt) dx.$$

Prove $F(t)$ is continuous in R. Is $F(t)$ uniformly continuous in R? Prove your conclusion.

Proof: Since $f \in L^1(R^1)$, for any $\epsilon > 0$, there exists a step function $\varphi(x) = \sum_{i=1}^n a_i \chi(E_i)$ vanishes outside $[-N, N]$ such that $\int_{-\infty}^{\infty} |f(x) - \varphi(x)| < \epsilon$. Let $a = \max\{a_1, a_2, \ldots, a_n\}$.

Now consider $\sin xt$. For any $\epsilon > 0$, there exists a $\delta = \frac{\epsilon}{4aN^2}$, such that for any $|\triangle t| < \delta$, we have

$$\begin{aligned}
|\sin xt - \sin[x(t+\triangle t)]| &= |2\cos(xt + xt + x\triangle t)\sin(xt - xt - x\triangle t)| \\
&\le 2|\sin(-x\triangle t)| = 2| - \sin(x\triangle t)| \\
&= 2|\sin(x\triangle t)| \le 2|x\triangle t| \\
&= 2|x||\triangle t| \le 2N\tfrac{\epsilon}{4aN^2} \\
&= \tfrac{\epsilon}{2aN}.
\end{aligned}$$

Then we have

$$\begin{aligned}
|F(t) - F(t+\triangle t)| &= |\int_{-\infty}^{\infty}(f(x)sin(xt) - f(x)sin(x(t+\triangle t)))dx| \\
&= |\int_{-\infty}^{\infty} f(x)[sin(xt) - sin(x(t+\triangle t))]dx| \\
&\le |\int_{-\infty}^{\infty}[f(x) - \varphi(x)][sin(xt) - sin(x(t+\triangle t))]dx| + |\int_{-\infty}^{\infty} \varphi(x)[sin(xt) - sin(x(t+\triangle t))]dx| \\
&\le \int_{-\infty}^{\infty} |f(x) - \varphi(x)|dx + \int_{-N}^{N} |\varphi(x)[sin(xt) - sin[x(t+\triangle t)]]|dx \\
&\le \epsilon + \int_{-N}^{N} |a\tfrac{\epsilon}{2aN}|dx \\
&\le 2\epsilon.
\end{aligned}$$

Therefore $F(t)$ is uniformly continuous on R.

□

4.2.6: You should indicate that your a depends on the given epsilon. Some of the inequalities are not correct.

See a correction below (note that $|[sin(xt) - sin(x(t + \triangle t))]| \le |sin(xt)| + |sin(x(t + \triangle t)) \le 2$**).**

$$
\begin{aligned}
|F(t) - F(t + \triangle t)| &= |\int_{-\infty}^{\infty}(f(x)sin(xt) - f(x)sin(x(t + \triangle t)))dx| \\
&= |\int_{-\infty}^{\infty} f(x)[sin(xt) - sin(x(t + \triangle t))]dx| \\
&\le |\int_{-\infty}^{\infty}[f(x) - \varphi(x)][sin(xt) - sin(x(t + \triangle t))]dx| + |\int_{-\infty}^{\infty} \varphi(x)[sin(xt) - sin(x(t + \triangle t))]dx| \\
&\le 2\int_{-\infty}^{\infty}|f(x) - \varphi(x)|dx + \int_{-N}^{N}|\varphi(x)[sin(xt) - sin[x(t + \triangle t)]]|dx \\
&\le 2\epsilon + \int_{-N}^{N}|a\frac{\epsilon}{2aN}|dx \\
&\le 3\epsilon.
\end{aligned}
$$

4.2.7 *(Aug, 2004)(Apr, 2004)(May, 1996)*
If $f(x)$ is integrable over R, then

$$\lim_{\lambda\to\infty}\int_R f(x)\cos(\lambda x)dx = 0.$$

Proof: Since $f(x)$ is integrable over R, for any $\epsilon > 0$, there exists a step function $\varphi(x) = \sum_{i=1}^n a_i\chi(E_i)$ defined on $[-M, M]$ and vanishes outside with that $\int_R |f(x) - \varphi(x)| < \epsilon$. Let $a = \max\{a_1, a_2, \ldots, a_n\}$ and $N = \left[\frac{a}{\epsilon}\right]$. Then for any $\lambda > N$, we have

$$
\begin{aligned}
\int_R f(x)\cos(\lambda x)dx &= \int_R[f(x) - \varphi(x)]\cos(\lambda x)dx + \int_R \varphi(x)\cos(\lambda x)dx \\
&\le \int_R[f(x) - \varphi(x)]dx + \int_{-M}^{M}\sum_{i=1}^n a_i\chi(E_i)\cos(\lambda x)dx \\
&\le \int_R |f(x) - \varphi(x)|dx + \int_M^M a\cos(\lambda x)dx \\
&< \epsilon + \frac{a}{\lambda}\sin\lambda x|_{-M}^{M} \\
&\le \epsilon + \frac{2a}{\lambda} \\
&\le 3\epsilon.
\end{aligned}
$$

Therefore $\lim_{\lambda\to\infty}\int_R f(x)\cos(\lambda x)dx = 0$.

□

4.2.8 *(Apr, 2005)*

(a) State Fatou's Lemma.

(b) Show by an example that the strict inequality in Fatou's Lemma is possible.

(c) Show that Fatou's Lemma can be derived from the Monotone Convergence Theorem.

Proof: (a) See 4.1.6.

(b) Define $f_n = n\chi([0, \frac{1}{n}])$. Then for any $\epsilon = \frac{1}{k} > 0$, and any $x \in [\epsilon, 1]$, let $N = k$, we have $|f_n(x) - 0| = 0$ for any $n > N$. Since ϵ is arbitrary, we have $f_n(x) \to f(x) = 0$ a.e. on $[0, 1]$. But $\int f_n = 1$ for any real number n, and $\int f = 0$.

(c) Define $h_n(x) = \inf_{i\ge n}\{f_i(x)\}$. Then h_n is an increasing sequence of functions with $h_n(x) \to f(x)$ a.e. By the Monotone Convergence Theorem, $\int f = \lim\int h_n \le \underline{\lim}\int f_n$.

□

4.2.9 *(Aug, 2003)*
Is the product of two integrable functions from R to R integrable?

Proof: It's not. Let $f(x) = x^{-\frac{1}{2}}$ with $x \in (0,1)$. Since $f(x)$ is measurable, $f(x) \geq 0$ and $\int_0^1 f(x)dx = 2$, we have $f(x)$ is integrable. But $\int_0^1 f(x)^2 = \int_0^1 x^{-1} > \sum_{k=1}^{\infty} \frac{1}{k} = \infty$, so the product is not an integrable function.

