On the Lagrangian description of absolutely continuous curves in the Wasserstein space on the line; well-posedness for the Continuity Equation

Mohamed Amsaad* Adrian Tudorascu†

December 24, 2014

Abstract

The Lagrangian description of absolutely continuous curves of probability measures on the real line is analyzed. Whereas each such curve admits a Lagrangian description as a well-defined flow of its velocity field, further conditions on the curve and/or its velocity are necessary for uniqueness. We identify two seemingly unrelated such conditions that ensure that the only flow map associated to the curve consists of a time-independent rearrangement of the generalized inverses of the cumulative distribution functions of the measures on the curve. At the same time, our methods of proof yield uniqueness within a certain class for the curve associated to a given velocity, i.e. they provide uniqueness for the solution of the continuity equation within a certain class of curves.

1 Introduction

Consider the problem

$$\partial_t X(t,x) = v(t, X(t,x)), \quad \text{under } X(0,x) = X_0(x),\ x \in I, \tag{$Flow$}$$

where I is the interval $(0,1)$ and $X_0 : I \to \mathbb{R}$, $v : (0,T) \times \mathbb{R} \to \mathbb{R}$ are given functions. Note that the solution X is written as a function of two variables in order to account not only for the time-variable but also for the initial value prescribed for X. If $X_0 \equiv \mathrm{Id}$ in I and the solution exists and is unique for all $x \in I$, loosely speaking, the function X is called the *classical flow* of v. The terminology comes primarily from Fluid Dynamics: if v stands for the velocity of fluid flow, then $X(t,x)$ accounts for the position at time t of the fluid particle that was initially

*Department of Mathematics, West Virginia University, Morgantown, WV 26506 (`mamsaad@math.wvu.edu`)

†Department of Mathematics, West Virginia University, Morgantown, WV 26506 (`adriant@math.wvu.edu`)

Key words: Continuity Equation, Lagrangian Flow, Optimal Transport, Wasserstein metric, Wasserstein space. AMS code: 34A12, 34A34, 35A02, 35A24, 35F20, 35Q35, 60E05, 76A02.

$(t = 0)$ at position x (if $X_0 \equiv \mathrm{Id}$) or, more generally, $X_0(x)$. This equation is the basis for the Lagrangian description of fluid flow (where the trajectory of a particle is observed through time). If $\mathcal{L}^1$ is the Lebesgue measure on $\mathbb{R}$ and $\mu_t := X_{t\#}\mathcal{L}^1|_I$ (i.e. $\mu_t(B) = \mathcal{L}^1(X_t^{-1}(B))$ for all Borel sets $B \subset \mathbb{R}$), then we get the Eulerian description of the distribution $X_{0\#}\chi$ (where $\mathcal{L}^1|_I =: \chi$) as it is transported by the velocity v: this means (μ, v) solves the *Continuity Equation*

$$\partial_t \mu + \partial_y(\mu\, v) = 0 \quad \text{in } (0,T) \times \mathbb{R} \tag{CE}$$

from Fluid Mechanics in the sense of distributions. It is worth mentioning that $(Flow)$ and (CE) are closely connected to the Transport Equation

$$\partial_t F + v\, \partial_y F = 0 \;\; \text{in } (0,T) \times \mathbb{R}. \tag{$Trans$}$$

Indeed, the cumulative distribution function F_t of ρ_t formally satisfies the above equation (see Proposition 3.2 below).

If $v = v(t, y)$ is sufficiently smooth, one can formally take the derivative of both sides of the equation with respect to x to verify that the spatial derivative $u = \partial_x X(t,x)$ satisfies

$$\partial_t u(t,x) = \partial_y v(t, X(t,x))\, u(t,x), \quad u(0,x) = X_0'(x). \tag{1.1}$$

The solution procedure shows that

$$u(t,x) = X_0'(x) \exp\left(\int_0^t \partial_y v(s, X(s,x))\, ds \right).$$

Whereas we have not really found an explicit solution (because $\partial_y v$ depends on X), we can conclude that $X_0'(x) > 0$ implies $\partial_x X(t,x) > 0$ for all $(t,x) \in (0,T) \times I$, which shows that for every $t \in (0,T)$, the function $x \mapsto X(t,x)$ is strictly increasing, with positive slope everywhere. This, when referring to (CE), is in agreement with the intuition that "nice" velocities v preserve in time the order of the positions of particles on the real line, or, equivalently, the characteristics do not cross. If v is not smooth enough to justify our little calculation above, then the order need not be preserved through time. This has fundamental implications in scalar Conservation Laws, where the crossing of characteristics is responsible for the formation of shocks. Our goal in this paper is to study conditions under which $x \mapsto X(t,x)$ stays monotone nondecreasing for all times $t \in [0,T)$ if X_0 is itself monotone nondecreasing. The meaning of "conditions" is quite vague at this point; however, they shall not be imposed on the velocity v, but rather on the Eulerian flow (CE) whose Lagrangian characterization is provided by $(Flow)$. Under the minimal assumptions we shall impose on μ, the velocity v will be uniquely determined by μ from (CE) (in some precise sense). Thus, our endeavor will be to study $(Flow)$ under the constraint that $X_{t\#}\chi = \mu_t$ be given for all $t \in [0,T]$ in the form of an absolutely continuous curve of probability measures [3].

DiPerna & Lions [7], Ambrosio [1] (also, see [5] for a good survey on such problems) have addressed the questions of existence, uniqueness and stability for *regular* Lagrangian flows, i.e. solutions X of $(Flow)$ such that $X_{t\#}\mathcal{L}^d \ll C\mathcal{L}^d$ for some positive real constant independent of $t \in [0,T]$. In the Sobolev case (there is a similar version if only certain BV regularity on

v is assumed), the almost (minor improvements are available [5]) state-of-the-art uniqueness result covers only velocities

$$v \in L^\infty((0,T)\times\mathbb{R}^d;\mathbb{R}^d)\cap L^1([0,T];W^{1,p}(\mathbb{R}^d;\mathbb{R}^d)) \quad \text{for some } p>1.$$

It is proved that if v is this regular, then its regular Lagrangian flow, if it exists, then it is unique. We had originally planned a complete departure from that setting, as our initial goal was to investigate the uniqueness of the Lagrangian flow associated to a given absolutely continuous path of probability measures (see Definitions 1.1 and 2.1 below). Thus, no explicit conditions on the velocity v were to be made, even though any assumption on the curve of measures will implicitly reflect on the (unique) velocity associated to it. We have managed to stay true to this goal only in part; see the main result Theorem 3.12, which covers the case of higher integrability for the densities on the curve. However, while analyzing the case of continuous densities (Theorem 3.4, the other main result) we discovered that a continuous velocity would not only yield uniqueness for the Lagrangian flow associated to the curve, but it would also give uniqueness for the continuity equation (CE) within the class of absolutely continuous curves of probabilities [3]. Thus, in Theorem 3.4 we also make the continuity assumption on v.

We would like to make it clear that we only deal with solutions of (CE) that are absolutely continuous curves of probability measures: it is easy to construct examples of velocities v and distributional solutions of (CE) originating at a given probability ρ_0 but whose masses change in time, and/or become negative. Indeed, one may take any smooth and positive density ρ_0 that vanishes at $\pm\infty$ and set $\rho(t,y) := (1-t)|1-t|\rho_0(y)$: then ρ is a distributional solution for (CE) with $v(t,y) = 2F_0(y)/[(1-t)\rho_0(y)]$ if $t\in[0,1)\cup(1,\infty)$ and $v(t,\cdot)\equiv 0$ if $t=1$ (F_0 is the cumulative distribution function of ρ_0). Note that while $\rho(0,\cdot)$ is a smooth probability density, the mass of $\rho(t,\cdot)$ decreases in time and $\rho(t,\cdot)$ even goes below zero for $t>1$. Such solutions of (CE) are non-physical in Conservation Laws or Thermodynamics (for example), where this equation is also used to express precisely conservation of mass for ρ; furthermore, as ρ usually denotes a physical quantity (such as material density or absolute temperature), it is also important in applications that it does not pointwise go below zero.

It is convenient to introduce the definition of *Flow of a Borel map*, which is the Lagrangian flow restricted to the one-dimensional case. We take $I := (0,1)$, which ensures the measure $X_\#\mathcal{L}^1|_I = X_\#\chi$ is a probability measure, but everything we achieve in this paper can be trivially extended to any bounded interval.

Definition 1.1 (Flow of a Borel map). *Let $v:[0,T]\times\mathbb{R}\longrightarrow\mathbb{R}$ and $X_0: I\to\mathbb{R}$ be Borel functions. We say that $X:[0,T]\times I\longrightarrow\mathbb{R}$ is a* flow map *for v if*

(i) *the map $[0,T]\ni t\to X(t,x)$ is absolutely continuous for a.e. $x\in I$;*

(ii) *$\partial_t X(\cdot,x) = v(\cdot, X(\cdot,x))$ for a.e. $x\in I$.*

Furthermore, we say that $X:[0,T]\times I\longrightarrow\mathbb{R}$ is a flow map for v starting at X_0 *if, beside* (i), (ii)*, the following is satisfied:*

(iii) *$X(0,x) = X_0(x)$ for a.e. $x\in I$.*

We have seen earlier that *any* solution X of $(Flow)$ will be nondecreasing in x provided that X and v are regular enough to justify (1.1). It is, however, this regularity that might not be there, which will enable solutions that are not nondecreasing in x (see Example 1.2 below). Examples 1.4 and 1.5 however, illustrate that there are even irregular v for which only nondecreasing solutions exist. By one-dimensional Optimal Transport, it is known that there is a bijective correspondence between the closed, convex cone of monotone nondecreasing functions in $L^2(I)$ and the metric space $\mathcal{P}_2(\mathbb{R})$ endowed with the quadratic Wasserstein distance (see, e.g., [10]). This implies [10] (due to the uniqueness of the velocity for a given AC^2 curve) there is a bijective correspondence between paths $M \in \mathcal{H} := H^1(0,T;L^2(I))$ consisting of monotone nondecreasing maps M_t and curves $\mu \in AC^2(0,T;\mathcal{P}_2(\mathbb{R}))$, via $M_{t\#}\chi = \mu_t$. It was also proved in [10] that M satisfies $(Flow)$ in the sense of Definition 1.1. Thus, the initial value problem $(Flow)$ admits a solution in $\mathcal{H}$ if and only if v is the velocity associated with some curve $\mu \in AC^2(0,T;\mathcal{P}_2(\mathbb{R}))$. We shall see that one can obtain such a curve from $(Flow)$ by defining $\mu_t := X_{t\#}\chi$; thus, we shall see that if X_0 is nondecreasing, then *as soon as* $(Flow)$ *admits a solution* $X \in \mathcal{H}$*, it automatically admits the solution* $M \in \mathcal{H}$ *consisting of the monotone rearrangements of the maps* X_t. Thus, it comes as a very natural question to investigate when $(Flow)$ has *only* spatially nondecreasing solutions.

A study of the regularity of the monotone rearrangements of maps belonging to a time-continuous family of maps was performed by Loeper in [11]. It is proved that the distributional time derivative is a signed measure. This is very weak by comparison to what we achieve here, but it covers a much more general case: arbitrary spatial dimension, and the family of maps is not necessarily the flow of a map.

Example 1.2. Let us consider the (Borel) function $v : [0,2] \times \mathbb{R} \to \mathbb{R}$ given by

$$v(t,y) = \frac{y}{t-1} \quad \text{if } t \in [0,2]\backslash\{1\}, \quad v(1,\cdot) \equiv 0.$$

Clearly, v is analytic everywhere except on the fiber $\{1\} \times \mathbb{R}$. It is easy to see that $X(t,x) := (1-t)x$ satisfies $X \in \mathcal{H}$ (it is, in fact, analytic and bounded in $(0,2) \times I$) and is a solution for $(Flow)$ for this particular v. Note that X_t is nondecreasing only if $t \in [0,1]$, and strictly decreasing otherwise. However, by setting

$$M(t,x) = (1-t)x \text{ if } t \in [0,1] \text{ and } M(t,x) = (1-t)(1-x) \text{ if } t \in (1,2],$$

we observe that $M \in \mathcal{H}$ is also a solution with $M_0(x) = x$ for all $x \in I$. The maps M_t are the monotone rearrangements of the maps X_t. So, this is an example where $(Flow)$ has other solutions, beside monotone non-decreasing ones. It also illustrates that once a solution exists, there will also exist a monotone non-decreasing solution, as discussed above.

The obvious issue with Example 1.2 is that, whereas for $t > 1$ we get decreasing solutions, the solution X_t is still nondecreasing (and, thus, coincides with M_t) for all $t \in [0,1]$. It is not difficult to modify the example in such a way that the time threshold $t = 1$ is replaced by an arbitrarily small positive time. However, the natural question is whether it is true that for any solution $X \in W^{1,q}(0,T;L^p(I))$ of $(Flow)$ with $X_0 \equiv \mathrm{Id}$ there exists a time horizon $\bar{t} > 0$ such that X_t is nondecreasing for all $t \in [0,\bar{t}]$. As the next example will show, this is false.

Following a suggestion by Alberto Bressan, we have constructed a velocity v and its flow X such that for any $\varepsilon > 0$ there exists a time $0 < t_\varepsilon < \varepsilon$ for which X_{t_ε} is not nondecreasing.

Example 1.3. Let $v : [0,1] \times I \to \mathbb{R}$ given by $v(t,y) = 2\,\mathrm{sgn}(2^{-n/2} - t)\sqrt{1 - 2^{-n} - y}$ for all positive integers n, all $1 - 2^{-n+1} < y \leq 1 - 2^{-n}$ and all $t \in [0,1]$, i.e.

$$v(t,y) = \begin{cases} 2\sqrt{1-2^{-n}-y} & \text{if } 1-2^{1-n} < y \leq 1-2^{-n},\ \ 0 \leq t < 2^{-n/2}, \\ -2\sqrt{1-2^{-n}-y} & \text{if } 1-2^{1-n} < y \leq 1-2^{-n},\ \ 2^{-n/2} \leq t \leq 1. \end{cases}$$

If $n \geq 2$ is an integer, define the function $X : [0,1] \times I \to I$ by

$$X(t,x) = \begin{cases} 1-2^{-n} - (t - \sqrt{1-2^{-n}-x})^2 & \text{if } 1-2^{1-n} < x < 1-2^{-n},\ 0 \leq t < \sqrt{1-2^{-n}-x} \\ 1-2^{-n} & \text{if } 1-2^{1-n} < x < 1-2^{-n}-2^{-n-1},\ \sqrt{1-2^{-n}-x} \leq t \leq 1 \\ 1-2^{-n} & \text{if } 1-2^{-n}-2^{-n-1} < x \leq 1-2^{-n},\ \sqrt{1-2^{-n}-x} \leq t < 2^{-n/2} \\ 1-2^{-n} - (t-2^{-n/2})^2 & \text{if } 1-2^{-n}-2^{-n-1} < x \leq 1-2^{-n},\ 2^{-n/2} \leq t < 2^{1-n/2} \\ 1-2^{1-n} & \text{if } 1-2^{-n}-2^{-n-1} < x \leq 1-2^{-n},\ 2^{1-n/2} \leq t \leq 1. \end{cases}$$

If $n = 1$ we remove the last branch from the above definition and cut t off at 1 in the penultimate branch.