□

4.2.10 *(Aug, 2003)*
Show that

$$\lim_{n\to\infty} \int_0^1 \frac{x^n + 1}{x^n + 2}$$

exists and find its value.

Proof: Since $\int_0^1 \frac{x^n+1}{x^n+2} = \int_0^1 \frac{x^n+2-1}{x^n+2} = 1 - \int_0^1 \frac{1}{x^n+2}$ and $f_n(x) = \frac{1}{x^n+2}$ with $x \in [0,1]$ is an increasing sequence of nonnegative measurable function bounded by 1/2, by Monotone Convergence Theorem, we have $\lim \int_0^1 \frac{1}{x^n+2} = \int_0^1 \lim \frac{1}{x^n+2} = 1/2$. And so

$$\lim_{n\to\infty} \int_0^1 \frac{x^n + 1}{x^n + 2} = 1 - \lim_{n\to\infty} \int_0^1 \frac{1}{x^n + 2} = 1/2.$$

□

4.2.11 *(Aug, 2003)*
Let $\{f_n\}$ *be a sequence of measurable functions on* $[0,1]$*. Describe the three concept of convergence stated in (1)-(3) and give any implications between them. The implication must be proved. The lack of implication must be supported by a counter example.*

(1) $f_n \to 0$ *in measure as* $n \to \infty$.

(2) $f_n \to 0$ *a.e. as* $n \to \infty$.

(3) $\|f_n\|_1 \to 0$ *as* $n \to \infty$.

Proof:

(1) $f_n \to 0$ in measure as $n \to \infty$ means for any $\epsilon > 0$, there exists an N, such that for each $n > N$, we have $m\{x : |f_n(x) - 0| \geq \epsilon\} < \epsilon$.

(2) $f_n \to 0$ a.e. as $n \to \infty$ means the set of points in E which is not converges to 0 has measure 0.

(3) $\|f_n\|_1 \to 0$ as $n \to \infty$ means for any $\epsilon > 0$, there exists an N, such that for each $n > N$, we have $|\|f_n(x)\| - 0| < \epsilon$.

(1) $\nRightarrow$ (2)

Let $n = k + 2^m$ with $0 \leq k < 2^m$ and $m = 1, 2, 3\cdots$. Then $n = 1, 2, 3, \cdots$. Define $f_n(x) = 1$ when $n = k + 2^m$ and $x \in (k2^{-m}, (k+1)2^{-m}]$. Otherwise $f_n(x) = 0$.

Then for any $1 > \epsilon > 0$, there exists $2^{-m_0} < \epsilon$, let $N = 2^{m_0}$, then we have $m\{x : |f_n(x) - 0| \geq \epsilon\} < \epsilon$ for any $n > N$. So $f_n \to 0$ in measure.

But for any $x \in (0,1]$, let $\epsilon_0 = 1/2$, for any $N > 0$, there exists $2^{m_1} > N$, such that x is in one of the interval $(k2^{-m_1}, (k+1)2^{-m_1}]$ with $0 \leq k < 2^{m_1}$. Assume that this interval is $(k_0 2^{-m_1}, (k_0+1)2^{-m_1}]$ and so $|f_{k0+2^{m_1}} - 0| = 1 > \epsilon_0$. Then $f_n \nrightarrow 0$ anywhere in $(0,1]$.

4.2.11

(1) $\not\Rightarrow$ (2): You need to show, by definition, that $m\{f_n \nrightarrow 0\} > 0$. In this example, you have $m\{f_n \nrightarrow 0\} = 2^{-m} \to 0$.

Here is a question: how do we describe the set $A = \{f_n \nrightarrow 0\}$?

$x \in A$ **iff there exists a $1/m > 0$, and for any $N > 0$, there exists an $n \geq N$ such that $|f_n(x)| \geq 1/m$. This might suggests that:**

Can we show that (you may need to modify it)

$$A = \cup_{m=1}^{\infty} \cup_{N=1}^{\infty} \cup_{n \geq N} \{|f_n(x)| \geq 1/m\}?$$

Then, ask the question (the same idea of thinking by definitions), can we show that $m(A) = 0$? If not, how do we construct a correct example from the proof? Remember, you need an example with $m(A) > 0$.

Since the set $A = \{f_n n \to 0\} = \{\}$

(2) $\Rightarrow$ (1)

If $f_n \to 0$ a.e., by 3.2.16, we have for any $\epsilon > 0$ and $\delta > 0$, there exists an N and $A \subset [0,1]$ with $mA < \delta$, such that for any $n > N$ and $x \in \tilde{A}$, we have $|f_n(x)| < \epsilon$. Let $\delta = \epsilon$. Then $\{x : |f_n(x)| \geq \epsilon\} \subseteq A$, that means $m\{x : |f_n(x)| \geq \epsilon\} < \epsilon$.

(2) $\not\Rightarrow$ (3)

Define $f_n(x) = n\chi([0, 1/n])$, by 4.2.8, $f_n \to 0$ a.e., but $\|f_n\|_1 \to 1$.

(3) $\not\Rightarrow$ (2)

We could use the example given in (1) $\not\Rightarrow$ (2). We have $\|f_n\|_1 \to 0$ but $f_n \nrightarrow 0$ a.e..

(1) $\not\Rightarrow$ (3)

We could use the example given in (2) $\not\Rightarrow$ (3). We have $f_n \to 0$ a.e. hence $f_n \to 0$ in measure, but $\|f_n\|_1 \to 1$.

(3) $\Rightarrow$ (1)

By way of contradiction, assume that $f_n \nrightarrow 0$ in measure, then there exists an $\epsilon_0 > 0$, such that for any N, there exists an $n > N$ with $m\{x; |f_n(x)| \geq \epsilon_0\} \geq \epsilon_0$. But then $\|f_n\|_1 \geq \epsilon_0^2 > 0$, contradicts to $\|f_n\|_1 \to 0$.

□

4.2.12 *(Aug, 2003)*

Suppose that f is a continuous function on $[0,1]$ for which

$$\int_{[0,1]} f(t)t^n dt = 0, \qquad \forall n = 1, 2, 3, \ldots$$

Show that f is the zero function.

Proof: Since f is a continuous function on $[0,1]$, assume that $|f|$ is bounded by M. And for any $\epsilon > 0$, there exists a polynomial $p(x)$ such that $|f(x) - p(x)| < \epsilon/M$ for any $x \in [0,1]$. And since $\int_{[0,1]} f(t)t^n dt = 0$, for $\forall n = 1, 2, 3, \ldots$, we have

$$\int_{[0,1]} f(x)p(x) = 0 = \int_{[0,1]} f(x)[p(x) - f(x)] + \int_{[0,1]} f(x)^2.$$

Since $|\int_{[0,1]} f(x)[p(x) - f(x)]| < \epsilon$, we have $|\int_{[0,1]} f(x)^2| < \epsilon$. Then $f(x)^2 = 0$ a.e., and so $f(x) = 0$ a.e..Since f is a continuous function with $f(x) = 0$ a.e., $f(x)$ is the zero function.