Note that $X(0,x) = x$ for all $x \in I$ and $(Flow)$ holds in the classical sense for all $x \in I \backslash \{1 - 2^{-n}\ :\ n \text{ positive integer }\}$. If $x = 1$, then $(Flow)$ is satisfied in the classical sense for $t \in [0,1]$. If $x = 1 - 2^{-n}$ for some integer $n \geq 2$, then $X(\cdot, x)$ is continuous in t on $[0,1]$. So is $\partial_t X(\cdot, x)$ except at $t_n = 2^{1-n/2}$ where it has a jump discontinuity. The differential equation $(Flow)$ is satisfied in the classical sense on both sides of t_n, so it is satisfied in the integral sense over the whole time interval $[0,1]$.

Thus, $(Flow)$ is satisfied in the integral sense for all $x \in I$ and all $t \in [0,1]$. Also, note that on each interval $I_n := (1-2^{-n+1}, 1-2^{-n})$ we have the following property: for all $t \in [2^{-n/2}, 1]$ the map $X(t,\cdot)$ maps the left half of the interval to a single value, namely $1 - 2^{-n}$. Then it maps the right half to a single value as well, namely $1 - 2^{-n} - (t - 2^{-n/2})^2$, which lies strictly below $1 - 2^{-n}$ for $t \in (2^{-n/2}, 1]$ (see the interrupted line vs the solid line in Figure 1). Thus, $X(t,\cdot)$ is *not* Lebesgue a.e. equal to a nondecreasing function over I_n for any integer $n \geq 2$ and any $t \in (2^{-n/2}, 1]$. Since $2^{-n/2}$ approaches zero as $n \to \infty$, we deduce that, for arbitrarily small $t > 0$, the function $X(t,\cdot)$ is *not* Lebesgue a.e. equal to a nondecreasing function over $(0,1)$.

Since both X and v are bounded, we infer $X \in W^{1,\infty}(0,1; L^2(I)) \subset \mathcal{H}$, so all the requirements on X are satisfied.

Finally, note that Example 1.3 can be easily modified to make $(Flow)$ satisfied in the classical sense for all $t \in (0,1)$ (see Figure 1). Indeed, one may replace the fourth branch in the definition of X above by the quadratic $u_n(t) = a_n t^2 + b_n t + c_n$ such that $u_n(2^{-n/2}) = 1 - 2^{-n}$, $\partial_t u_n(2^{-n/2}) = 0$ and $u_n(1) = 1 - 2^{1-n}$. For $a_n = -1/(2^{n/2}-1)^2, b_n = 2^{1-n/2}/(2^{n/2}-1)^2, c_n = 1 - 2^{1-n} + [1 - 2^{1-n/2}/(2^{n/2}-1)^2]$, and $t \in [2^{-n/2}, 1]$ this function will satisfy the differential equation $\partial_t u_n = -\sqrt{4a_n(u_n - c_n) + b_n^2}$. Thus, it suffices to replace the formula in the second branch of the definition of v above by $f(y) = -[2/(2^{n/2}-1)]\sqrt{1-2^{-n}-y}$ if $1 - 2^{1-n} < y \leq 1 - 2^{-n}$ and $2^{-n/2} \leq t \leq 1$.

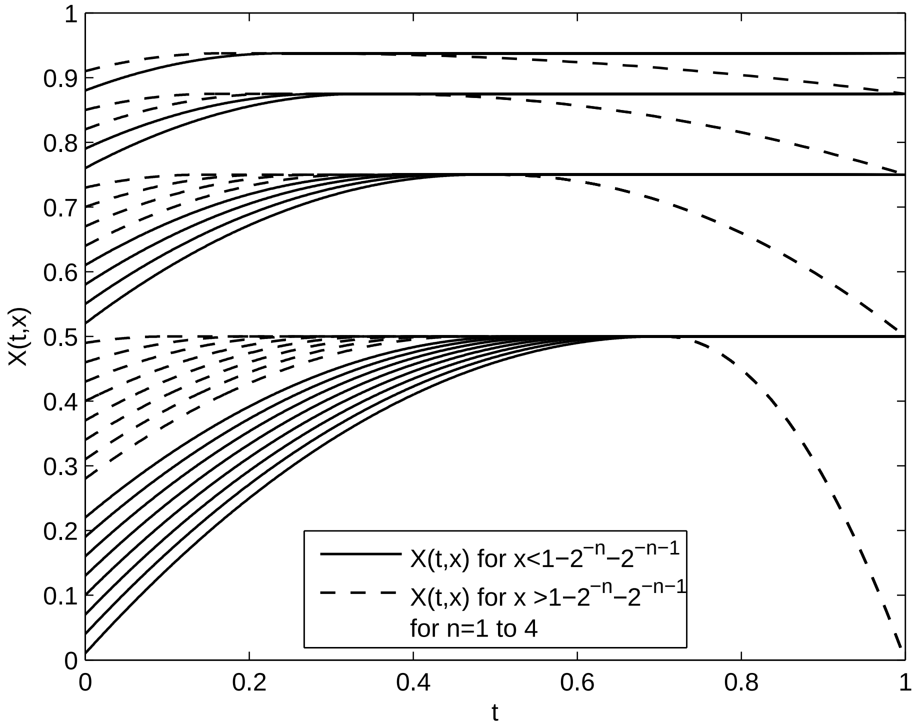


Figure 1: $(Flow)$ satisfied in the classical sense for Example 1.3.

Not only does Example 1.3 provide an instance where the non-uniqueness of the flow (albeit classical) manifests instantaneously for $t > 0$ (as we have seen there are solutions other than the monotone ones for arbitrarily small positive time), but it also gives instantaneous mass concentrations (Dirac deltas) in the measures $X_{t\#}\chi$ (as a result of X_t developing flat portions instantaneously). The following example shows that this can occur even if the flow is unique (i.e. consisting of monotone nondecreasing maps).

Example 1.4. Let $0 < T \leq 1/2$ and $v : [0,T] \times \mathbb{R} \to \mathbb{R}$ given by

$$v(t,y) = 0 \quad \text{if} \quad y \leq 0 \quad \text{and} \quad v(t,y) = \frac{y-1}{1-t} \quad \text{if} \quad y > 0.$$

One can prove that

$$X(t,x) = 0 \quad \text{if} \quad 0 \leq x \leq t \quad \text{and} \quad X(t,x) = \frac{x-t}{1-t} \quad \text{if} \quad t < x \leq 1$$

is the only solution to $(Flow)$. Note that X_t develops a flat portion as soon as $t > 0$.

Furthermore, as the following example shows, it is also possible that there are *two different monotone solutions* X and Y for $(Flow)$. The example is constructed along the lines of the classical $v(t,x) = \sqrt{|x|}$, which is often used to illustrate non-uniqueness for the initial value problem $\dot{x} = \sqrt{|x|}$, $x(0) = 0$. This example helps justify why the constraint "$X_{t\#}\chi$ is prescribed" is used here as a criterion to sort out between solutions. In [1], the author uses this example towards the same goal. The criterion chosen there is different from ours: it is the *regularity* of the flow, as described earlier.

Example 1.5. Let $v(t,y) = 2\sqrt{|y-1|}$, $\quad T = 1$ and note that

$$X^\varepsilon(t,x) = \begin{cases} 1 + (t - \sqrt{1-x})\left|t - \sqrt{1-x}\right| & \text{if } t \in [0, \sqrt{1-x}] \\ 1 & \text{if } t \in [\sqrt{1-x}, \varepsilon + \sqrt{1-x}] \\ 1 + (t - \varepsilon - \sqrt{1-x})\left|t - \varepsilon - \sqrt{1-x}\right| & \text{if } t \in [\sqrt{1-x} + \varepsilon, 1] \end{cases} \tag{1.2}$$

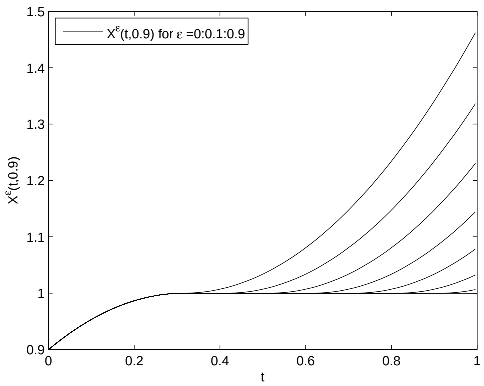
satisfies (*Flow*) for any $0 \le \varepsilon < 1$ (see Figure 2) (the various ranges of t that do not make sense in the above definition are not to be used: for example, when $x = 0$ we only use the first branch to get $X^\varepsilon(t,0) = 2t - t^2$ for all $t \in [0,1]$, or, when $\varepsilon + \sqrt{1-x} \ge 1$ the function is defined only on two pieces, namely $[0, \sqrt{1-x}]$ and $[\sqrt{1-x}, 1]$). The maps X_t^ε are all nondecreasing (one needs to carefully write $X^\varepsilon(t,\cdot)$ for fixed t as a function of x to check that), yet different for different ε; see Figure 2 (c). One can easily check that the solution X^ε is classical (C^1 in time) and, since it is bounded on $(0,T) \times I$, (*Flow*) provides a bound on $\dot{X}^\varepsilon$ as well. Thus, $X^\varepsilon \in \mathcal{H}$ trivially. In terms of measures $\mu_t := X_{t\#}\chi$, this means that there are infinitely (in fact, a continuum; one for each $0 \le \varepsilon < 1$; see Figure 2 (a),(b)) many curves $\mu \in AC^2(0,1;\mathcal{P}_2(\mathbb{R}))$, all originating at $\mu_0 = \chi$ and sharing the same velocity v.

As far as uniqueness for (*Flow*) goes, Example 1.5 shows that in general there might be more than one curve whose velocity v is. This accounts for a non-uniqueness "mechanism", which we now dub *multiple-curve* (MC) condition on the velocity v. Thus, Example 1.5 provides a v satisfying (MC) (or, we could say "v is (MC)"). If v is not (MC), then we say v is (SC) (single-curve). Examples of v with (SC) are abundant; any continuous v that is also Lipschitz in the y-variable would do, as this implies pointwise existence and uniqueness for $\dot{x}(t) = v(t, x(t))$ for *any* initial $x = x(0)$. But this regularity is not necessary, as we remark next.

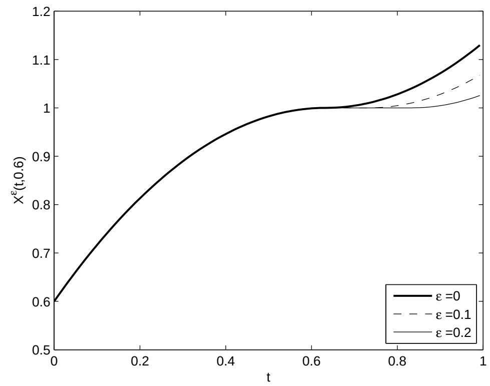
Remark 1.6. *We have seen that the velocity from Example 1.4 produces a single flow map X. If we use it to compute the curve $\mu = X_\#\chi$, we get $\mu_t = t\delta_0 + (1-t)\chi$ (the convex interpolation between the Dirac mass at 0 and χ), and one can check $\mu \in AC^2(0,T;\mathcal{P}_2(\mathbb{R}))$. Thus, v is (SC) (even though it is discontinuous) and its (unique) curve develops atoms as soon as $t > 0$.*

Thus, if a velocity v is (MC), there is no hope for a uniqueness result for (*Flow*), as each curve of probabilities associated with v will produce its own distinct solution (consisting of the optimal maps). The question of whether the family of optimal maps is the only solution becomes pertinent again once we prescribe the curve associated with v. This is yet another motivation for undertaking the present analysis.

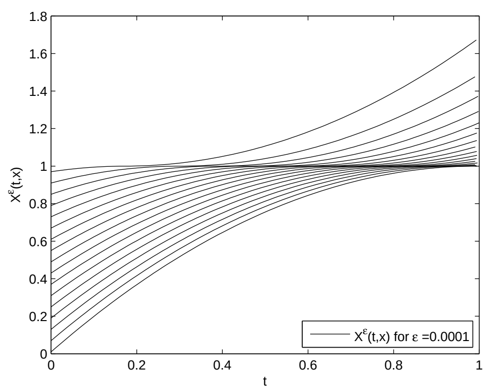
Our *main contribution* is identifying sufficient conditions on a path $\mu \in AC^q(0,T;\mathcal{P}_p(\mathbb{R}))$ to ensure that the path $M \in W^{1,q}(0,T;L^p(I))$ consisting of the optimal maps such that $M_{t\#}\chi = \mu_t$ is the unique solution to (*Flow*) which pushes χ forward to μ. As a byproduct



(a) One flow with specific range of ε.



(b) Another flow with various values of ε.



(c) Different flows with one value of ε.

Figure 2: $(Flow)$ has different monotone solutions for Example 1.5.

of our approach, uniqueness for the continuity equation (CE) also arises within a precisely defined class of solutions; within that class, the velocity v will be an (SC) velocity.

As we have already pointed out, the idea of treating these first-order differential equations as depending on the parameter given by the initial data is not new [5], [1], [4]. This should come as no surprise, given that such equations describe the flow of a scalar field, which is the reason why they are instrumental to the Lagrangian description of particle motion. However, the proposed approach is novel in that it is based on the connection between the flow equation and one-dimensional Optimal Transport. In our opinion, exploring the close link between $(Flow)$, (CE), and $(Trans)$ is worth pursuing before more complicated, higher-order PDE's are studied. The main idea here stems precisely from the analysis of ODE's of the type $(Flow)$ performed by taking into account the dependence of solutions on initial data.

If the velocity v is a "nice" function, the whole theory is simple and well-understood. Otherwise, the misleading simplicity of (CE) does not warn the researcher of the hazards involved with its analysis. Its connection with $(Flow)$ offers a glimpse into that, since $(Flow)$ can be too general to be handled easily. These issues have been recognized and have been the object of deep mathematical works [1], [4] etc. Our present work fills some knowledge gap even in the simplest of cases (spatial dimension one).

The paper is organized as follows: Section 2 begins by briefly recalling the definition of some objects essential to our presentation, such as Wasserstein distance/space, absolutely continuous curves of probability measures, generalized inverse etc. Then a necessary and sufficient condition on the regularity Lagrangian flow is proved in order for the associated path of measures to possess AC-regularity. The main result of Section 2 is Theorem 2.11 which shows that, quite generally, AC paths consisting of probabilities absolutely continuous with respect to the Lebesgue measure admit Lagrangian descriptions provided by the family of optimal maps. Section 3 focuses on two different types of conditions under which the said Lagrangian description is unique: Theorem 3.4 explores the case of continuous densities and velocities by a direct method (loosely connected to the narrative from Section 2), whereas Theorem 3.12 is more deeply indebted to the results from Section 2 as it analyzes the case where the densities enjoy some precisely quantified integrability (no conditions imposed on the velocity in this case). Theorem 3.4 also provides a uniqueness result for the continuity equation (CE) within a reasonably general class of solutions. Theorem 3.12 does the same, albeit in a more restrictive setting. In both cases, the uniqueness results for the Lagrangian flow and the uniqueness results for the continuity equation are obtained concomitantly from our method of proof. We conclude with Section 4, where we discussed open problems and possible applications.

2 Lagrangian Flows associated to $AC^q(0,T;\mathcal{P}_p(\mathbb{R}))$ curves

If $1 \le p < \infty$ and $d \ge 1$ is an integer, we denote by $\mathcal{P}_p(\mathbb{R}^d)$ the p-Wasserstein space on $\mathbb{R}^d$, i.e. the set of Borel probabilities on $\mathbb{R}^d$ with finite p-moment and endowed with the p-Wasserstein metric

$$W_p(\mu,\nu) := \inf\Big\{ \Big(\mathbb{E}[|X-Y|^p]\Big)^{\frac{1}{p}} \ : \ \mathrm{law}(X)=\mu,\ \mathrm{law}(Y)=\nu \Big\}.$$

Also, in this paper $\mathcal{P}_p^{ac}(\mathbb{R}^d)$ stands for the set of all Borel probability measures $\mu \in \mathcal{P}_p(\mathbb{R}^d)$ which are absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^d$. Let us begin by recalling the definition of $AC^q(0,T;\mathcal{P}_p(\mathbb{R}^d))$ (see [3]):

Definition 2.1. *If $1 \le p < \infty$ and $1 \le q \le \infty$, a path $[0,T] \ni t \mapsto \mu_t \in \mathcal{P}_p(\mathbb{R}^d)$ is said to lie in $AC^q(0,T;\mathcal{P}_p(\mathbb{R}^d))$ provided that there exists $\beta \in L^q(0,T)$ such that*

$$W_p(\mu_s,\mu_t) \le \int_s^t \beta(\tau)d\tau \text{ for all } 0 \le s \le t \le T.$$

It is proved in [3] that if $1 < p < \infty$, then any such curve admits a Borel velocity $v : (0,T)\times\mathbb{R}^d \to \mathbb{R}^d$ of minimal norm in the sense that:

$$(\mu, v) \text{ satisfies } (CE) \text{ in the sense of distributions in } (0,T)\times\mathbb{R}^d \tag{2.1}$$

$$v(t,\cdot) \in L^p(\mu(t,\cdot);\mathbb{R}^d) \text{ for a.e. } t \in (0,T) \text{ and } (0,T) \ni t \mapsto \|v(t,\cdot)\|_{L^p(\mu(t,\cdot);\mathbb{R}^d)} \in L^q(0,T); \tag{2.2}$$

$$\int_0^T \|v(t,\cdot)\|_{L^p(\mu(t,\cdot);\mathbb{R}^d)} dt \text{ is minimal among all velocities satisfying (2.1) and (2.2).} \tag{2.3}$$

Moreover, it is showed in [3] that this "velocity of minimal norm" is unique, in the sense that if v_1 and v_2 satisfy (1), (2), (3) above for a.e. $t \in (0,T)$, then $v_1(t,\cdot) \equiv v_2(t,\cdot)$ μ_t a.e.

Note that $AC^q(0,T;\mathcal{P}_p(\mathbb{R}^d)) \subset AC^1(0,T;\mathcal{P}_1(\mathbb{R}^d))$ for all $1 \le p < \infty$ and $1 \le q \le \infty$.

More can actually be said if $d = 1$; the proof of the proposition below is borrowed, with minor modifications, from [12]. It shows that on the line there is at most one integrable (in the sense specified below) velocity, the minimality condition on its norm being redundant.

Theorem 2.2. *Consider a path $\mu \in AC^1(0,T;\mathcal{P}_1(\mathbb{R}))$ for some $0 < T < \infty$. Then there exists at most one Borel velocity v for μ such that $v \in L^1(\mu)$ (as a function of both time and space) for a.e. $t \in (0,T)$. More precisely, if $v_1,\ v_2 : (0,T) \times \mathbb{R} \to \mathbb{R}$ are Borel maps such that $v_i \in L^1(\mu)$ for $i = 1,2$, and such that both (μ, v_1) and (μ, v_2) satisfy (CE) in the sense of distributions, then for Lebesgue a.e. $t \in (0,T)$ we have $v_1(t,\cdot) \equiv v_2(t,\cdot)$ in the $\mu(t,\cdot)$–a.e. sense.*

Proof: By subtracting (CE) written for both (μ, v_1) and (μ, v_2) and by taking test functions $\varphi(t,y) = \xi(t)\zeta(y)$, the equations above readily yield

$$\int_{\mathbb{R}} u(t,y)\zeta'(y)\,\mu(t,dy) = 0 \text{ for a.e. } t \in (0,T) \text{ and any } \zeta \in C^1_c(\mathbb{R}),$$

where $u := v_1 - v_2$. Fix $\varepsilon > 0$ and $\phi \in C_c(\mathbb{R})$. If $\phi = 0$ on $[R, +\infty)$, consider, for each natural number $n > R$, the function

$$\Phi_n(y) := \begin{cases} \int_{-\infty}^y \phi(z)\,dz & \text{if } y < n, \\ \omega(y-n) & \text{if } n \le y \le n+1, \\ 0 & \text{if } y > n+1 \end{cases} \tag{2.4}$$

where $\omega \in C^1[0,1]$ such that $\omega(0) = \int_{-\infty}^R \phi(z)\,dz$, $\omega(1) = 0$ and $\omega'(0) = 0 = \omega'(1)$. Clearly, $\Phi_n \in C^1_c(\mathbb{R})$. Thus,

$$\int_{\mathbb{R}} u(t,y)\phi(y)\,\mu(t,dy) + \int_n^{n+1} u(t,y)\omega'(y-n)\,\mu(t,dy) = 0 \text{ for a.e. } t \in (0,T)\ .$$

We have $|\omega'(y-n)| \le \|\omega'\|_{L^\infty(0,1)} =: C$ for all $n > R$ and all $y \in (n, n+1)$. Since $u(t,\cdot) \in L^1(\mu(t,\cdot))$ and $\mu(t,\cdot)$ is a Borel probability for Lebesgue a.e. $t \in (0,T)$, we conclude that for such t and any $\varepsilon > 0$ we have

$$\left| \int_{\mathbb{R}} u(t,y)\phi(y)\,\mu(t,dy) \right| \le \varepsilon$$

if n is sufficiently large. Due to the arbitrariness of ε and ϕ, the proof is concluded. QED.

The following is an obvious consequence.

Corollary 2.3. *Let $1 \leq p < \infty$ and $1 \leq q \leq \infty$ and $\mu \in AC^q(0,T;\mathcal{P}_p(\mathbb{R}))$ be given. Then, there exists at most one Borel map $v : (0,T) \times \mathbb{R} \to \mathbb{R}$ such that (μ, v) satisfies* (2.1) *(with $d = 1$) and*

$$v \in L^1(\mu), \text{ i.e. } \int_0^T \int_{\mathbb{R}} |v(t,y)|\mu(t,dy)dt < \infty. \tag{2.5}$$

This uniqueness result enables us to make the following definition:

Definition 2.4. *Let $1 \leq p < \infty$ and $1 \leq q \leq \infty$ and $\mu \in AC^q(0,T;\mathcal{P}_p(\mathbb{R}))$ be given. If it exists, the Borel map $v : (0,T) \times \mathbb{R} \to \mathbb{R}$ such that* (2.1) *and* (2.5) *are satisfied is called the L^1–velocity associated to μ.*

If $p = 1$, then even when $d = 1$ and $\rho(t,\cdot) \in \mathcal{P}_1^{ac}(\mathbb{R})$ for all $t \in (0,T)$, a velocity v satisfying (2.1) and (2.5) may not exist.

Example 2.5. Let $M : (0,1) \times (0,1) \to \mathbb{R}$ be the family of optimal maps given by:

$$M(t,x) = \begin{cases} x & \text{if } x \in [0, 1-t), \\ 1+x & \text{if } x \in [1-t, 1] \end{cases}$$

for all $t \in (0,1)$. Also, $M(0,x) = x$ and $M(1,x) = 1 + x$ for all $x \in [0,1]$. Then we can easily compute the curve $\rho(t,\cdot) = M_{t\#}\chi$ to obtain $\rho(t,\cdot) = \chi_{[0,1-t]} + \chi_{[2-t,2]}$, which shows that $\rho(t,\cdot) \in \mathcal{P}_p^{ac}(\mathbb{R})$ for all $p \geq 1$ and for all $t \in [0,1]$. However, for all $0 \leq s \leq t \leq 1$

$$W_p(\rho_s, \rho_t) = \left(\int_0^1 |M(s,x) - M(t,x)|^p \, dx\right)^{\frac{1}{p}} = \left(\int_{1-t}^{1-s} 1 \, dx\right)^{\frac{1}{p}} = |t-s|^{\frac{1}{p}},$$

and this is bounded by $\int_s^t \beta(\tau)d\tau$ for some $\beta \in L^1(0,1)$ if and only if $p = 1$ (in which case we may take $\beta \equiv 1 \in L^\infty(0,1)$). Thus, $\rho \in AC^\infty(0,1;\mathcal{P}_1^{ac}(\mathbb{R}))$ but $\rho \notin AC^q(0,1;\mathcal{P}_p(\mathbb{R}))$ for any $1 < p < \infty$ and any $1 \leq q \leq \infty$.

Next, assume that the L^1–velocity v associated to ρ exists. Then, for all $\varphi \in C_c^1(\mathbb{R})$, the function $t \mapsto \int_{\mathbb{R}} \varphi(y)\rho(t,dy)$ is absolutely continuous on $[0,1]$ and

$$\frac{d}{dt}\int_{\mathbb{R}} \varphi(y)\rho(t,dy) = \int_{\mathbb{R}} v(t,y)\varphi'(y)\rho(t,dy) \text{ at a.e. } t \in (0,1),$$

i.e.

$$-\varphi(1-t) + \varphi(2-t) = \int_0^{1-t} v(t,y)\varphi'(y)\,dy + \int_{2-t}^2 v(t,y)\varphi'(y)\,dy$$

for a.e. $t \in [0,1]$. Take $\varphi \in C_c^1(\mathbb{R})$ such that $\varphi \equiv 1$ on $[0, 5/8]$ and $\varphi \equiv 2$ on $[11/8, 2]$. Then the above equality must be satisfied, in particular, at a.e. $t \in (3/8, 5/8)$; this yields $1 = 0$, a contradiction. In conclusion, we have produced an example of a curve lying in the "most regular" subset of the $AC^q(0,1;\mathcal{P}_1(\mathbb{R}))$ families of curves (namely, $AC^\infty(0,1;\mathcal{P}_1^{ac}(\mathbb{R}))$), and yet whose L^1-velocity does not exist.

Example 2.6. Let $f(z) = (1 - \ln z)^{-1}$, so that $f'(z) = z^{-1}(1 - \ln z)^{-2}$ for $z \in (0,1)$. Then $f \in L^\infty(0,1)$ and $f' \in L^p(0,1)$ if and only if $p = 1$, so that $f \in W^{1,1}(0,1)$ but $f \notin W^{1,p}(0,1)$ for any $p > 1$. Set $g(x) := \min\{-f'(x), -e/4\}$ and note that g is continuous on $(0,1]$, increasing on $(0, e^{-1})$ and constant on $[e^{-1}, 1]$. Just like f', we have $g \in L^p(0,1)$ if and only if $p = 1$. Finally, let $M(t,x) := f(t)g(x)$ for all $(t,x) \in [0,1] \times (0,1)$ ($f(0) = 0$ in the right-limit sense) to see that the curve $[0,T] \ni t \mapsto \rho(t,\cdot) =: M(t,\cdot)_{\#}\chi$ lies in $AC^1(0,1;\mathcal{P}_1(\mathbb{R}))$ but not in $AC^q(0,1;\mathcal{P}_p(\mathbb{R}))$ for any $1 \le p < \infty$ and any $1 < q \le \infty$. Furthermore, the flat portions in the graphs of $M(t,\cdot)$ yield Dirac masses in the measures $\rho(t,\cdot)$ for all $t \in [0,T]$ (while the increasing portions show that these measures are not purely discrete). So, $\rho \notin AC^1(0,1;\mathcal{P}_1^{ac}(\mathbb{R}))$. However, in spite of its very basic $AC^1(0,1;\mathcal{P}_1(\mathbb{R}))$ regularity, one can easily see that $v(t,y) = f'(t)y/f(t)$ if $t \in (0,1]$ is the L^1-velocity of ρ. To recapitulate, here we have a curve with no better than $AC^1(0,1;\mathcal{P}_1(\mathbb{R}))$ regularity, but for which the L^1-velocity exists.

Remark 2.7. *We have included the condition "the L^1-velocity v is assumed to exist if $p = 1$" in the upcoming statements, whenever the results apply to $AC^1(0,1;\mathcal{P}_1^{ac}(\mathbb{R}))$ curves with their velocities. Examples 2.5 and 2.6 show that this condition is neither redundant nor vacuous in the case $p = 1$.*

The most general problem ($Flow$) discussed here assumes only Borel regularity on v and we only study solutions X belonging to some time-Sobolev spaces $W^{1,q}(0,T;L^p(I))$. The reason for this extra-requirement on the object introduced in Definition 1.1 will become clear in this section, where we prove that any map X as in Definition 1.1 which also satisfies $X_{t\#}\chi = \rho_t$ for all $t \in [0,T]$ for some $\rho \in AC^q(0,T;\mathcal{P}_p(\mathbb{R}))$ must, in fact, lie in $W^{1,q}(0,T;L^p(I))$. This also means that the time-derivative along paths $X(\cdot,x) \in W^{1,1}(0,T)$ which is denoted by $\partial_t X(\cdot,x) \in L^1(0,T)$ coincides $\mathcal{L}^2$-a.e. with the functional derivative $\dot{X} \in L^q(0,T;L^p(I))$ of X. This definition turns out to be equivalent to requiring that $X \in W^{1,q}(0,T;L^p(I))$ satisfy

$$X(t,x) = X(s,x) + \int_s^t v(\tau, X(\tau,x))\,d\tau \quad \text{for } \mathcal{L}^1\text{-a.e } x \in I \text{ and every } 0 \le s \le t \le T.$$

On the other hand, we shall prove that a solution X of ($Flow$) in the sense of Definition 1.1 belongs to $W^{1,q}(0,T;L^p(I))$ if and only if $[0,T] \ni t \mapsto X_{t\#}\chi =: \rho_t$ belongs to $AC^q(0,T;\mathcal{P}_p(\mathbb{R}))$. This has the important consequence that if a map $X \in W^{1,q}(0,T;L^p(I))$ solves ($Flow$), then so does M, where M_t is the monotone rearrangement of X_t for all $t \in [0,T]$. The theorem we prove next establishes the connection between the regularity of an absolutely continuous curve of probabilities and that of its Lagrangian description. Note that there is no claim at this point that said description be unique; this result applies to *any* flow map associated to the given curve.

Theorem 2.8. *If $X : [0,T] \times I \longrightarrow \mathbb{R}$ is a Lagrangian flow map associated to the Borel map $v : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$, then the following are equivalent:*

(1) *$X_{\#}\chi = \rho \in AC^q(0,T;\mathcal{P}_p(\mathbb{R}))$ and v is its unique L^1-velocity field (or, simply, the* velocity field associated to *ρ). In fact,*

$$(0,T) \ni t \mapsto \|v_t\|_{L^p(\rho_t)} \in L^q(0,T). \tag{2.6}$$

(2) *$X \in W^{1,q}(0,T;L^p(I))$, in which case $\partial_t X$ coincides a.e. in $(0,T)\times I$ with the functional derivative $\dot X$ of X.*

Proof. (1)$\Longrightarrow$(2). Note that

$$\int_0^T \|v_t\|^q_{L^p(\rho_t)}\,dt = \int_0^T \left(\int_I |v(t,X(t,x))|^p\,dx\right)^{\frac{q}{p}} dt,$$

so $\partial_t X \in L^q(0,T;L^p(I)) \subset L^1((0,T)\times I)$. Now,

$$|X(t,x)| \le |X_0(x)| + \int_0^t |v(s,X(s,x))|\,ds$$

$$\Longrightarrow \|X(t,\cdot)\|_{L^p(I)} \le \|X_0(x)\|_{L^p(I)} + \int_0^t \|\partial_s X(s,\cdot)\|_{L^p(I)} ds \quad \text{for all} \quad t\in[0,T].$$

Thus, $X \in L^\infty(0,T;L^p(I)) \subset L^1((0,T)\times I)$. Since $X(\cdot,x)\in W^{1,1}(0,T)$ for a.e. $x\in I$, we have

$$\int_0^T \partial_t\varphi(t,x)X(t,x)\,dt = -\int_0^T \varphi(t,x)\partial_t X(t,x)\,dt \quad \text{for all} \quad \varphi\in C^1_c((0,T)\times I).$$

We use $X, \partial_t X \in L^1((0,T)\times I)$ to conclude

$$\int_0^T\int_I \partial_t\varphi(t,x)X(t,x)\,dx\,dt = -\int_0^T\int_I \varphi(t,x)\partial_t X(t,x)\,dx\,dt \quad \text{for all} \quad \varphi\in C^1_c((0,T)\times I).$$

Thus, $X\in W^{1,q}(0,T;L^p(I))$ and $\partial_t X = \dot X$ a.e. in $(0,T)\times I$.