□

4.2.12: Need a reason why there exists such a poly. I told you this solution before. You need an approximation theorem here:

And for any $\epsilon > 0$, there exists a polynomial $p(x)$ such that $|f(x) - p(x)| < \epsilon/M$ for any $x \in [0,1]$.

4.2.13 *(Apr, 2002)*

Let $\{g_n\}$ be a sequence of nonnegative integrable function which converges to an integrable function g. Let f_n be a sequence of measurable functions such that $|f_n| < g_n$ and $f_n \to f$ a.e.. if

$$\int_0^1 g(x)dx = \lim \int_0^1 g_n(x)dx,$$

then

$$\int_0^1 f(x)dx = \lim \int_0^1 f_n(x)dx.$$

Proof: Since Let f_n is a sequence of measurable functions with $|f_n| < g_n$, we have $\int f_n \leq \int g_n < \infty$, so f_n is integrable. And so f is integrable since $\int f \leq \underline{\lim} f_n < \infty$. And so $g_n - f_n$ is nonnegative. By Fatou's Lemma, we have

$$\int g - f \leq \underline{\lim} \int g_n - f_n.$$

Then

$$\int g - \int f \leq \underline{\lim} \int g_n - \overline{\lim} \int f_n.$$

That is

$$\int f \geq \overline{\lim} \int f_n.$$

Again by Fatou's Lemma, since $f_n + g_n$ is nonnegative, we have

$$\int g + f \leq \underline{\lim} \int g_n + f_n.$$

Then

$$\int f \leq \underline{\lim} \int f_n.$$

So $\int_0^1 f(x)dx = \lim \int_0^1 f_n(x)dx$.

□

4.2.14 *(Aug, 2001)*

Let f be nonnegative and measurable on E, and $E_n = \{x \in E : f(x) \geq n\}$. Show that if $\sum_{n=1}^{\infty} n \cdot mE_n < \infty$, then f is integrable on E, but the converse is not true.

Proof: Let $f(x) = 1/2$ with $E = R$, that is $mE = \infty$, then $E_n = \emptyset$ for all n, so $\sum_{n=1}^{\infty} n \cdot mE_n = 0 < \infty$. But $\int_R f = \infty$, so f is not integrable on R.

If $mE < \infty$. Let $E_0 = E$ and $F_n = E_n \setminus E_{n+1}$ with $n = 0, 1, 2 \dots$. Then $F_n = \{x \in E : n \leq f(x) < n+1\}$. Then

$$\begin{aligned} \int f &\leq mF_0 + 2mF_1 + 3mF_2 + \dots \\ &= mE - mE_1 + 2(mE_1 - mE_2) + 3(mE_2 - mE_3) + \dots \\ &= mE + mE_1 + mE_2 + mE_3 + \dots \\ &\leq mE + \textstyle\sum_{n=1}^{\infty} n \cdot mE_n \\ &< \infty. \end{aligned}$$

So f is integrable on E.

For the converse part, let $f(x) = x^{-1/2}$. Then $\int_{(0,1)} f(x)dx = 2$, so f is integrable over $(0, 1)$. But the set $E_n = (0, 1/n^2)$ for $n = 1, 2, 3 \dots$. Then we have $\sum_{n=1}^{\infty} n \cdot mE_n = \sum_{n=1}^{\infty} 1/n = \infty$.

□

4.2.15 *(Apr, 2001)*

Let f_n be a sequence of integrable functions over E such that $0 \leq f_{n+1} \leq f_n$ hold almost everywhere for each n. Show that $f_n \downarrow 0$ holds almost everywhere if and only if $\int f_n dx \downarrow 0$.

Proof: $\Rightarrow$: Since f_1 is integrable, $f_n \downarrow 0$ a.e. and $0 \leq f_{n+1} \leq f_n$, by Lebesgue Convergence Theorem, we have $\int f_n dx \downarrow 0$.

$\Leftarrow$:

For any fixed $\epsilon > 0$, let $E_n = \{x : |f_n| < \epsilon\}$ with $n = 1, 2, 3 \cdots$. Then we have $E_1 \subseteq E_2 \subseteq E_3 \dots$. If $mE_n \nrightarrow mE$, then there exists an $\epsilon_0 > 0$, for any N, there exists an $n > N$, such that

$$mE - mE_n = m(E \setminus E_n) = m\{x : |f_n(x)| \geq \epsilon\} > \epsilon_0,$$

then $\int_E f_n \geq \epsilon \cdot \epsilon_0 > 0$, which is a contradiction to $\int f_n dx \downarrow 0$.

So $mE_n \to mE$. That means for any $\delta > 0$, there exists an N, such that for any $n > N$, we have

$$mE - mE_n = m\{x : |f_n(x)| \geq \epsilon\} < \delta.$$

Since ϵ and δ are arbitrary, and $E_1 \subseteq E_2 \subseteq E_3 \cdots$, we have if for any $y \notin \{x : |f_N(x)| \geq \epsilon\}$, $|f_n(y)| < \epsilon$ for any $n > N$. And since the set $\{x : |f_n(x)| \geq \epsilon\}$ is smaller than any δ, we have $f_n \downarrow 0$ a.e..

□

4.2.16 *(Aug, 2000)*

Suppose $\{f_n\}$ is a sequence of nonnegative measurable functions, and $\sum_{n=1}^{\infty} f_n$ converges a.e. on E. Show that

$$\int_E \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int_E f_n.$$

Proof: Let $\sum_{n=1}^{\infty} f_n$ converges to f a.e. on E. Since $\{f_n\}$ is nonnegative and measurable, then $g_n = \sum_{k=1}^{n} f_n$ is a increasing sequence of nonnegative measurable function with $g_n \to f$ a.e.. By Monotone Convergence Theorem, we have

$$\int_E \sum_{n=1}^{\infty} f_n = \int_E f = \lim \int_E g_n = \lim \int_E \sum_{k=1}^{n} f_n = \lim \sum_{k=1}^{n} \int f_n = \sum_{k=1}^{\infty} \int f_n.$$

□

4.2.16: Need a reason why f exists:

Since $\{f_n\}$ is nonnegative and measurable, then $g_n = \sum_{k=1}^n f_n$ is a increasing sequence of nonnegative measurable function with $g_n \to f$ a.e..