(2)$\Longrightarrow$(1). To prove $\rho\in AC^q(0,T;\mathcal{P}_p(\mathbb{R}))$ note that the fact that $(X_s\times X_t)_{\#}\chi$ is a transport plan between ρ_s and ρ_t implies

$$W_p(\rho_s,\rho_t) \le \|X_s - X_t\|_{L^p(I)} \le \int_s^t \|\dot X(\tau,\cdot)\|_{L^p(I)} d\tau \quad \text{for all} \quad 0\le s\le t\le T. \tag{2.7}$$

According to [3], the fact that the map $[0,T]\ni t\mapsto \|\dot X(t,\cdot)\|_{L^p(I)}$ lies in $L^q(0,T)$ implies $\rho\in AC^q(0,T;\mathcal{P}_p(\mathbb{R}))\subset AC^1(0,T;\mathcal{P}_1(\mathbb{R}))$. It is now enough to prove (2.6) (which implies (2.4)) and the fact that (ρ,v) satisfies (CE) as distributions. Property (2.6) follows immediately (via $X_{t\#}\chi=\rho_t$ and the a.e. identification $\partial_t X=\dot X$) from

$$\int_0^T \|\dot X(t,\cdot)\|^q_{L^p(I)} dt = \int_0^T \left(\int_I |v(t,X(t,x))|^p\,dx\right)^{q/p} dt = \int_0^T \|v_t\|^q_{L^p(\rho_t)}\,dt. \tag{2.8}$$

Since any $\varphi\in C^1_c(I)$ is Lipschitz, we have that for a.e. $x\in I$ the function $[0,T]\ni t\mapsto \varphi(X(t,x))$ lies in $W^{1,1}(0,T)$ and its a.e. derivative is $\varphi'(X(t,x))\partial_t X(t,x)$. Or, equivalently, for a.e. $x\in I$, we have (we use $(Flow)$ for the second equality)

$$\begin{aligned}\int_0^T \dot\xi(t)\varphi(X(t,x))dt &= -\int_0^T \xi(t)\varphi'(X(t,x))\partial_t X(t,x)dt\\ &= -\int_0^T \xi(t)\varphi'(X(t,x))v(t,X(t,x))dt \text{ for all } \xi\in C^1_c(0,T).\end{aligned}$$

But both integrands are in $L^1((0,T)\times I)$, so by Fubini's Theorem the above equality can be integrated in x, then we can change the order of integration to get

$$\int_0^T \dot{\xi}(t)\int_I \varphi(X(t,x))dxdt = -\int_0^T \xi(t)\int_I \varphi'(X(t,x))v(t,X(t,x))dxdt \text{ for all } \xi\in C_c^1(0,T).$$

Since $X_{t\#}\chi=\rho_t$, the above translates into

$$\int_0^T \dot{\xi}(t)\int_{\mathbb{R}} \varphi(y)\rho_t(dy)\,dt = -\int_0^T \xi(t)\int_{\mathbb{R}} \varphi'(y)v(t,y)\rho_t(dy)\,dt. \tag{2.9}$$

This proves that v is, indeed, a velocity for ρ. $\square$

Remark 2.9. *Due to the uniqueness of the L^1-velocity v (when it exists) for a given $\rho\in AC^1(0,T;\mathcal{P}_1(\mathbb{R}))$, every time we refer to a Lagrangian flow map X for v under the constraint $X_{t\#}\chi=\rho_t$, we may simply call it a* Lagrangian flow map associated to ρ.

It has been known since Fréchet [9] that any Borel map S defined on (in our case) I can be monotonically rearranged over I, i.e. there exists a nondecreasing map $M: I\to\mathbb{R}$ such that $\chi(S^{-1}(B))=\chi(M^{-1}(B))$ for all Borel sets $B\subset\mathbb{R}$. In other words, there exists a nondecreasing function M such that the Lebesgue measure of the pre-images of any Borel set through S and M coincide. This function satisfies

$$M(x)=\inf\{y\in\mathbb{R}\ :\ F(y)>x\} \text{ for all } x\in(0,1),$$

where F is the cumulative distribution function of the Borel probability measure $\mu:=S_{\#}\chi$, i.e. M is the *generalized inverse* of F. It also turns out that F is the generalized inverse of M.

Remark 2.10. *If μ has no atoms, then $F_{\#}\mu=\chi$ optimally, M is strictly increasing and $F\circ M\equiv \mathrm{Id}$. We also have that if $\mu\ll\mathcal{L}^1$, then $M\circ F\equiv\mathrm{Id}$ on the support of μ.*

We continue with the main result of this section.

Theorem 2.11. *Let $\rho\in AC^q(0,T;\mathcal{P}_p^{ac}(\mathbb{R}))$ for some $1\le p<\infty$ and $1\le q\le\infty$. If $p=1$, assume also that the L^1–velocity of ρ exists. Denote by $M(t,\cdot)$ the optimal map pushing forward χ to $\rho(t,\cdot)$. Then $M\in W^{1,q}(0,T;L^p(I))$ and it is a flow map associated to ρ.*

Proof. Since

$$\begin{aligned}
\|M(t,\cdot)\|_{L^p(I)}^p &= \int_{\mathbb{R}}|y|^p\rho(t,y)\,dy = W_p^p(\rho_t,\delta_0)\\
&\le 2^{p-1}\left(W_p^p(\rho_t,\rho_0)+\int_{\mathbb{R}}|y|^p\rho_0(y)\,dy\right)\\
&\le 2^{p-1}\left[\left(\int_0^T\|v_t\|_{L^p(\rho_t)}\,dt\right)^p+\int_{\mathbb{R}}|y|^p\rho_0(y)\,dy\right]<\infty \quad \text{for all } t\in[0,T]
\end{aligned}$$

and

$$\int_0^T \left(\int_I |v(t,M(t,x))|^p\,dx\right)^{q/p} dt = \int_0^T \|v_t\|^q_{L^p(\rho_t)}\,dt < \infty,$$

it suffices to prove that $v(t, M(t, x))$ is the distributional time-derivative of M to obtain the desired thesis.

Consider a standard mollifier $\eta^\varepsilon(y) = \eta(y/\varepsilon)/\varepsilon$ for $0 < \varepsilon < 1$ and let

$$\rho^\varepsilon(t,\cdot) = \eta^\varepsilon * \rho(t,\cdot), \quad E^\varepsilon(t,\cdot) = \eta^\varepsilon * [v(t,\cdot)\,\rho(t,\cdot)]\,.$$

Here, $\eta \in C_c^\infty(\mathbb{R})$ is supported in $[-1, 1]$, nonnegative, even and $\int_{\mathbb{R}} \eta = 1$. Thus, for fixed $y \in \mathbb{R}, z \mapsto \eta^\varepsilon(z - y)$ is smooth and supported in $[y - \varepsilon, y + \varepsilon]$. So, it can be used as a test function in (CE) to deduce that

$$[0,T] \ni t \mapsto \int_{\mathbb{R}} \eta^\varepsilon(z-y)\,\rho_t(z)dz = \rho^\varepsilon(t,y)$$

is absolutely continuous and

$$\partial_t \rho^\varepsilon(t,y) = \int_{\mathbb{R}} \partial_z[\eta^\varepsilon(z-y)]\,v(t,z)\,\rho(t,z)\,dz = -\int_{\mathbb{R}} (\eta^\varepsilon)'(y-z)\,v(t,z)\,\rho(t,z)\,dz = -\,\partial_y E^\varepsilon(t,y)$$

for a.e. $t \in [0, T]$. Now let $F^\varepsilon(t, \cdot)$ and $F(t, \cdot)$ be the cumulative distribution functions of $\rho^\varepsilon(t, \cdot)$ and $\rho(t, \cdot)$, respectively. Note that since $\rho(t, \cdot) \in L^1(\mathbb{R})$, we have that

$$\rho^\varepsilon(t,\cdot) \xrightarrow[\varepsilon\to 0^+]{} \rho(t,\cdot) \quad \text{strongly in } L^1(\mathbb{R}),$$

which implies

$$|F^\varepsilon(t,y) - F(t,y)| \le \int_{-\infty}^y |\rho^\varepsilon(t,z) - \rho(t,z)|\,dz \xrightarrow[\varepsilon\to 0^+]{} 0 \tag{2.10}$$

uniformly in $y \in \mathbb{R}$. Also, $F^\varepsilon(t, \cdot)$ is smooth with $\partial_y F^\varepsilon(t, y) = \rho^\varepsilon(t, y)$ for all t and y.
Since $\rho \in AC^1(0, T; \mathcal{P}_1(\mathbb{R}))$, we deduce that

$$\int_{\mathbb{R}} |y|\,\rho(t,y)\,dy \le C < \infty \quad \text{for all } t.$$

We also have that $t \mapsto F^\varepsilon(t, y)$ is absolutely continuous for a.e. $y \in \mathbb{R}$, with $\partial_t F^\varepsilon(t, y) = -E^\varepsilon(t, y)$. Indeed, this comes as a consequence of $\partial_t \rho^\varepsilon(t, y) = -\partial_y E^\varepsilon(t, y)$. In order to prevent some integrability issues, we also introduce a cut-off function in y, namely, $\xi_k \in C_c^1(\mathbb{R})$ such that $\xi_k(y) = y$ if $|y| \le k$, $\xi_k(y) = 0$ if $|y| \ge 3k$ and $|\xi_k(y)| \le \min\{2|y|, k + 1\}$, $|\xi_k'(y)| \le 1$ for all $y \in \mathbb{R}$. Let $\varphi \in C_c^1(I)$ and note that

$$[0,T] \ni t \mapsto \xi_k(y)\,\varphi(F^\varepsilon(t,y))\,\rho^\varepsilon(t,y)$$

is also absolutely continuous for a.e. $y \in \mathbb{R}$, with

$$\begin{aligned}
\frac{\partial}{\partial t}\Big[\xi_k(\varphi \circ F^\varepsilon)\rho^\varepsilon\Big] &= \xi_k(y)\,\varphi'(F^\varepsilon(t,y))\,\partial_t F^\varepsilon(t,y)\,\rho^\varepsilon(t,y) + \xi_k(y)\,\varphi(F^\varepsilon(t,y))\,\partial_t \rho^\varepsilon(t,y) \\
&= -\Big[\xi_k(y)\,\varphi'(F^\varepsilon(t,y))\,E^\varepsilon(t,y))\,\rho^\varepsilon(t,y) + \xi_k(y)\,\varphi(F^\varepsilon(t,y))\,\partial_y\,E^\varepsilon(t,y)\Big],
\end{aligned}$$

i.e., for any $\zeta \in C^1_c(0,T)$, we have:

$$\int_0^T \dot\zeta(t)\xi_k(y)\,\varphi(F^\varepsilon(t,y))\,\rho^\varepsilon(t,y)\,dt = \int_0^T \zeta(t)\Big[\xi_k(y)\,\varphi'(F^\varepsilon(t,y))\,E^\varepsilon(t,y))\,\rho^\varepsilon(t,y) \\ + \xi_k(y)\,\varphi(F^\varepsilon(t,y))\,\partial_y E^\varepsilon(t,y)\Big]\,dt \quad \text{for a.e. } y\in\mathbb{R}. \tag{2.11}$$

We would like to integrate the above equality in y over $\mathbb{R}$, then integrate by parts over $\mathbb{R}$ the last term in the right hand side; for this we need to show both sides are integrable over $\mathbb{R}$. First,

$$\left|\int_0^T \dot\zeta(t)\,\xi_k(y)\,\varphi(F^\varepsilon(t,y))\,\rho^\varepsilon(t,y)\,dt\right| \le 2\|\dot\zeta\|_\infty\,\|\varphi\|_\infty \int_0^T |y|\,\rho^\varepsilon(t,y)\,dt,$$

and we continue by noticing that

$$\begin{aligned}\int_{\mathbb{R}}|y|\,\rho^\varepsilon(t,y)\,dy &= \int_{\mathbb{R}}|y|\int_{\mathbb{R}}\eta^\varepsilon(y-z)\,\rho(t,z)\,dz\,dy\\ &= \int_{\mathbb{R}}\left(\int_{\mathbb{R}}|y|\,\eta^\varepsilon(y-z)\,dy\right)\rho(t,z)\,dz\\ &\le \int_{\mathbb{R}}\left(\int_{\mathbb{R}}\Big(|z|+|y-z|\Big)\eta^\varepsilon(y-z)\,dy\right)\rho(t,z)\,dz\\ &= \int_{\mathbb{R}}|z|\,\rho(t,z)\,dz + \int_{\mathbb{R}}\rho(t,z)\,dz\int_{\mathbb{R}}|y|\,\eta^\varepsilon(y)\,dy\\ &\le \int_{\mathbb{R}}|z|\,\rho(t,z)\,dz + C\,\varepsilon, \quad \text{where } C := \int_{-1}^1 |y|\,\eta(y)\,dy.\end{aligned}$$

Thus, $\displaystyle\int_{\mathbb{R}}|\xi_k(y)|\,\rho^\varepsilon(t,y)\,dy$ is bounded by a finite constant which is independent of t and $0<\varepsilon<1$. So, $(t,y)\mapsto \xi_k(y)\,\varphi(F^\varepsilon(t,y))\,\rho^\varepsilon(t,y)$ is in $L^\infty(0,T;L^1(\mathbb{R}))$, with bounds independent of $\varepsilon\in(0,1)$ and k. As for the right hand side of (2.11), we see that:

$$|E^\varepsilon(t,y)| \le \int_{\mathbb{R}}\eta^\varepsilon(y-z)\,|v(t,z)|\,\rho(t,z)\,dz \le \frac{1}{\varepsilon}\max_{\mathbb{R}}|\eta|\|v_t\|_{L^1(\rho_t)}.$$

Thus,

$$\int_{\mathbb{R}}\Big|\xi_k(y)\,\varphi'(F^\varepsilon(t,y))\,E^\varepsilon(t,y)\,\rho^\varepsilon(t,y)\Big|\,dy \le \frac{C}{\varepsilon}\,\|v_t\|_{L^1(\rho_t)}\,\|\varphi'\|_\infty,$$

where we absorbed the uniform bound on $\displaystyle\int_{\mathbb{R}}|\xi_k(y)|\,\rho^\varepsilon(t,y)\,dy$ proved above in the constant C. Note that the right hand side of this inequality lies in $L^1(0,T)$, so

$$(t,y)\mapsto \xi_k(y)\,\varphi'(F^\varepsilon(t,y))\,E^\varepsilon(t,y)\,\rho^\varepsilon(t,y)$$

is in $L^1((0,T)\times\mathbb{R})$ (even though, in this case, the bound may be of order ε^{-1}).

Finally, the last term in (2.11) is $\xi_k(y)\,\varphi(F^\varepsilon(t,y))\,\partial_y E^\varepsilon(t,y)$ and it satisfies

$$\begin{aligned}\int_{\mathbb{R}}\Big|\xi_k(y)\,\varphi(F^\varepsilon(t,y))\,\partial_y E^\varepsilon(t,y)\Big|\,dy &\le (k+1)\|\varphi\|_\infty \int_{\mathbb{R}}\int_{\mathbb{R}} \big|(\eta^\varepsilon)'(y-z)\big|\;|v(t,z)|\,\rho(t,z)\,dz\,dy\\ &= (k+1)\|\varphi\|_\infty \int_{\mathbb{R}}\left(\int_{\mathbb{R}} \big|(\eta^\varepsilon)'(y-z)\big|\;dy\right)|v(t,z)|\,\rho(t,z)\,dz\\ &= \varepsilon^{-1}(k+1)\|\varphi\|_\infty\|\eta'\|_{L^1(\mathbb{R})}\|v(t,\cdot)\|_{L^1(\rho(t,\cdot))},\end{aligned}$$

so it lies in $L^1((0,T)\times\mathbb{R})$ as well. Since $[0,T]\ni t\mapsto \|v(t,\cdot)\|_{L^1(\rho(t,\cdot))}$ lies in $L^1(0,T)$, we deduce that (2.11) can be integrated with respect to y over $\mathbb{R}$ and Fubini's Theorem may be applied to discover, after a spatial integration by parts of the last term in the right hand side (which leads to the cancelation of the first term in the right hand side), that

$$\int_0^T \dot\zeta(t)\int_{\mathbb{R}} \xi_k(y)\,\varphi(F^\varepsilon(t,y))\,\rho^\varepsilon(t,y)\,dy\,dt = -\int_0^T \zeta(t)\int_{\mathbb{R}} \xi_k'(y)\,\varphi(F^\varepsilon(t,y))\,E^\varepsilon(t,y))\,dy\,dt, \tag{2.12}$$

with integrands in $L^1((0,T)\times\mathbb{R})$.

Next, we let $\varepsilon\to 0^+$ and use the uniform convergence of $F^\varepsilon(t,\cdot)$ to $F(t,\cdot)$ and the $L^1(\mathbb{R})$-convergence of $\rho^\varepsilon(t,\cdot)$ to $\rho(t,\cdot)$, along with that of $E^\varepsilon(t,\cdot)$ to $v(t,\cdot)\rho(t,\cdot)$ to infer that for each $t\in[0,T]$ we have (since ξ_k is continuous and compactly supported)

$$U^\varepsilon(t) := \int_{\mathbb{R}} \xi_k(y)\,\varphi(F^\varepsilon(t,y))\,\rho^\varepsilon(t,y)\,dy \xrightarrow[\varepsilon\to 0^+]{} \int_{\mathbb{R}} \xi_k(y)\,\varphi(F(t,y))\,\rho(t,y)\,dy$$

and

$$V^\varepsilon(t) := \int_{\mathbb{R}} \xi_k'(y)\,\varphi(F^\varepsilon(t,y))\,E^\varepsilon(t,y)\,dy \xrightarrow[\varepsilon\to 0^+]{} \int_{\mathbb{R}} \xi_k'(y)\,\varphi(F(t,y))\,v(t,y)\rho(t,y)\,dy.$$

By some well-known convolution properties of L^1-functions (note also that $\rho(t,\cdot)$ is nonnegative, as a probability density), we have

$$|U^\varepsilon(t)|\le (k+1)\|\varphi\|_\infty \text{ and } |V^\varepsilon(t)|\le (k+1)\|\varphi\|_\infty\|v(t,\cdot)\|_{L^1(\rho(t,\cdot))}.$$

Next we let $\varepsilon\to 0^+$ in (2.12) and use Dominated Convergence over $[0,T]$ to get

$$\int_0^T \dot\zeta(t)\int_{\mathbb{R}} \xi_k(y)\,\varphi(F(t,y))\,\rho(t,y)\,dy\,dt = -\int_0^T \zeta(t)\int_{\mathbb{R}} \xi_k'(y)\,\varphi(F(t,y))\,v(t,y)\,\rho(t,y)\,dy\,dt.$$

But $|\xi_k(y)|\le 2|y|$, $\|\xi_k'\|_\infty\le 1$ and $\xi_k(y)$ and $\xi_k'(y)$ converge pointwise to y and 1, respectively, for all $y\in\mathbb{R}$. Recall that the first moment of $\rho(t,\cdot)$ is bounded uniformly with respect to t, and that $v\rho\in L^1((0,T)\times\mathbb{R})$. Thus, we may let $k\to\infty$ and use Dominated Convergence on $[0,T]\times\mathbb{R}$ to get

$$\int_0^T \dot\zeta(t)\int_{\mathbb{R}} y\,\varphi(F(t,y))\,\rho(t,y)\,dy\,dt = -\int_0^T \zeta(t)\int_{\mathbb{R}} \varphi(F(t,y))\,v(t,y)\,\rho(t,y)\,dy\,dt. \tag{2.13}$$

We now use the fact that

$$\rho(t,\cdot) \ll \mathcal{L}^1 \text{ implies } F(t, M(t,x)) = x \quad \text{for a.e. } x \in (0,1), \tag{2.14}$$

and that $M_{t\#}\chi = \rho_t$ to conclude:

$$\int_0^T \dot{\zeta}(t) \int_0^1 \varphi(x)\, M(t,x)\, dx\, dt = -\int_0^T \zeta(t) \int_0^1 v(t, M(t,x))\, \varphi(x)\, dx\, dt$$

for all $\zeta \in C_c^1(0,T)$, $\varphi \in C_c^1(I)$. Thus, the distributional time-derivative of $M(t,x)$ is $v(t, M(t,x))$. Of course, the last displayed equality and the uniform $L^q - L^p$ bounds obtained in the first paragraph of this proof also imply that for a.e. $x \in I$ the function $t \mapsto M(t,x)$ is absolutely continuous on $[0,T]$ and its a.e. time derivative is $v(\cdot, M(\cdot, x))$, so M is as in Definition 1.1. □

Now let $X_0 : I \to \mathbb{R}$ such that $X_{0\#}\chi =: \rho_0 \ll \mathcal{L}^1$. If F_0 is the c.d.f. of ρ_0 and $M_0 : I \to \mathbb{R}$ is the optimal map pushing χ forward to ρ_0, then $g_0 := F_0 \circ X_0$ is the Lebesgue a.e. unique map such that $g_{0\#}\chi = \chi$ and $X_0 = M_0 \circ g_0$. This is the *polar decomposition* [4] of X_0.

Corollary 2.12. *Let $\rho \in AC^q(0,T; \mathcal{P}_p^{ac}(\mathbb{R}))$ for some $1 \le p < \infty$ and $1 \le q \le \infty$. If $p = 1$ assume also that the L^1–velocity of ρ exists. Then for any $X_0 : I \to \mathbb{R}$ such that $X_{0\#}\chi = \rho_0$ there exists a flow map $X \in W^{1,q}(0,T; L^p(I))$ associated to ρ that starts at X_0. More precisely, X can be chosen such that $X_t = M_t \circ g_0$ for all $t \in [0,T]$, where M is the family of optimal maps associated to ρ, and g_0 is the measure-preserving map such that $X_0 = M_0 \circ g_0$.*

Proof. Let $X_0 = M_0 \circ g_0$ be the polar decomposition of X_0 as recalled above. There exists a Borel set $A \subset I$ such that $\chi(A) = 1$ and

$$M(t,z) = M_0(z) + \int_0^t v(s, M(s,z))ds \text{ for all } t \in [0,T] \text{ and all } z \in A.$$

But $1 = \chi(A) = \chi(g_0^{-1}(A))$ due to $g_{0\#}\chi = \chi$, so $g_0(x) \in A$ for $\mathcal{L}^1$-a.e. $x \in I$. Thus,

$$M(t, g_0(x)) = M_0(g_0(x)) + \int_0^t v(s, M(s, g_0(x)))ds \text{ for all } t \in [0,T] \text{ and } \mathcal{L}^1\text{-a.e. } x \in A.$$

Thus, $X_t := M_t \circ g_0$ satisfies $(Flow)$ with $X(0,\cdot) \equiv X_0$. □

3 Uniqueness for the Lagrangian flow and for the Continuity Equation

In this section we identify sufficient conditions for the uniqueness of the Lagrangian flow.

3.1 Continuous Case

Lemma 3.1. *If $I \subset \mathbb{R}$ is an open interval, then $C_c^1(I)$ is separable with respect to the $C^1(\bar{I})$ topology.*

Proof. Assume first that $I = (a,b)$ is bounded and fix $\varphi \in C_c(I)$ with $\bar{\varphi} := \fint_I \varphi = 1$. Define the operator $\mathcal{S} : C_c(I) \longrightarrow C_c^1(I)$ by

$$\mathcal{S}\xi(x) = \int_a^x \xi(z)\,dz - \bar{\xi}\int_a^x \varphi(z)\,dz$$

(note that, indeed, this acts between the specified spaces).
We next see that $\mathcal{S}$ is onto by checking that $\mathcal{S}(f') = f$ for any $f \in C_c^1(I)$. It is also linear and continuous with respect to the sup norm. Indeed, since $(\mathcal{S}\xi)'(x) = \xi(x) - \bar{\xi}\varphi(x)$, we have

$$\|\mathcal{S}\xi\|_\infty + \|(\mathcal{S}\xi)'\|_\infty \le C\,\|\xi\|_\infty \quad \text{for some } C < \infty \text{ independent of } \xi \in C_c(I).$$

Thus, the separability of $C_c(I)$ with the sup norm implies the separability of $C_c^1(I)$ with respect to the $C_c^1(\bar{I})$-norm.

If I is unbounded, we can write it as a countable union of bounded intervals to conclude. □

Proposition 3.2. *Let I, J be two open intervals, J be unbounded below and $U = (\rho, w) \in L^1(I \times J; \mathbb{R}^2)$ with* div $U = 0$ *in the sense of distributions. Then, for a.e. $y \in J$ the function $F(t,y) = \int_{-\infty}^y \rho(t,z)\,dz$ lies in $W^{1,1}(I)$ and $\dot{F}(\cdot,y) = -w(\cdot,y)$. Furthermore, if $J \ni y \mapsto w(\cdot,y)$ is continuous in $L^1_{loc}(I)$-weak and $\rho(t,\cdot) \in L^1(J)$ for all $t \in I$, then the above conclusions hold* for all *$y \in J$.*

Proof. For every $\xi \in C_c^1(I)$ and $\zeta \in C_c^1(J)$ we have

$$\int_I \dot{\xi}(t)\left(\int_J \rho(t,z)\zeta(z)\,dz\right)dt = -\int_I \xi(t)\left(\int_J w(t,z)\zeta'(z)\,dz\right)dt. \tag{3.1}$$

Fix $k \in \mathbb{Z}, k \ge 2$ and consider:

$$\zeta_k(z) = \begin{cases} 0 & \text{if} \quad z \in (-\infty, -k-1], \\ z+k+1 & \text{if} \quad z \in (-k-1, -k], \\ 1 & \text{if} \quad z \in (-k, y], \\ -kz+ky+1 & \text{if} \quad z \in (y, y+\frac{1}{k}], \\ 0 & \text{if} \quad z > y+\frac{1}{k}, \end{cases}$$

which is continuous, compactly supported and piecewise linear on $\mathbb{R}$. Let $\{\zeta^n\}_{n\ge 1} \subset C_c^1(J)$ be such that $\zeta^n \xrightarrow[n\to\infty]{} \zeta_k$ uniformly, and

$$(\zeta^n)' \xrightarrow[n\to\infty]{} \zeta_k' \quad \text{everywhere except at } z = -k-1, -k, y, y+\tfrac{1}{k}$$

(where ζ_k is not differentiable) and such that $\left\|(\zeta^n)'\right\|_\infty \le 2\,k$. Then we can pass to the limit in (3.1) with $\zeta \equiv \zeta^n$ to get

$$\int_I \dot\xi(t) \int_J \rho(t,z)\zeta_k(z)\,dz\,dt = -\int_I \xi(t)\left[\int_{-k-1}^{-k} w(t,z)\,dz - k\int_y^{y+\frac{1}{k}} w(t,z)\,dz\right] dt.$$

Since $\rho, w \in L^1(I\times J)$, we can pass to the limit in

$$\int_I \left[\dot\xi(t)\int_J \rho(t,z)\zeta_k(z)\,dz + \xi(t)\int_{-k-1}^{-k} w(t,z)\,dz\right] dt$$

by Dominated Convergence to get (as $k \to \infty$):

$$\lim_{k\to\infty}\int_I \fint_y^{y+\frac{1}{k}} w(t,z)\xi(t)\,dz\,dt = \int_I \dot\xi(t)F(t,y)\,dt \tag{3.2}$$

By Lemma 3.1, there exists a sequence $\{\xi_n\}_{n\ge1}\subset C_c^1(I)$ dense in $C_c^1(I)$ with respect to the $C^1(\bar I)$ topology. Fix such $\xi \equiv \xi_n$ in (3.2) above. The fact that

$$z \mapsto \int_I w(t,z)\xi_n(t)\,dt \in L^1(J)$$

implies there exists a sequence of measurable subsets $\mathcal{A}_n$ of J with $\mathcal{L}^1\left(J\backslash\mathcal{A}_n\right) = 0$ such that every $y \in \mathcal{A}_n$ is a Lebesgue point for this mapping. Thus, at any such y we have

$$\lim_{k\to\infty}\int_I \fint_y^{y+\frac{1}{k}} w(t,z)\xi_n(t)\,dz\,dt = \lim_{k\to\infty}\fint_y^{y+\frac{1}{k}}\int_I w(t,z)\xi_n(t)\,dt\,dz = \int_I \xi_n(t)w(t,y)\,dt. \tag{3.3}$$

Let $\mathcal{A} = \bigcap\limits_{n\ge1}\mathcal{A}_n$, so that $\mathcal{L}^1\left(J\backslash\mathcal{A}\right) = 0$ and (3.3) holds for all $y\in\mathcal{A}$ and all $n\ge1$. By (3.2), we get

$$\int_I \xi_n(t)w(t,y)\,dt = \int_I \dot\xi_n(t)F(t,y)\,dt \quad \text{for all } n\ge1 \text{ and all } y\in\mathcal{A}.$$

By the density described above and the fact that $F(\cdot,y)\in L^1(I)$ and $w(\cdot,y)\in L^1(J)$ for a.e. $y\in J$, we deduce

$$\int_I \dot\xi(t)F(t,y)\,dt = \int_I \xi(t)w(t,y)\,dt \quad \text{for a.e. } y\in J \text{ and all } \xi\in C_c^1(0,T).$$

Thus, we conclude that the function $t\mapsto F(t,y)\in W^{1,1}(I)$ for a.e. $y\in J$ and its distributional derivative is $\dot F(\cdot,y) = -w(\cdot,y)$. In particular,

$$F(b,y) - F(a,y) = -\int_a^b w(t,y)dt \tag{3.4}$$

for all $a,\ b\in I$ with $a\le b$ and a.e. $y\in J$, and we will use this to prove the second statement. Pick an arbitrary $y_0\in J$ and consider a sequence $\{y_n\}_{n\ge1}\subset J$ such that $y_n\to y_0$ and (3.4) holds for $y=y_n$ for all $n\ge1$. Then we get that (3.4) holds for y_0 as well by passing to the limit as $n\to\infty$. Indeed, $\rho(t,\cdot)\in L^1(J)$ for all $t\in I$ implies $F(t,\cdot)$ is continuous on J for all $t\in I$. To pass to the limit in the right hand side we use that $w(\cdot,y_n)$ converges to $w(\cdot,y_0)$ weakly in $L^1(a,b)$. Thus, (3.4) holds for all $a,\ b\in I$ and all $y\in J$. $\square$

Remark 3.3. *Since $\rho(t,\cdot) \in L^1(J)$ for a.e. $t \in I$, we infer $F(t,\cdot) \in L^\infty(J)$ with spatial derivative $\partial_y F(t,\cdot) = \rho(t,\cdot) \in L^1(J)$. If $w = v\rho$ for some Borel map $v = v(t,y)$ and $\dot{X}(t,x) = v(t, X(t,x))$ in some well-defined sense, then a formal calculation reveals*

$$\begin{aligned} \partial_t[F(t,X(t,x))] &= \dot{F}(t,X(t,x)) + \partial_y F(t,X(t,x))\dot{X}(t,x) \\ &= -w(t,X(t,x)) + \rho(t,X(t,x))v(t,X(t,x)) = 0. \end{aligned} \tag{3.5}$$

So, provided that $\rho(0,\cdot) = \chi$ and $X(0,\cdot) = \mathrm{Id}_I$, we deduce $F(t,X(t,x)) = x$ for all $t \in [0,T]$ and a.e. $x \in I$. This fact has far reaching consequences, as we shall see below.