4.2.17 *(Aug, 1997) Suppose $\{f_n\}$ and f are measurable functions and $f_n \to f$ a.e. in E with $mE < \infty$. Show that there exists a sequence of measurable sets $\{E_k\}_{k=1}^\infty$ such that*

$$\bigcup_{k=0}^\infty E_k = E,\ mE_0 = 0, \text{ and } f_n \to f \text{ uniformly on each } E_k \text{ for } k = 1, 2, \cdots .$$

Proof: Since $f_n \to f$ a.e., by Egoroff's Theorem, we have for $\delta_k = 1/k$, there exists a measurable set $F_k \subseteq E$ with $mF_k < \delta_k$ such that f_n converges to f uniformly on $\widetilde{F_k} = E_k$. Let $E_0 = \cap_{k=1}^\infty F_k$. Then $mE_0 = 0$ and

$$E = \bigcap_{k=1}^\infty F_k \cup \widetilde{\bigcap_{k=1}^\infty F_k} = E_0 \cup \bigcup_{k=1}^\infty \widetilde{F_k} = \bigcup_{k=0}^\infty E_k.$$

□

4.2.18 *(May, 1996) Give one example of Lebesgue integrable function $f(x)$ on $[0,1]$ which is not Riemann integrable. Explain why.*

Proof: Define $f(x) = \begin{cases} 1 & x \notin [0,1] \cap Q \\ 0 & x \in [0,1] \cap Q \end{cases}$. Then $f(x)$ is discontinuous at every point in $[0,1]$, so it is not Riemann integrable. But it is Lebesgue integrable, since f can write down as a simple function $f(x) = c_1\chi(E_1) + c_2\chi(E_2)$ with $c_1 = 1$, $c_2 = 0$, $E_1 = [0,1] \setminus Q$ and $E_2 = [0,1] \cap Q$, where E_1 and E_2 are measurable. So $\int f = c_1 mE_1 + c_2 mE_2 = 1 < \infty$. Therefore $f(x)$ is Lebesgue integrable.

□

4.2.19 *(Aug, 2005)*

If E is a measurable set with $m(E) < \infty$, and $\{f_n\}$ is a sequence of measurable function on E, then

$$f_n \to 0 \text{ in measure} \Longleftrightarrow \int_E \frac{|f_n|}{1+|f_n|} \to 0.$$

Proof: $\Longrightarrow$: Since $f_n \to 0$ in measure, we have for any $\epsilon > 0$, there exists an N, such that for any $n > N$, the set $A_n = \{x : |f_n(x) - 0| \geq \epsilon/mE\}$ is with measure smaller than ϵ/mE. Now we have

$$\begin{aligned} \int_E \tfrac{|f_n|}{1+|f_n|} &= \int_{A_n} \tfrac{|f_n|}{1+|f_n|} + \int_{E\setminus A_n} \tfrac{|f_n|}{1+|f_n|} \\ &\leq \int_{A_n} 1 + \int_{E\setminus A_n} \tfrac{\epsilon/mE}{1+|f_n|} \\ &\leq \int_{A_n} 1 + \int_{E\setminus A_n} \epsilon/mE \\ &< 2\epsilon. \end{aligned}$$

So $\int_E \frac{|f_n|}{1+|f_n|} \to 0$.

$\Longleftarrow$: By way of contradiction. Assume that $f_n \nrightarrow 0$ in measure, then there exists an $\epsilon_0 > 0$, for any N, there exists an $n > N$ such that the set $A = \{x : |f_n(x) - 0| \geq \epsilon_0\}$ is with measure $mA \geq \epsilon_0$. Then we have $\int_E \frac{|f_n|}{1+|f_n|} = \int_{E\setminus A} \frac{|f_n|}{1+|f_n|} + \int_A \frac{|f_n|}{1+|f_n|} \geq \int_A \frac{\epsilon_0}{1+|f_n|}$???

4.2.19: $\Longleftarrow$:

Your using the definition is the right direction. What is missing is a little bit calculus:

The function $h(x) = x/(1+x)$ is an increasing function (compute the derivative to see $h'(x) = 1/(1+x)^2 > 0$). So the proof goes like this: (I am using your wording. Also pay attention to English: where did I change your wording?)

By way of contradiction. Assume that $f_n \nrightarrow 0$ in measure. Then (not then) there exists an $\epsilon_0 > 0$, such that for any N, there exists an $n > N$ such that the set $A = \{x : |f_n(x)| \geq \epsilon_0\}$ is with measure $mA \geq \epsilon_0$.

Then we have $\int_E \frac{|f_n|}{1+|f_n|} = \int_{E\setminus A} \frac{|f_n|}{1+|f_n|} + \int_A \frac{|f_n|}{1+|f_n|} \geq \int_A \frac{\epsilon_0}{1+\epsilon_0} \geq 0$. (you add the details).

4.2.20 *(Apr, 2005)*
Suppose f is a non-negative integrable function on $[0,1]$. If

$$\int_0^1 f^n = \int_0^1 f \quad \textit{for all } n = 1, 2, \ldots,$$

then $f(x)$ must be the characteristic function of some measurable set $E \subset [0,1]$ except for a measure 0 set.

If there exists an $\epsilon > 0$, the set $\{x : f(x) > 1+\epsilon\}$ has positive measure, then $f^n \to \infty$ and so $\int_0^1 f^n \to \infty$, a contradiction to $\int_0^1 f^n = \int_0^1 f$ and f is integrable.

Also, if the set $\{x : \epsilon < f(x) < 1-\epsilon\}$ has positive measure, then $f^n \to 0$ and so $\int_0^1 f^n \to 0$. Then contradicts to $\int_0^1 f^n = \int_0^1 f$ for any $n = 1, 2, \ldots$.

Since f is a non-negative integrable function on $[0,1]$, it is measurable and only takes 0 and 1 except for a measure 0 set, so $f(x)$ must be the characteristic function of some measurable set $E \subset [0,1]$ except for a measure 0 set.

□

New begin from Aug 1st

Chapter 5

Differentiation and Integration

5.1 Notes

Theorem 5.1.1 *Let f be an increasing real-valued function defined in the interval $[a,b]$. Then f is differentiable a.e., the derivative f' is measurable, and*

$$\int_a^b f'(x)dx \leq f(b) - f(a).$$

Let f be a real-valued function defined on the interval $[a,b]$, and let $a = x_0 < x_1 < \cdots < x_k = b$ be any subdivision of $[a,b]$. Define

$$p = \sum_{i=1}^{k} [f(x_i) - f(x_{i-1})]^+$$

$$t = \sum_{i=1}^{k} [f(x_i) - f(x_{i-1})]^-$$

$$t = p + n = \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})|,$$

where we use r^+ to denote r, if $r \geq 0$ and 0, if $r < 0$, and set $r^- = |r| - r^+$. We have $f(b) - f(a) = p - n$. Set

$$P = \sup p,$$

$$N = \sup n,$$

$$T = \sup t,$$

where we take the suprema over all possible subdivisions of $[a,b]$. We call P, N, T the positive negative, and total variation of f over $[a,b]$. We sometimes write $T_a^b(f)$ to denote the total variation of f on $[a,b]$. If $T < \infty$, we say f is of bounded variation over $[a,b]$ or abbreviated by $f \in BV$.

Lemma 5.1.2 *If f is of bounded variation on $[a,b]$, then*

$$T_a^b = P_a^b + N_a^b$$

and

$$P_a^b - T_a^b = f(b) - f(a).$$

Proof: Since for any subdivision, we have $p = n + f(b) - f(a)$, taking suprema of both sides, we have $P_a^b - N_a^b = f(b) - f(a)$.

Since $t = p + n = 2n + f(b) - f(a)$, taking suprema of both sides, we have $T_a^b = 2N_a^b + f(b) - f(a)$, and by what we have proved, $N_a^b = -f(b) + f(a) + P_a^b$. So

$$T_a^b = 2N_a^b + f(b) - f(a) = N_a^b + f(b) - f(a) - f(b) + f(a) + P_a^b = P_a^b + N_a^b.$$

Theorem 5.1.3 *A function f is of bounded variation on $[a, b]$ if and only if it is the difference of two monotone real-valued functions on $[a, b]$.*

Proof: ”$\Rightarrow$: ” Define $g(x) = P_a^x(f)$ and $h(x) = N_a^x(f)$. Then g and h are monotone increasing functions on $[a, b]$. By 5.1.2 and since $P_a^x + f(a)$ is a monotone increasing function, we have $f(x) = P_a^x - N_a^x + f(a)$, the difference of two monotone increasing functions.

”$\Leftarrow$: ” If $f = g - h$ where g and h are two monotone increasing functions on $[a, b]$, then for any subdivision, we have

$$\sum |f(x_i) - f(x_{i-1})| \leq \sum g(x_i) - g(x_{i-1}) + \sum h(x_i) + h(x_{i-1}) = g(b) - g(a) + h(b) + h(a) < \infty.$$

So $T_a^b < g(b) - g(a) + h(b) + h(a) < \infty$, then f is of bounded variation.

Corollary 5.1.4 *If f is of bounded variation on $[a, b]$, then $f'(x)$ exists a.e. in $[a, b]$.*

Theorem 5.1.5 *Let f be an integrable function in $[a, b]$, and suppose that*

$$F(x) = F(a) + \int_a^x f(t)dt.$$

Then $F'(x) = f(x)$ a.e. in $[a, b]$.

A real-valued function f defined on $[a, b]$ is said to be absolutely continuous on $[a, b]$ if for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\sum_{i=1}^{n} |f(x_i') - f(x_i)| < \epsilon$$

for every finite collection $\{(x_i', x_i)\}$ of nonoverlapping intervals with

$$\sum_{i=1}^{n} |x_i' - x_i| < \delta.$$

Theorem 5.1.6 *If f is absolutely continuous on $[a, b]$, then it is of bounded variation on $[a, b]$.*

Theorem 5.1.7 *If f is absolutely continuous on $[a, b]$ and $f'(x) = 0$ a.e., then f is constant.*

Theorem 5.1.8 *A function F is an indefinite integral if and only if F is absolutely continuous.*

5.2 Exercises

5.2.1 *(Aug, 2005)*
Determine if the following is true or false:
If f is absolutely continuous in $[0,1]$, then f is of bounded variation in $[0,1]$.

Proof: By the definition of absolutely continuous, we have for $\epsilon = 1$, there exists a $\delta > 0$, such that

$$\sum_{i=1}^{n} |f(x_i') - f(x_i)| < 1$$

for every finite collection $\{(x_i', x_i)\}$ of nonoverlapping intervals with

$$\sum_{i=1}^{n} |x_i' - x_i| < \delta.$$

Then for any subdivision of $[0,1]$, we can split them (by adding new division point) into $K = \left[\frac{1}{\delta}\right] + 1$ sets of intervals with each length smaller than δ. So $T_0^1(f) \le K$, f is of bounded variation on $[0,1]$.

□

5.2.2 *(May, 2009)(Apr, 2005)*
Let $f(x)$ be monotone increasing on $[0,1]$ with $f(0) = 0$ and $f(1) = 1$. If the set $\{f(x); x \in [0,1]\}$ is dense in $[0,1]$, show that f is a continuous function on $[0,1]$. Is it absolutely continuous on $[0,1]$? Prove your conclusion.

Proof: We have proved f is continuous in 2.1.25. Let f be the Cantor ternary function in $[0,1]$, then f is monotone increasing and continuous. Also $f'(x) = 0$ a.e.. If it is absolutely continuous, by 5.1.8, we have $\int_0^1 f'(x) = f(1) - f(0)$. But $\int_0^1 f'(x) = 0$ not equal to $f(1) - f(0) = 1$. So f may not be absolutely continuous on $[0,1]$.

□

5.2.3 *(Aug, 2004)(Apr, 2004)*
Let f, g be two absolutely continuous functions on $[0,1]$. Prove that fg is absolutely continuous on $[0,1]$. Is it also true if the interval is replaced by $(-\infty,\infty)$? Justify your conclusion.

Proof: Since f, g are absolutely continuous $[0,1]$, it is continuous and so bounded. Assume that f is bounded by K, g is bounded by M. By the definition of absolutely continuous, we have for any ϵ, there exists δ_1 and δ_2, such that

$$\sum_{i=1}^{n} |f(x_i') - f(x_i)| < \epsilon/M \qquad \sum_{i=1}^{n} |g(y_i') - g(y_i)|/K$$

for any nonoverlapping intervals $\{(x_i', x_i)\}$ and $\{(y_i', y_i)\}$ with

$$\sum_{i=1}^{n} |x_i' - x_i| < \delta_1 \qquad \sum_{i=1}^{n} |y_i' - y_i| < \delta.$$

Then let $\delta = \min\{\delta_1, \delta_2\}$. We have

$$\begin{aligned}
\sum_{i=1}^{n} |f(x_i')g(x_i') - f(x_i)g(x_i)| &\le \sum_{i=1}^{n} |f(x_i')g(x_i') - f(x_i')g(x_i)| + \sum_{i=1}^{n} |f(x_i')g(x_i) - f(x_i)g(x_i)| \\
&\le \sum_{i=1}^{n} K|g(x_i') - g(x_i)| + \sum_{i=1}^{n} M|f(x_i') - f(x_i)| \\
&< 2\epsilon
\end{aligned}$$

for any nonoverlapping intervals $\{(x_i', x_i)\}$ with $\sum_{i=1}^{n} |x_i' - x_i| < \delta$.