The following statement makes three distinct claims: first, the joint continuity in time-space of v ensures that the continuity equation (CE) has at most one solution $\rho \in AC^1(0,T;\mathcal{P}_1^{ac}(\mathbb{R}))$ whose density $\rho = \rho(t,x)$ is jointly continuous in time-space. Secondly, if such continuous solution exists, then it is also unique within the larger class $AC^1(0,T;\mathcal{P}_1^{ac}(\mathbb{R}))$ (no continuity of densities imposed). Finally, the Lagrangian description of such jointly continuous solution (if it exists) is unique (it is precisely the one provided by Corollary 2.12).

Theorem 3.4. *Let $AC^1_{cont}(0,T;\mathcal{P}_1^{ac}(\mathbb{R}))$ be the set of all $\rho \in AC^1(0,T;\mathcal{P}_1^{ac}(\mathbb{R}))$ such that $\rho \in C([0,T]\times\mathbb{R})$. If $v \in C([0,T]\times\mathbb{R})$, then there exists at most one curve $\rho \in AC^1_{cont}(0,T;\mathcal{P}_1^{ac}(\mathbb{R}))$ originating at a given probability density $\rho_0 \in C(\mathbb{R})$ whose velocity v is. Furthermore, if such a curve exists, then it is also the unique $AC^1(0,T;\mathcal{P}_1^{ac}(\mathbb{R}))$ curve starting at ρ_0 and whose velocity v is. Finally, its only Lagrangian description $X \in W^{1,1}(0,T;L^1(\mathbb{R}))$ starting at a given X_0 (such that $X_{0\#}\chi = \rho_0$) is given by $X_t = M_t \circ g_0$. Here, M_t are the optimal maps such that $M_{t\#}\chi = \rho_t$ for all $t \in [0,T]$, and g_0 is the a.e. unique χ-preserving map such that $X_0 = M_0 \circ g_0$.*

Proof. Consider the curves $\rho \in AC^1_{cont}(0,T;\mathcal{P}_1^{ac}(\mathbb{R}))$ and $\tilde{\rho} \in AC^1(0,T;\mathcal{P}_1^{ac}(\mathbb{R}))$ such that $\rho(0,\cdot) = \rho_0 = \tilde{\rho}(0,\cdot)$. Let, as usual, $F(t,\cdot)$ be the cumulative distribution function of $\rho(t,\cdot)$. Furthermore, let $\tilde{X} \in W^{1,1}(0,T;L^1(I))$ be a Lagrangian flow map associated with $\tilde{\rho}$; indeed, we know from Corollary 2.12 that we can, for example, take $\tilde{X} \equiv \tilde{M} \circ g_0$, where $\tilde{M}$ is the family of optimal maps such that $\tilde{M}_{t\#}\chi = \tilde{\rho}_t$ for all $t \in [0,T]$, and g_0 is the χ-preserving map from the polar decomposition of $X_0 = M_0 \circ g_0 = \tilde{M}_0 \circ g_0 = \tilde{X}(0,\cdot)$. Our strategy is to analyze the function

$$g(t,x) := F(t,\tilde{X}(t,x)), \text{ for all } (t,x) \in [0,T] \times I.$$

We will show that $g(\cdot,x) \in W^{1,1}(0,T)$ for a.e. $x \in I$ and $\dot{g}(\cdot,x) \equiv 0$. Since

$$g(0,x) = F(0,\tilde{X}(0,x)) = \tilde{F}(0,\tilde{X}(0,x)) = g_0(x) \text{ for a.e. } x \in I,$$

this will imply

$$g(t,x) = g_0(x) \text{ for all } t \in [0,T] \text{ and a.e. } x \in I. \tag{3.6}$$

Note that the above displayed equalities hold because $M_0 = \tilde{M}_0$ (since $\rho(0,\cdot) = \rho_0 = \tilde{\rho}(0,\cdot)$) and $\rho_0 \ll \mathcal{L}^1$ (see (2.14)). In fact, (2.14) gives that $F(t,M(t,\cdot)) = \mathrm{Id}$ a.e. in I for all $t \in [0,T]$, where M_t are the optimal maps pushing χ forward to $\rho(t,\cdot)$. Before we justify (3.6), let us show why that yields the desired thesis. The claims are:

$$g(t,x) = g_0(x) \text{ for all } t \in [0,T] \text{ and a.e. } x \in I \text{ implies } \rho(t,\cdot) = \tilde{\rho}(t,\cdot) \text{ for all } t \in [0,T] \tag{3.7}$$

and

$$\tilde{X}_t = M_t \circ g_0 \text{ for all } t \in [0,T], \tag{3.8}$$

i.e. $\rho = \tilde{\rho}$ and the Lagrangian flow is necessarily the one consisting of a time-independent rearrangement of the optimal maps pushing χ forward to ρ_t. Indeed, note that (3.6) implies

$$\tilde{F}_{t\#}\tilde{\rho}_t = \chi = g_{0\#}\chi = g_{t\#}\chi = F_{t\#}[\tilde{X}_{t\#}\chi] = F_{t\#}\tilde{\rho}_t \text{ for all } t \in [0,T].$$

Since both F_t and $\tilde{F}_t$ are nondecreasing, we infer (by the uniqueness of the optimal map pushing $\tilde{\rho}_t$ forward to χ) $F(t,y) = \tilde{F}(t,y)$ for $\tilde{\rho}_t$-a.e. $y \in \mathbb{R}$. Thus, $\partial_y F(t,y) = \partial_y \tilde{F}(t,y)$ for $\mathcal{L}^1$-a.e. y in the interior of the support of $\tilde{\rho}_t$, i.e. $\rho(t,\cdot) = \tilde{\rho}(t,\cdot)$ Lebesgue a.e. in the interior of the support of $\tilde{\rho}_t$. This means that both densities give rise to the same probability (note that the continuity of either density is not necessary here). Finally, since we now know $\tilde{X}_{t\#}\chi = \rho_t = M_{t\#}\chi$, we can write $\tilde{X}_t = M_t \circ s_t$ as the polar factorization of $\tilde{X}_t$. So,

$$g_0 = g_t = F_t \circ \tilde{X}_t = F_t \circ M_t \circ s_t = s_t,$$

which proves claim (3.8).

Now let us get on with the proof that $g(\cdot,x)$ is absolutely continuous. Fix $x \in (0,1)$ for which $t \mapsto \tilde{X}(t,x)$ is in $W^{1,1}(0,T)$, and so we have

$$\tilde{X}(t,x) = X_0(x) + \int_0^t \dot{\tilde{X}}(s,x)\,ds = X_0(x) + \int_0^t v(s,\tilde{X}(s,x))\,ds$$

for $t \in [0,T]$. This implies

$$|\tilde{X}(t,x)| \le |X_0(x)| + \|\dot{\tilde{X}}(\cdot,x)\|_{L^1(0,T)} =: C(x) < +\infty \text{ for all } t \in [0,T].$$

To show that $t \mapsto g(t,x)$ is absolutely continuous on $[0,T]$ (i.e. $g(\cdot,x) \in W^{1,1}(0,T)$) let us notice first that $g(\cdot,x) \in L^\infty(0,T)$, so it all amounts to proving that there exists $f \in L^1(0,T)$ such that for all $0 \le a \le b \le T$:

$$|g(b,x) - g(a,x)| \le \int_a^b f(t)\,dt.$$

As expected, we begin by estimating

$$\begin{aligned} |g(b,x) - g(a,x)| &= |F(b,\tilde{X}(b,x)) - F(a,\tilde{X}(a,x))| \\ &\le |F(b,\tilde{X}(b,x)) - F(b,\tilde{X}(a,x))| + |F(b,\tilde{X}(a,x)) - F(a,\tilde{X}(a,x))| \\ &=: E_1 + E_2. \end{aligned}$$

We have:

$$E_1 = \left| \int_{\tilde{X}(a,x)}^{\tilde{X}(b,x)} \rho(b,y)\,dy \right| \le \left(\max_{[0,T]\times[-C(x),C(x)]} \rho \right) |\tilde{X}(b,x) - \tilde{X}(a,x)|,$$

since $|\tilde{X}(a,x)|, |\tilde{X}(b,x)| \le C(x) < +\infty$ and $\rho \in C([0,T]\times\mathbb{R})$.
Let $\max\{\max_{[0,T]\times[-R,R]} \rho, \max_{[0,T]\times[-R,R]} |v|\} =: \mathscr{M}(R) < +\infty$ for all finite $R > 0$. Thus,

$$E_1 \le \mathscr{M}(C(x)) \int_a^b |\dot{\tilde{X}}(s,x)|\, ds. \tag{3.9}$$

We use Proposition 3.2 to estimate E_2:

$$|F(b,\tilde{X}(a,x)) - F(a,\tilde{X}(a,x))| \le \int_a^b |v(t,\tilde{X}(a,x))| \rho(t,\tilde{X}(a,x)) dt \le [\mathscr{M}(C(x))]^2 (b-a). \tag{3.10}$$

By (3.9) and (3.10) we conclude that for a.e. $x \in (0,1)$ the function $g(\cdot,x)$ is absolutely continuous on $[0,T]$.

The next step is to prove that

$$\dot{g}(\cdot,x) \equiv 0, \quad \text{for a.e. } x \in (0,1).$$

Pick $t \in (0,T)$ where $\dot{\tilde{X}}(t,x)$ exists in the pointwise sense and let $h \in \mathbb{R}$ such that $-t/2 \le h \le (T-t)/2$. Set up the difference quotient:

$$\frac{g(t+h,x) - g(t,x)}{h} = \frac{F(t+h,\tilde{X}(t+h,x)) - F(t,\tilde{X}(t,x))}{h}.$$

<u>Case I:</u> If $\tilde{X}(t+h,x) = \tilde{X}(t,x)$ for all h such that $|h| \le \delta$ (for some $\delta > 0$), then

$$\begin{aligned}
\dot{g}(t,x) &= \lim_{h\to 0} \frac{F(t+h,\tilde{X}(t,x)) - F(t,\tilde{X}(t,x))}{h} \\
&= -\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} v(s,\tilde{X}(t,x)) \rho(s,\tilde{X}(t,x)) ds \\
&= -v(t,\tilde{X}(t,x)) \rho(t,\tilde{X}(t,x))
\end{aligned}$$

due to the continuity of $s \mapsto v(s,y)\rho(s,y)$ for all $y \in \mathbb{R}$. But $\tilde{X}(\cdot,x)$ is constant on $(t-\delta,t+\delta)$, so $v(t,\tilde{X}(t,x)) = \dot{\tilde{X}}(t,x) = 0$ implies $\dot{g}(t,x) = 0$.

<u>Case II:</u> There exists a sequence $\{h_n\}_n \subset [-t/2,(T-t)/2]$ such that $h_n \xrightarrow[n\to\infty]{} 0$ and $\tilde{X}(t+h_n,x) \ne \tilde{X}(t,x)$ for all $n \ge 1$.
Then:

$$\begin{aligned}
\frac{g(t+h_n,x) - g(t,x)}{h_n} &= \frac{F(t+h_n,\tilde{X}(t+h_n,x)) - F(t+h_n,\tilde{X}(t,x))}{\tilde{X}(t+h_n,x) - \tilde{X}(t,x)} \frac{\tilde{X}(t+h_n,x) - \tilde{X}(t,x)}{h_n} \\
&\quad + \frac{F(t+h_n,\tilde{X}(t,x)) - F(t,\tilde{X}(t,x))}{h_n} \\
&= E_1 + E_2.
\end{aligned}$$

We have seen that $E_2 \xrightarrow[n\to\infty]{} -v(t,\tilde{X}(t,x))\rho(t,\tilde{X}(t,x))$. So, it suffices to prove $E_1 \xrightarrow[n\to\infty]{} v(t,\tilde{X}(t,x))\rho(t,\tilde{X}(t,x))$. Note that:

$$E_1 = \fint_{\tilde{X}(t,x)}^{\tilde{X}(t+h_n,x)} \rho(t+h_n,y)\,dy\ \frac{\tilde{X}(t+h_n,x)-\tilde{X}(t,x)}{h_n}.$$

Since $\dot{\tilde{X}}(t,x)$ exists in the pointwise sense (due to our initial choice of t), we have

$$\frac{\tilde{X}(t+h_n,x)-\tilde{X}(t,x)}{h_n} \xrightarrow[n\to\infty]{} \dot{\tilde{X}}(t,x) = v(t,\tilde{X}(t,x)).$$

As for $\fint_{\tilde{X}(t,x)}^{\tilde{X}(t+h_n,x)} \rho(t+h_n,y)\,dy$, we use the fact that the restriction of ρ to $[t-\delta,t+\delta]\times[-C(x),C(x)]$ is uniformly continuous, so $\tilde{X}(t+h_n,x) \xrightarrow[n\to\infty]{} \tilde{X}(t,x)$ implies

$$\lim_{n\to\infty} \fint_{\tilde{X}(t,x)}^{\tilde{X}(t+h_n,x)} \rho(t+h_n,y)\,dy = \rho(t,\tilde{X}(t,x)).$$

This concludes the proof. □

Remark 3.5. *The assumptions on v can be weakened, as it can be seen from the proof. Indeed, we can only require that v is locally essentially bounded, $w := v\rho$ satisfy the conditions from Proposition 3.2, and the map $\rho v = \rho(t,y)v(t,y)$ is continuous on $[0,T]$ for all $y\in\mathbb{R}$.*

Before coming up with an application (Corollary 3.8), we need the following:

Proposition 3.6. *Let $v : [0,T]\times\mathbb{R}\to\mathbb{R}$ satisfy:*

(i) $v\in C([0,T]\times\mathbb{R})$;

(ii) For all $t\in[0,T]$, $v(t,\cdot)\in C^1(\mathbb{R})$;

(iii) There exists $\lambda\in L^1(0,T)$ *such that* $|\partial_y v(t,y)|\le\lambda(t)$ *for all* $(t,y)\in[0,T]\times\mathbb{R}$.