It is not true if the interval is replaced by $(-\infty, \infty)$. Let $f(x) = g(x) = x$. Then for any $\epsilon > 0$, let $\delta = \epsilon$, for any finite nonoverlapping intervals $\{(x_i', x_i)\}$ with $\sum_{i=1}^{n} |x_i' - x_i| < \delta$, we have that $\sum_{i=1}^{n} |f(x_i') - f(x_i)| = \sum_{i=1}^{n} |x_i' - x_i| < \epsilon$.

But $f(x)g(x) = x^2$ is not absolutely continuous. Since for $\epsilon_0 = 1$ and any $\delta > 0$, there exists $x \in R$ with $|1/x| < \delta$ such that $|(x + 1/x)^2 - x^2| = 2 + 1/x^2 > \epsilon_0$.

□

5.2.4 *(Aug,2004)*

Let $f(x) = x \cos \frac{\pi}{x}$ *for* $0 < x \le 1$ *and* $f(0) = 0$.

(1) Is f *continuous on* $[0, 1]$*?*

(2) Is f *uniformly continuous on* $[0, 1]$*?*

(3) Is f *absolutely continuous on* $[0, 1]$*?*

Justify your conclusion.

Proof: (1) For a fixed $x \in (0, 1]$ and any ϵ, let $\delta_1 = \frac{1}{2}x$, $\delta_2 = \frac{x}{2\pi}\epsilon$, $\delta_3 = \epsilon$ and $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, for any $y \in (x - \delta, x + \delta)$, we have

$$\begin{aligned}
|\cos \tfrac{\pi}{x} - \cos \tfrac{\pi}{y}| &= |2 \sin 1/2(\tfrac{\pi}{x} + \tfrac{\pi}{y}) \sin 1/2(\tfrac{\pi}{x} - \tfrac{\pi}{y})| \\
&\le 2|\sin 1/2(\tfrac{\pi}{x} - \tfrac{\pi}{y})| \\
&< 2|1/2(\tfrac{\pi}{x} - \tfrac{\pi}{y})| \\
&= \pi \tfrac{|x-y|}{xy} \\
&\le 2\pi \tfrac{|x-y|}{x^2} \\
&< \tfrac{\epsilon}{x}.
\end{aligned}$$

And so

$$\begin{aligned}
|f(x) - f(y)| &= |x \cos \tfrac{\pi}{x} - y \cos \tfrac{\pi}{y}| \\
&\le |x \cos \tfrac{\pi}{x} - x \cos \tfrac{\pi}{y}| + |x \cos \tfrac{\pi}{y} - y \cos \tfrac{\pi}{y}| \\
&= |x||\cos \tfrac{\pi}{x} - \cos \tfrac{\pi}{y}| + |\cos \tfrac{\pi}{y}||x - y| \\
&< \epsilon + \epsilon.
\end{aligned}$$

Then f is continuous on $[0, 1]$.

(2) Since $[0, 1]$ is a closed bounded interval, we have f is also uniformly continuous on $[0, 1]$.

(3) f is not absolutely continuous on $[0, 1]$. By 5.2.1, we want to prove $f \notin BV$. Since $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$, for any $M > 0$, there exists $N > 0$, such that for any $n > N$, we have $\sum_{k=1}^{N} \frac{1}{k} > M$. Let $0 < 1/N < 1/(N - 1) < \cdots < 1$

be a subdivision of $[0,1]$, then the total variation for this subdivision is

$$\begin{aligned} & |f(0) - f(1/N)| + |f(1/N) - f(1/(N-1)| + \cdots + |f(1/2) - f(1)| \\ > \; & |f(1/N) - f(1/(N-1))| + \cdots + |f(1/2) - f(1)| \\ = \; & |1/N \cos \tfrac{\pi}{1/N} - 1/(N-1) \cos \tfrac{\pi}{1/(N-1)}| + |1/(N-1) \cos \tfrac{\pi}{1/(N-1)} - 1/(N-2) \cos \tfrac{\pi}{1/(N-2)}| \\ & + \cdots + |1/2 \cos \tfrac{\pi}{1/2} - \cos \pi| \\ > \; & 1/N + 1/(N-1) + \cdots + 1 \\ > \; & M. \end{aligned}$$

Then $f \notin BV$.

□

5.2.5 *(Apr, 2002)*
Prove that a function F is an indefinite integral $\Longleftrightarrow$ it is absolutely continuous.

$\Rightarrow$: By 4.1.9.
$\Leftarrow$: If f is absolutely continuous, it is also of bounded variation, then

$$F(x) = F_1(x) - F_2(x)$$

where F_1 and F_2 are monotone increasing functions. Then $F' = F_1' - F_2'$ exists a.e. and so

$$\int F'(x) = \int F_1'(x) - F_2'(x) \leq \int F_1(x) + F_2(x) \leq F_1(b) + F_2(b) - F_1(a) - F_2(a) < \infty.$$

Then F' is integrable. Let $G(x) = \int_a^x F'(t)dt$, then G is absolutely continuous and so is $f = F - G$.
???

5.2.6 *(Aug, 2001)*
Let $f(x,y)$ be a bounded function on the unit square $Q = (0,1) \times (0,1)$. Suppose for each y, that f is a measurable function of x. For each $(x,y) \in Q$, let the partial derivative $\frac{\partial f}{\partial x}$ exists. Under the assumption that $\frac{\partial f}{\partial y}$ is bounded in Q, prove that

$$\frac{d}{dy}\int_0^1 f(x,y)dx = \int_0^1 \frac{\partial f(x,y)}{\partial y}dx.$$

Proof: Since

$$\frac{\int_0^1 f(x,y+h)dx - \int_0^1 f(x,y)dx}{h} = \int_0^1 \frac{f(x,y+h) - f(x,y)}{h}dx,$$

and

$$\frac{f(x,y+h) - f(x,y)}{h} \to \frac{\partial f(x,y)}{\partial y} \quad as \quad h \to 0.$$

If $\frac{\partial f}{\partial y}$ is bounded, by bounded convergence theorem, we have

$$\frac{d}{dy}\int_0^1 f(x,y)dx = \int_0^1 \frac{\partial f(x,y)}{\partial y}dx.$$

□

5.2.7 *(Aug, 2001)(Aug, 1997)(May, 1996)*
Prove:

(1) If f is absolutely continuous on $[a, b]$, then for any set $E \subset [a, b]$ with $mE = 0$, there holds $m(f(E)) =$ 0.

(2) For a continuous and increasing function f on $[a, b]$, if $m(f(E)) = 0$ for every E in $[a, b]$ with $mE = 0$, then f is absolutely continuous on $[a, b]$.

Proof: (1) Since $E \subset [a, b]$ with $mE = 0$, we have for any $\delta > 0$, there exists a open covering $O = \cup O_i$ with $mO < \delta$ to cover E in $[a, b]$ and and $f(O)$ covers $f(E)$. WLOG, O_i are disjoint. By the definition of absolutely continuous, for any $\epsilon > 0$, there exists a $\delta > 0$, such that $\sum_{i=1}^{n} mf(O_i) < \epsilon$ for any finite n with $\sum_{i=1}^{n} mO_i < \delta$. So it also holds for $n = \infty$.

(2)???

5.2.8 *(Jan, 2001)*

(1) Prove that if f is absolutely continuous on $[a, b]$, then f is continuous and of bounded variation on $[a, b]$.

(2) Let $f(x) = x \sin \frac{1}{x}$ for $0 < x \leq 1$ and $f(0) = 0$. Show that f is continuous on $[0, 1]$ but not of bounded variation on $[0, 1]$.

Proof: (1) See 5.2.1.

(2) See 5.2.4.

5.2.9 *(Aug, 2000)*
Show that a function satisfying a Lipschitz condition on $[a, b]$ is absolutely continuous. (Note: A function f is said to satisfying a Lipschitz condition on $[a, b]$ if there is a constant M such that $|f(x) - f(y)| \leq M|x - y|$ for all x, y in $[a, b]$.)

Proof: If satisfying a Lipschitz condition on $[a, b]$, then for any $\epsilon > 0$, let $\delta = \epsilon/M$, we have

$$\sum_{i=1}^{n} |f(x_i') - f(x_i)| \leq \sum_{i=1}^{n} M|x_i' - x_i| < \epsilon$$

for any nonoverlapping interval $\{(x_i', x_i)\}$ with $\sum_{i=1}^{n} |x_i' - x_i| < \delta$. Then f is absolutely continuous in $[a, b]$.

□

5.2.10 *(may, 1996)*
Is there a monotone function on $[0, 1]$ which is discontinuous at all rational points? Prove your conclusion.

Proof: Define $f(x) = 1$ when x is a rational number in $[0, 1]$, and be 0 otherwise. And let $g(x)$ be the total variation of $f(x)$, that means $g(x) = T_0^x f(t)$. Then $g(x)$ is a monotone function on $[0, 1]$ which is discontinuous at all rational points.

Here is a monotone function f on $[0, 1]$ which is discontinuous at every rational points.

Let $p_1, p_2, ..., p_n, ...$ be an infinite sequence of positive numbers such that $\sum_{n=1}^{\infty} p_n < \infty$ is convergent (for example, $p_n = 2^{-n}$), and let $a_1, a_2, ..., a_n, ...$ be a sequence that listing all the rational numbers in $[0, 1]$.

Define $f : (-\infty, \infty) \mapsto (-\infty, \infty)$ **as follows:** $f(x) = 0$ **for all** $x < 0$**. For any** $x \geq 0$**, let** $A(x) = \{a_i : a_i \leq x\}$**, and define** $f(x) = \sum_{a_i \in A(x)} p_i$**. Since** $p_n > 0$ **for all** $n \geq 1$**,** $f(x)$ **is an increasing function. If** $x = a_n$ **is a rational number in** $[0, 1]$**, then**

$$\lim_{x \to a_n^+} f(x) - \lim_{x \to a_n^-} f(x) = p_n > 0,$$

and so $f(x)$ **is not continuous at** $x = a_n$**.**

Chapter 6

The Classical Banach Space

6.1 Notes

A measurable function defined on $[0,1]$ is said to belong to the space $L^p = L^p(E)$ if $\int_E |f|^p < \infty$, $0 < p < \infty$. For a function $f \in L^p$, we define

$$\|f\|_p = \left(\int_E |f|^p\right)^{1/p}.$$

And we define L^∞ the space of all bounded measurable functions on $[0,1]$, and

$$\|f\|_\infty = \inf\{M : m(t : f(t) > M) = 0\}.$$

Two important inequalities:

If f and g are in L^p with $1 \le p <\le \infty$, then so is $f + g$ and

$$\|f + g\|_p \le \|f\|_p + \|g\|_p.$$

If p and q are nonnegative extended real numbers with that $\frac{1}{p} + \frac{1}{q} = 1$, and if $f \in L^p$ and $g \in L^q$, then $f \cdot g \in L^1$ and

$$\int |fg| \le \|f\|_p \|g\|_q.$$

A sequence $\{f_n\}$ in a normed linear linear space is said to converge to an element f in the space if given $\epsilon > 0$, there is an N such that for all $n > N$, we have $\|f_n - f\| < \epsilon$. And we denoted it by $f_n \to f$.

A normed linear space is called **complete** if every Cauchy sequence $\{f_n\}$ in the space, there is an element f such that $f_n \to f$. A complete normed linear space is called **Banach space**.

6.2 Excercises

6.2.1 *(Aug, 2004)*

Let $\{f_n\}$ be a sequence of real Lebesgue measurable functions on $[0,1]$. If for any real $g(x) \in L^2[0,1]$, the sequence of real numbers

$$\int_0^1 g(x) f_n(x) dx$$

converges. Does $f_n(x)$ converges to a function $f(x)$ in $L^2[0,1]$? Justify your conclusion.

Proof: Define $f_n(x) = \sqrt{n}\chi_{[0,1/n]}$. Then for any $g(x) \in L^2[0,1]$, we have for any ϵ and δ, there exists a $A \subset [0,1]$ with $mA < \delta$ such that $\int_A |g|^2 < \epsilon^2$. And there exists an N, such that for any $n > N$, we have $1/n < \delta$, then

$$\begin{aligned} \int_0^1 g(x)f_n(x)dx &= \int_0^{1/n} g(x)f_n(x)dx \\ &\le \left(\int_0^{1/n} |g(x)|^2\right)^{1/2}\left(\int_0^{1/n} |f_(x)|^2\right)^{1/2} \\ &< \epsilon. \end{aligned}$$

Now suppose that $f_n \to f$ in L^2, then for any ϵ, there exists an N such that for any $n \ge N$, we have

$$||f_N(x) - f||_2 <= \int_0^1 |f_N - f|^2 = \int_0^{1/N} |f_N - f|^2 + \int_{1/N}^1 |f|^2 < \epsilon.$$

So $\int_0^{1/N} |f_N - f|^2 < \epsilon$, that means $f_N \to f$ a.e. in R. But for $n = 2N$, we can also get that $f_2N \to f$ which is a contradiction. So such f does not exist.