Then, for any positive probability density $\rho_0\in C(\mathbb{R})\cap\mathcal{P}_1(\mathbb{R})$, v is the L^1-velocity of a curve $\rho\in AC^1_{cont}(0,T;\mathcal{P}_1^{ac}(\mathbb{R}))$ originating at ρ_0.

Proof. Let us begin by noticing that $\rho_0>0$ everywhere implies M_0 is continuous, strictly increasing on $(0,1)$, and $M_0(0+)=-\infty$, $M_0(1-)=\infty$. Also, M_0 is the true inverse of the c.d.f. F_0 of ρ_0. This shows that both $F_0\in C^1(\mathbb{R})$ and $M_0\in C^1(0,1)$.

By the classical theory, for each $x\in I$ the initial value problem $(Flow)$ with $X_0(x)=M_0(x)$ admits a (unique) solution $X(t,x)$. Fix $x\in I$ and $-x<h<1-x$, $h\neq 0$ and let

$$Y_h(t,x) := \frac{X(t,x+h)-X(t,x)}{h},$$

so that it satisfies

$$\dot{Y}_h(t,x) = \frac{1}{h}[v(t,X(t,x+h)) - v(t,X(t,x))] = f_h(t,x)Y_h(t,x), \tag{3.11}$$

where

$$f_h(t,x) := \int_0^1 \partial_y v(t,(1-\tau)X(t,x) + \tau X(t,x+h))d\tau.$$

Thus,

$$Y_h(t,x) = \frac{M_0(x+h) - M_0(x)}{h} \exp\left[\int_0^t f_h(s,x)ds\right],$$

which gives, in particular,

$$|X(t,x+h) - X(t,x)| \leq e^{\|\lambda\|_{L^1(0,T)}}|M_0(x+h) - M_0(x)|. \tag{3.12}$$

We get from this that $X(t,\cdot)$ is continuous in I. Next, we have, for all $\tau \in [0,1]$,

$$\lim_{h\to 0} \partial_y v(t,(1-\tau)X(t,x) + \tau X(t,x+h)) = \partial_y v(t,X(t,x))$$

by the continuity of $\partial_y v(t,\cdot)$ and $X(t,\cdot)$. Due to (iii), we have that for $\mathcal{L}^1$-a.e. $t \in (0,T)$ and all $\tau \in [0,1]$

$$|\partial_y v(t,(1-\tau)X(t,x) + \tau X(t,x+h))|, \quad |\partial_y v(t,X(t,x))| \leq \lambda(t) < \infty.$$

Thus, we use Dominated Convergence to integrate in τ and get $f_h(t,x) \xrightarrow[h\to 0]{} f(t,x)$ for a.e. $t \in (0,T)$. Since $\lambda \in L^1(0,T)$, we use Dominated Convergence again (for the integrals in t this time) to infer

$$\lim_{h\to 0} \|f_h(\cdot,x) - f(\cdot,x)\|_{L^1(0,T)} = 0. \tag{3.13}$$

Consider now

$$Y(t,x) := M_0'(x) \exp\left[\int_0^t \partial_y v(s,X(s,x))ds\right], \tag{3.14}$$

i.e. the unique solution of

$$\dot{Y}(t,x) = \partial_y v(t,X(t,x))Y(t,x), \quad Y(0,x) = M_0'(x). \tag{3.15}$$

Set $D_h(t,x) := Y_h(t,x) - Y(t,x)$ to get, according to (3.11) and (3.15),

$$\dot{D}_h(t,x) = f(t,x)D_h(t,x) + R_h(t,x),$$

where $f(t,x) := \partial_y v(t,X(t,x))$ and

$$\begin{aligned} R_h(t,x) &:= Y_h(t,x)\int_0^1 [\partial_y v(t,(1-\tau)X(t,x) + \tau X(t,x+h)) - \partial_y v(t,X(t,x))]d\tau \\ &= Y_h(t,x)[f_h(t,x) - f(t,x)]. \end{aligned}$$

The solution procedure yields

$$D_h(t,x) = D_h(0,x)\exp\left[\int_0^t f(s,x)ds\right] + \int_0^t R_h(s,x)\exp\left[\int_s^t f(u,x)du\right]ds,$$

which implies

$$|D_h(t,x)| \le e^{\|\lambda\|_{L^1(0,T)}}\left[|D_h(0,x)| + \int_0^T |R_h(t,x)|dt\right].$$

But (3.12) and (3.13) give

$$\begin{aligned}\int_0^T |R_h(t,x)|dt &= \int_0^T |Y_h(t,x)||f_h(t,x)-f(t,x)|dt\\ &\le e^{\|\lambda\|_{L^1(0,T)}}\left|\frac{M_0(x+h)-M_0(x)}{h}\right|\|f_h(\cdot,x)-f(\cdot,x)\|_{L^1(0,T)} \xrightarrow[h\to 0]{} 0\end{aligned}$$

for all $x \in I$ (since $M_0 \in C^1(0,1)$). Since

$$D_h(0,x) = \frac{M_0(x+h)-M_0(x)}{h} - M_0'(x) \xrightarrow[h\to 0]{} 0 \text{ for all } x \in I,$$

we deduce $Y_h(t,x) \xrightarrow[h\to 0]{} Y(t,x)$ for all $(t,x) \in [0,T]\times I$, which means that for all $t \in [0,T]$

$$X(t,\cdot) \text{ is differentiable at all } x \in I \text{ and } \partial_x X(t,x) = Y(t,x). \tag{3.16}$$

The formula (3.14) for Y clearly shows (in light of the hypothesis (iii) and the continuity of $X(t,\cdot)$) that $Y(t,\cdot)$ is continuous in I, so we get that $X(t,\cdot) \in C^1(0,1)$ for all $t \in [0,T]$. Since M_0 is strictly increasing in I, we also get from (iii) and (3.14) that

$$e^{-\|\lambda\|_{L^1(0,T)}}M_0'(x) \le \partial_x X(t,x) \le e^{\|\lambda\|_{L^1(0,T)}}M_0'(x),$$

which yields

$$e^{-\|\lambda\|_{L^1(0,T)}}[M_0(y)-M_0(x)] \le X(t,y)-X(t,x) \le e^{\|\lambda\|_{L^1(0,T)}}[M_0(y)-M_0(x)] \tag{3.17}$$

for all $0 < x \le y < 1$. It immediately follows that $X(t,\cdot)$ is strictly increasing in I and $X(t,0+) = -\infty$ and $X(t,1-) = \infty$. If $\rho(t,\cdot) := X(t,\cdot)_{\#}\chi$, we have $\rho(t,X(t,x))\partial_x X(t,x) = 1$ for all $x \in I$, i.e.

$$\rho(t,y) = \frac{\exp\left[-\int_0^t \partial_y v(s,X(s,F(t,y)))ds\right]}{M_0'(F(t,y))} > 0 \tag{3.18}$$

in light of (3.14) (here, $F(t,\cdot)$ is the true inverse of $X(t,\cdot)$ or, equivalently, the c.d.f. of $\rho(t,\cdot)$). Clearly, $\rho(0,\cdot) = \rho_0$.

We claim that ρ defined above belongs to $AC^1_{cont}(0,T;\mathcal{P}_1^{ac}(\mathbb{R}))$ and v is its L^1-velocity. First, we would like to apply Theorem 2.8 to prove $\rho \in AC^1(0,T;\mathcal{P}_1(\mathbb{R}))$ and that v is the required velocity map; according to said theorem, it is enough to prove that $X \in W^{1,1}(0,T;L^1(I))$. let $m_0 \in \mathbb{R}$ denote the first moment of ρ_0 and estimate

$$\begin{aligned}
\int_{\mathbb{R}} |y|\rho(t,y)dy &= \int_I |X(t,x)|dx \\
&\leq m_0 + \int_I |X(t,x) - M_0(x)|dx \\
&\leq m_0 + \int_0^t \int_I |v(s,X(s,x))|dxds \\
&\leq m_0 + \int_0^t |\lambda(s)| \int_I |X(s,x)|dxds + \int_0^t |v(s,0)|ds \\
&\leq C_0 + \int_0^t \lambda(s) \int_I |X(s,x)|dxds,
\end{aligned}$$

where $C_0 := m_0 + T\|v(\cdot,0)\|_{L^\infty(0,T)} < \infty$ (since $t \mapsto v(t,0)$ is continuous on $[0,T]$). Gronwall's Lemma now gives a uniform (with respect to $t \in [0,T]$) bound on the first moment of $\rho(t,\cdot)$ or, equivalently, on $\|X(t,\cdot)\|_{L^1(I)}$. Then,

$$\begin{aligned}
\int_I |\dot{X}(t,x)|dx &\leq \int_0^T \int_I |v(s,X(s,x))|dxds \\
&= \int_0^T \int_{\mathbb{R}} |v(s,y)|\rho(s,y)dyds \\
&\leq \int_0^T \lambda(s) \int_{\mathbb{R}} |y|\rho(s,y)dyds + \int_0^T |v(s,0)| \int_{\mathbb{R}} \rho(s,y)dyds \\
&\leq \|\lambda\|_{L^1(0,T)} \sup_{t\in[0,T]} \|X(t,\cdot)\|_{L^1(I)} + \|v(\cdot,0)\|_{L^\infty(0,T)} < \infty.
\end{aligned}$$

Thus, $X \in W^{1,\infty}(0,T;L^1(I)) \subset W^{1,1}(0,T;L^1(I))$, so $\rho \in AC^1(0,T;\mathcal{P}_1(\mathbb{R}))$ and v is its L^1-velocity map (by Theorem (2.8)).

It only remains to prove that $\rho \in C([0,T] \times \mathbb{R})$. The plan is to show first that F is (jointly) continuous in $[0,T]\times\mathbb{R}$, then use (3.18) to infer that ρ has the same property. Since $\rho(t,\cdot) \ll \mathcal{L}^1$ is a probability density, we have that $y \mapsto F(t,y)$ is *uniformly* continuous in $\mathbb{R}$, but we would like to show more: namely that $F(t,\cdot)$ is uniformly continuous in $\mathbb{R}$ *uniformly* with respect to $t \in [0,T]$. For this, denote by ω a modulus of continuity for F_0, i.e. $\omega : [0,\infty) \to [0,\infty)$ is continuous, increasing, $\omega(0) = 0$, and satisfies

$$\omega(y_2 - y_1) \geq F_0(y_2) - F_0(y_1) \text{ for all } y_1 \leq y_2 \in \mathbb{R}.$$

This is equivalent to

$$\omega(M_0(x_2) - M_0(x_1)) \geq x_2 - x_1 \text{ for all } x_1 \leq x_2 \in I.$$

Now let $\alpha := \exp\left(\|\lambda\|_{L^1(0,T)}\right) > 0$ and use (3.17), the above displayed inequality, and the monotonicity of ω to infer

$$\omega(\alpha[X(t,x_2) - X(t,x_1)]) \geq x_2 - x_1 \text{ for all } x_1 \leq x_2 \in I,$$

i.e. (upon letting $\omega_\alpha(r) := \omega(\alpha r)$)

$$\omega_\alpha(y_2 - y_1) \geq F(t, y_2) - F(t, y_1) \text{ for all } y_1 \leq y_2 \in \mathbb{R}.$$

But ω_α is also a modulus of continuity (independent of $t \in [0, T]$), so $F(t, \cdot)$ is uniformly continuous in $\mathbb{R}$, uniformly with respect to $t \in [0, T]$. The continuity of $[0, T] \ni t \mapsto F(t, y)$ for all $y \in \mathbb{R}$ will also be needed in order to infer that F is continuous in $[0, T] \times \mathbb{R}$. To prove that, assume there exists $y \in \mathbb{R}$ such that $F(\cdot, y)$ is not continuous at some $t \in [0, T]$. Thus, there exists a sequence $\{t_n\}_n \subset [0, T]$ and $\delta > 0$ such that $t_n \to t$ and $|F(t_n, y) - F(t, y)| > \delta$ for all $n \geq 1$; so, it is either $F(t_n, y) - F(t, y) > \delta$ or $F(t, y) - F(t_n, y) > \delta$ for a subsequence (not relabeled) of $\{t_n\}_n$. Assume the former. Since $F(t_n, y) > \delta + F(t, y)$, we deduce $1 > \delta + F(t, y) > \delta > 0$, so $\delta + F(t, y)$ is in the domain of $X(t_n, \cdot)$. We know $X(t_n, \cdot)$ is strictly increasing in I for all $n \geq 1$, so $X(t_n, F(t_n, y)) > X(t_n, F(t, y) + \delta)$, i.e. $y > X(t_n, F(t, y) + \delta)$ for all $n \geq 1$. But $X(\cdot, x)$ is continuous on $[0, T]$ for all $x \in I$ (satisfies $(Flow)$ for all $x \in I$), so we can pass to the limit as $n \to \infty$ in the last inequality to deduce, after using again that $X(t, \cdot)$ is strictly increasing on I,

$$y \geq X(t, F(t, y) + \delta) > X(t, F(t, y)) = y,$$

a contradiction. If, instead, $F(t, y) - F(t_n, y) > \delta$, we rewrite it as $F(t, y) - \delta > F(t_n, y)$, which implies $1 > F(t, y) - \delta > 0$, i.e. $F(t, y) - \delta$ is in the domain of $X(t_n, \cdot)$. Since $X(t_n, \cdot)$ is strictly increasing in I for all integers $n \geq 1$, we deduce $X(t_n, F(t, y) - \delta) > X(t_n, F(t_n, y)) = y$ for all integers $n \geq 1$. As before, in the limit we find

$$y = X(t, F(t, y)) > X(t, F(t, y) - \delta) \geq y,$$

a contradiction. Thus, in light of the uniform continuity of $F(t, \cdot)$ holding uniformly with respect to t, we get $F \in C([0, T] \times \mathbb{R})$, as desired.