6.2.2 *(Apr, 2004)*
Let $f \in L^\infty[0,1]$, *show that*

$$\lim_{n\to\infty} ||f||_{L^n[0,1]} = ||f||_{L^\infty[0,1]}.$$

Proof: Assume that $||f||_{L^\infty[0,1]} = M$, then f is bounded by M except on a measure 0 set E. Then

$$\begin{aligned} \lim_{n\to\infty} ||f||_{L^n[0,1]} &= \lim_{n\to\infty}(||f||_{L^n[E]} + ||f||_{L^n[\tilde{E}]}) \\ &= 0 + \lim_{n\to\infty} ||f||_{L^n[\tilde{E}]} \\ &\le (\int_0^1 |M|^n)^{1/n} \\ &\le M. \end{aligned}$$

For any ϵ, the set $E = \{x : |f(x)| > M - \epsilon\}$ has measure greater $a > 0$. Then we have

$$\begin{aligned} \left(\int_0^1 |f|^n\right)^{1/n} &= \left(\int_E |f|^n + \int_{\tilde{E}} |f|^n\right)^{1/n} \\ &\ge \left(\int_E |f|^n\right)^{1/n} \\ &\ge ((M-\epsilon)^n mE)^{1/n} \\ &= (M-\epsilon)mE^{1/n}. \end{aligned}$$

Let $n \to 0$, we have

$$\lim_{n\to\infty} ||f||_{L^n[0,1]} \ge M - \epsilon.$$

So $\lim_{n\to\infty} ||f||_{L^n[0,1]} = ||f||_{L^\infty[0,1]}$.

6.2.3 *(Aug, 2000)*
Suppose $\{f_n\}$ *and* f *are function in* $L^p[0,1](p \ge 1)$, *and* $f_n \to f$ *a.e.. Show that* $\{f_n\}$ *converges to* f *in* L^p $\Longleftrightarrow ||f_n||_p \to ||f||_p$.

Proof: $\Rightarrow$: If $\{f_n\}$ converges to f in L^p, then for any $epsilon$, there exists an N such that for any $n > N$, we have $||f - f_n||_p < \epsilon$. Then

$$\Big|||f||_p - ||f_n||_p\Big| \le \Big|||f - f_n + f_n||_p - ||f_n||_p\Big| \le \Big|||f - f_n||_p + ||f_n||_p - ||f_n||_p\Big| = ||f - f_n||_p < \epsilon.$$

$\Leftarrow$: ???

6.2.4 *(Aug, 1997)*
Use the Hölder inequality to establish the generalized Hölder inequality: Let $p_i > 1$ *with* $\sum_{i=1}^m \frac{1}{p_i} = 1$. *Then* $\|\prod_{i=1}^m f_i\|_1 \le \prod_{i=1}^m \|f_i\|_{p_i}$ *for any* $f_i \in L^{p_i}(0,1)$.

Proof: We prove it by induction. Since it is true for $n = 2$, suppose it is true for any $n \le k$, for $n = k+1$, we have $\sum_{i=1}^{k+1} \frac{1}{p_i} = \sum_{i=1}^{k} \frac{1}{p_i} + \frac{1}{p_{k+1}} = 1$. Let $\sum_{i=1}^{k} \frac{1}{p_i} = \frac{1}{t}$, then

$$\|\prod_{i=1}^k f_i\|_1 \cdot \|f_{k+1}\|_1 \le \|\prod_{i=1}^k f_i\|_t \cdot \|f_{k+1}\|_{p_{k+1}}.$$

Since $\sum_{i=1}^k \frac{t}{p_i} = \sum_{i=1}^k \frac{1}{p_i/t} = 1$, we have

$$\|\prod_{i=1}^k f_i\|_t = \left(\prod_{i=1}^k \int_0^1 |f_i|^t\right)^{1/t} \le \left(\prod_{i=1}^k \|f_i^t\|_{p_i/t}\right)^{1/t} = \prod_{i=1}^k \|f_i\|_{p_i}.$$

So $\|\prod_{i=1}^m f_i\|_1 \le \prod_{i=1}^k \|f_i\|_{p_i} \cdot \|f_i\|_{p_{k+1}} = \prod_{i=1}^{k+1} \|f_i\|_{p_i}$.

6.2.5 *(Aug, 2001)*
For $f \in L^p[a,b] (p > 1)$, *set* $f = 0$ *outside of* $[a,b]$ *and define*

$$f_h(x) = \frac{1}{2h}\int_{x-h}^{x+h} f(t)dt \text{ for } h > 0.$$

Show that

$$\|f_h\|_p \le \|f\|_p \qquad \text{and} \qquad \lim_{h\to 0^+} \|f_h - f\|_p = 0.$$

(Note: You can use the fact, without giving its proof, that for integrable ϕ, *there holds* $\int_a^b |\phi_h(x)|dx \le \int_a^b |\phi(x)|dx$.*)*

Proof: Since $p > 1$, there exists $q \ge 1$ such that $1/p + 1/q = 1$. Then

$$\|f_h\|_p^p = \int_a^b \left|\frac{1}{2h}\int_{x-h}^{x+h} f(t)dt\right|^p dx \le \int_a^b \left|\frac{1}{2h}(2h)^{1/q}\|f(t)\|_p\right|^p dx = \int_a^b \left(\int_{x-h}^{x+h} |f(t)|dt\right) dx$$

6.2.6 *(Apr, 2001)*

(1) Assume that $1 \le p < q \le \infty$. *If* f *is in* $L^q[0,1]$, *show that* $\|f\|_p \le \|f\|_q$.
(2) Let f *be in* $L^2[0,1]$. *Show that* $\lim_{p\to 1^+} \|f\|_p = \|f\|_1$.
(3) Let f *be a bounded measurable function on* $[0,1]$. *Show that* $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.

6.2.7 *(May, 1996)*
Let $\{f_n\}$ *be a sequence of functions in* $L^2(0,1)$, *which converges almost everywhere to* 0 *and satisfies* $\|f_n\|_{L^2} \le 1$. *Can we conclude that* $\forall g \in L^2(0,1)$,

$$\lim_{n\to\infty} \int g(x) f_n(x) dx = 0?$$

Prove your conclusion.

Proof: Use the example of f_n in 6.2.1, and let $g(x) = 1$ in $(0,1)$. Then $f_n \to 0$ a.e., but

$$\lim_{n\to\infty} \int g(x) f_n(x) dx = 1.$$