So, if $t_n \to t$ and $y_n \to y$, we can use the continuity of $x \mapsto X(s, x)$ and of $y \mapsto \partial_y v(s, y)$ (the latter, according to (ii)) to deduce

$$\partial_y v(s, X(s, F(t_n, y_n))) \xrightarrow[n \to \infty]{} \partial_y v(s, X(s, F(t, y))) \text{ for each } s \in [0, T].$$

We use Dominated Convergence in light of (iii) to move on to

$$\int_0^t \partial_y v(s, X(s, F(t_n, y_n)))ds \xrightarrow[n \to \infty]{} \int_0^t \partial_y v(s, X(s, F(t, y)))ds.$$

But (iii) also gives

$$\left| \int_t^{t_n} \partial_y v(s, X(s, F(t_n, y_n)))ds \right| \leq \left| \int_t^{t_n} \lambda(s)ds \right| \xrightarrow[n \to \infty]{} 0,$$

which, in light of the previously displayed convergence, implies

$$\int_0^{t_n} \partial_y v(s, X(s, F(t_n, y_n)))ds \xrightarrow[n \to \infty]{} \int_0^t \partial_y v(s, X(s, F(t, y)))ds.$$

This, along with (3.18) and the fact that M_0' is continuous on I (note that $F(t, y) \in I$ for all $t \in [0, T]$ and all $y \in \mathbb{R}$, i.e. the values 0 and 1 are achieved only as the asymptotic limits of $F(t, \cdot)$ at $-\infty$ and ∞, respectively), implies the continuity of ρ in $[0, T] \times \mathbb{R}$. $\square$

Remark 3.7. *Note that the assumptions on v made in the statement of Proposition 3.6 do not include the continuity of $\partial_y v(t,y)$ in time.*

In light of Theorem 3.4, Proposition 3.6 gives sufficient conditions on the velocity to render it an (SC) velocity with respect to $\rho \in AC^1(0,T;\mathcal{P}_1^{ac}(\mathbb{R}))$, i.e. there is a unique curve in $\rho \in AC^1(0,T;\mathcal{P}_1^{ac}(\mathbb{R}))$ with that velocity. Yet another way to look at it:

Corollary 3.8. *For any v and ρ_0 satisfying the assumptions in Proposition 3.6, there is a unique solution $\rho \in AC^1(0,T;\mathcal{P}_1^{ac}(\mathbb{R}))$ of (CE) with initial $\rho(0,\cdot) = \rho_0$. Furthermore, this solution lies in $C([0,T]\times\mathbb{R})$ and is everywhere positive at all times.*

Remark 3.9. *It is not difficult to construct examples where Theorem 3.4 applies but Proposition 3.6 does not. For this, see example below. In this case all the conclusions of the above corollary hold in spite of v being much less regular.*

Example 3.10. Take $u : \mathbb{R} \to [1,2]$ to be continuous and nowhere differentiable (some Weierstrass function), then set, for example,

$$\rho(t,y) := \eta(t)\frac{u(y)}{y^4+t+1},$$

where $\eta(t)$ normalizes $\rho(t,\cdot)$ to a probability density over $\mathbb{R}$. Note that both η and $\dot\eta$ are bounded away from zero and infinity on $[0,T]$ for any $0<T<\infty$. Also, since $\rho \in C([0,T]\times\mathbb{R})$, in order to prove $\rho \in AC^1_{cont}(0,T;\mathcal{P}_1^{ac}(\mathbb{R}))$ we just need to check $\rho \in AC^1(0,T;\mathcal{P}_1^{ac}(\mathbb{R}))$. Set

$$g(t,y) := \frac{u(y)}{y^4+t+1}, \quad h(t,y) := \frac{u(y)}{(y^4+t+1)^2} = \frac{g(t,y)}{y^4+t+1},$$

so that, after some computations, we discover

$$\partial_t F(t,y) = \frac{H(t,\infty)}{G(t,\infty)}\left[\frac{G(t,y)}{G(t,\infty)} - \frac{H(t,y)}{H(t,\infty)}\right],$$

where

$$G(t,y) := \int_{-\infty}^{y} g(t,z)dz, \quad H(t,y) := \int_{-\infty}^{y} h(t,z)dz.$$

We can easily see that $G(\cdot,\infty)$ and $H(\cdot,\infty)$ are bounded on $[0,T]$ away from zero and infinity. Thus, the integrability of $\partial_t F$ is equivalent to that of

$$\frac{G(t,y)}{G(t,\infty)} - \frac{H(t,y)}{H(t,\infty)} = F(t,y) - \frac{H(t,y)}{H(t,\infty)}.$$

But $\tilde H(t,\cdot) := H(t,\cdot)/H(t,\infty)$ is the c.d.f. of the density $\tilde h(t,\cdot) := h(t,\cdot)/\int_{\mathbb{R}} h(t,z)dz$, which has uniformly (with respect to t) bounded first moment. Same is true about $\rho(t,y) = g(t,y)/\int_{\mathbb{R}} g(t,z)dz$. So,

$$\int_{\mathbb{R}} |\partial_t F(t,y)|dy \le \sup_{[0,T]}\left[\frac{H(t,\infty)}{G(t,\infty)}\right]\|F(t,\cdot) - \tilde H(t,\cdot)\|_{L^1(\mathbb{R})} = cW_1(\rho_t,\tilde h_t) \le C < \infty$$

for all $t \in [0,T]$, i.e. $\partial_t F \in L^\infty(0,T;L^1(\mathbb{R})) \subset L^1((0,T)\times\mathbb{R})$. Thus,

$$W_1(\rho_s,\rho_t) = \|M_s - M_t\|_{L^1(I)} = \|F(s,\cdot) - F(t,\cdot)\|_{L^1(\mathbb{R})} \leq \int_s^t \|\partial_t F(\tau,\cdot)\|_{L^1(\mathbb{R})} d\tau,$$

(see, e.g., [14] for the equality in the middle) and thus, since $\partial_t F \in L^\infty(0,T;L^1(\mathbb{R}))$, we deduce $\rho \in AC^\infty(0,T;\mathcal{P}_1(\mathbb{R})) \subset AC^1(0,T;\mathcal{P}_1^{ac}(\mathbb{R}))$. Since $\rho > 0$ everywhere, we use Proposition 3.2 and $(Trans)$ to reconstruct the velocity

$$v(t,y) = -\frac{\partial_t F(t,y)}{\rho(t,y)} \in C([0,T]\times\mathbb{R}) \ .$$

Since $\partial_t F$ is differentiable in y everywhere in $[0,T]\times\mathbb{R}$ and ρ is positive and nowhere differentiable in y, we infer v is not differentiable in y at each point (t,y) where $\partial_t F(t,y) \neq 0$. Fix $t \in [0,T]$ and assume there are $y_1 < y_2 \in \mathbb{R}$ such that $\partial_t F(t,y_1) = \partial_t F(t,y_2) = 0$, i.e. $F(t,y_i) = \tilde{H}(t,y_i)$ for $i = 1,\ 2$. Thus, there exists $y_1 < \bar{y} < y_2$ such that $\partial_y[F(t,\cdot) - \tilde{H}(t,\cdot)](\bar{y}) = 0$, i.e. $\rho(t,\bar{y}) = \tilde{h}(t,\bar{y})$, which is equivalent to $\bar{y}^4 + t + 1 = G(t,\infty)/H(t,\infty)$. This equation has at most two real solutions, so $v(t,\cdot)$ is differentiable at at most three points. Thus, (ρ, v) satisfies the assumptions of Theorem 3.4, while v violates those of Proposition 3.6.

3.2 Case of higher integrability

Here we show that uniqueness of the Lagrangian description of an absolutely continuous curve of probability measures may also be a consequence of some higher integrability enjoyed by the densities in space-time. The reader will note that the full power of Corollary 2.12 was not needed to prove Theorem 3.4 (we only needed existence of a Lagrangian map $\tilde{X}$ associated to $\tilde{\rho}$; its explicit nature, provided by Theorem 2.11, was irrelevant to the proof). The theorem below will use both Corollary 2.12 (to provide a Lagrangian description for $\tilde{\rho}$) and Theorem 2.11 (applied to ρ). We first present a helpful lemma.

Lemma 3.11. *Let $1 \leq p < \infty$, $1 < q \leq \infty$, and let $\rho \in AC^q(0,T;\mathcal{P}_p^{ac}(\mathbb{R}))$ with L^1-velocity v (assumed to exist if $p = 1$). Denote by M_t the optimal maps pushing χ forward to ρ_t for all $t \in [0,T]$ and define $M_t := M_0$ if $t < 0$ and $M_t := M_T$ if $t > T$. Then*

$$\lim_{h\to 0} \int_0^T \left\| \frac{M_{t+h} - M_t}{h} - \dot{M}_t \right\|_{L^p(I)}^q dt = 0. \tag{3.19}$$

Proof. By Theorem 2.11 we have that $M \in W^{1,q}(0,T;L^p(I))$, which implies

$$\left\| \frac{M(t+h,\cdot) - M(t,\cdot)}{h} - \dot{M}(t,\cdot) \right\|_{L^p(I)} \leq \fint_t^{t+h} \|\dot{M}(s,\cdot) - \dot{M}(t,\cdot)\|_{L^p(I)}\, ds,$$

so

$$\left\| \frac{M(t+h,\cdot) - M(t,\cdot)}{h} - \dot{M}(t,\cdot) \right\|_{L^p(I)} \xrightarrow[h\to 0]{} 0$$

for all $t \in (0,T)$ which are Lebesgue points for $[0,T] \ni t \mapsto \dot{M}(t,\cdot) \in L^p(I)$, i.e. for a.e. $t \in [0,T]$. Also, if $f(t) := \|\dot{M}(t,\cdot)\|_{L^p(I)}$ and $\mathcal{M}f$ is its Hardy-Littlewood maximal function, we get, for $t \in (0,T)$ and sufficiently small $|h|$,

$$\left\| \frac{M_{t+h} - M_t}{h} - \dot{M}_t \right\|_{L^p(I)} \leq f(t) + \mathcal{M}f(t).$$

The right hand side is guaranteed to lie in $L^q(0,T)$ only if $q > 1$. Thus, if $1 \leq p < \infty$ and $1 < q \leq \infty$ we get (3.19) by Dominated Convergence. QED.

Just as Theorem 3.4, the theorem below makes multiple claims; beside the uniqueness of the Lagrangian description we also have the uniqueness of solutions for (CE) within a certain class (see Remarks 3.14 and 3.15 below).

Theorem 3.12. *Let $1 \leq p < \infty$ and $1 < q < \infty$, and set $r := q^\star(1 + 1/p^\star)$, $s := 1 + p^\star$. If*

$$\rho \in AC^q(0,T;\mathcal{P}_p(\mathbb{R})) \cap L^r_{loc}(0,T;L^s(\mathbb{R})) =: \mathcal{U}[p,q],$$

let

$$\mathcal{S}[\rho] := \{\tilde{\rho} \in \mathcal{U}(p,q) \,:\, \tilde{\rho}(0,\cdot) = \rho(0,\cdot) \text{ and } (\exists)\ \lambda \in \mathbb{R} \text{ such that } \rho \leq \lambda\tilde{\rho} \text{ or } \tilde{\rho} \leq \lambda\rho\}. \tag{3.20}$$

(The inequalities above are to be understood in the $\mathcal{L}^2$–a.e. sense.) Then, for any $\rho \in \mathcal{U}[p,q]$, its L^1–velocity v (if $p = 1$ we assume v exists) is the velocity of no other curve in $\mathcal{S}[\rho]$. Furthermore, the only Lagrangian flow map $X \in W^{1,1}(0,T;L^1(\mathbb{R}))$ associated to some $\rho \in \mathcal{U}(p,q)$ starting at a given X_0 (such that $X_{0\#}\chi = \rho_0$) is given by $X_t = M_t \circ g_0$. Here, M_t are the optimal maps such that $M_{t\#}\chi = \rho_t$ for all $t \in [0,T]$, and g_0 is the a.e. unique χ-preserving map such that $X_0 = M_0 \circ g_0$.

Proof. Consider $\rho \in \mathcal{U}[p,q]$. Fix some $\tilde{\rho} \in \mathcal{S}[\rho]$ such that $\tilde{\rho} \leq \lambda\rho$ for some $\lambda \in \mathbb{R}$ (if such constant does not exist, then there exists $\lambda \in \mathbb{R}$ such that $\rho \leq \lambda\tilde{\rho}$; in this case one only needs to interchange the roles of ρ and $\tilde{\rho}$ in this proof) and assume $\tilde{X} : [0,T] \times I \longrightarrow \mathbb{R}$ is a flow map associated to $(\tilde{\rho}, v)$ ($\tilde{X}$ exists and lies in $W^{1,q}(0,T;L^p(I))$, according to Corollary 2.12). Let $g : [0,T] \times I \longrightarrow I$ defined by

$$g(t,x) := F(t,\tilde{X}(t,x)), \quad \text{where } F(t,y) := \int_{-\infty}^{y} \rho(t,z)\,dz.$$

We have $g_0 = F_0 \circ \tilde{X}_0 = F_0 \circ X_0$, which satisfies $g_{0\#}\chi = \chi$. Thus, note that we obtain the desired conclusions by proving that g is time-independent, then using the proved claim (3.7) in the present context. Clearly, $\tilde{X}_{t\#}\chi = \tilde{\rho}_t$ implies $F_{t\#}\tilde{\rho}_t = g_{t\#}\chi =: \vartheta_t$ for all $t \in [0,T]$, which leads to

$$\begin{aligned}
\int_0^1 \zeta(x)\vartheta_t(dx) = \int_0^1 \zeta(g_t(x))dx \ &= \int_{\mathbb{R}} \zeta(F_t(y))\tilde{\rho}_t(y)dy \\
&\leq \lambda \int_{\mathbb{R}} \zeta(F_t(y))\rho_t(y)dy = \lambda \int_0^1 \zeta(x)dx
\end{aligned}$$

for all nonnegative $\zeta \in C_c(I)$. Thus, ϑ_t is a Borel probability such that $\vartheta_t \ll \chi$ and (its density) $\vartheta_t \in L^\infty(I)$ with uniform bound with respect to $t \in [0,T]$. By Theorem 2.11, we know that the equality $\dot{M}(t,y) = v(t, M(t,y))$ between the distributional time-derivative of M (family of optimal maps $M_{t\#}\chi = \rho_t$) and $v_t \circ M$ holds $\mathcal{L}^2$-a.e. in $(0,T) \times (0,1)$. Thus, there exists a set of times $\mathcal{T} \subset (0,T)$ of full Lebesgue measure such that for each $t \in \mathcal{T}$ we have $\dot{M}(t,y) = v(t, M(t,y))$ for χ–a.e. $y \in (0,1)$. Let A_t denote the set of all such y's for a given $t \in \mathcal{T}$. Note that $1 = \vartheta_t(A_t) = \chi(g_t^{-1}(A_t))$ (due to $g_{t\#}\chi = \vartheta_t$). For any $x \in g_t^{-1}(A_t)$ we have $g_t(x) \in A_t$, so $\dot{M}(t, g(t,x)) = v(t, M(t, g(t,x)))$. Since $\mathcal{L}^1(g_t^{-1}(A_t)) = 1$, we conclude that

$$\text{for a.e. } t \in (0,T) \text{ we have } \dot{M}(t, g(t,x)) = v(t, M(t, g(t,x))) \quad \text{for a.e.} \quad x \in I. \tag{3.21}$$

But the support of $\tilde{\rho}_t$ is included in the support of ρ_t, which means, due to $\tilde{X}_{t\#}\chi = \tilde{\rho}_t$,

$$M_t \circ g_t = M_t \circ F_t \circ \tilde{X}_t \equiv \tilde{X}_t \text{ Lebesgue a.e. in } I. \tag{3.22}$$

Thus, according to (3.21), for Lebesgue a.e. $t \in [0,T]$ we have

$$\dot{M}(t, g(t,x)) = v(t, \tilde{X}(t,x)) \text{ for Lebesgue a.e. } x \in I,$$

which means

$$\dot{\tilde{X}}(t,x) = \dot{M}(t, g(t,x)) \text{ for Lebesgue a.e. } x \in I. \tag{3.23}$$

Now, fix an arbitrary $\varepsilon > 0$ sufficiently small. By (3.19), we have

$$\lim_{h\to 0} \int_\varepsilon^{T-\varepsilon} \left\| \frac{M_{t+h} - M_t}{h} - \dot{M}_t \right\|_{L^p(I)}^q dt = 0.$$

Since $g_{t\#}\chi = \vartheta_t$ and $\vartheta \in L^\infty((0,T)\times I)$, we deduce

$$\lim_{h\to 0} \int_\varepsilon^{T-\varepsilon} \left\| \frac{M_{t+h}\circ g_t - M_t \circ g_t}{h} - \dot{M}_t \circ g_t \right\|_{L^p(I)}^q dt = 0,$$

which means, in view of (3.22) and (3.23),

$$\lim_{h\to 0} \int_\varepsilon^{T-\varepsilon} \left\| \frac{M_{t+h}\circ g_t - \tilde{X}_t}{h} - \dot{\tilde{X}}_t \right\|_{L^p(I)}^q dt = 0.$$

But we also have $\tilde{X} \in W^{1,q}(0,T; L^p(I))$, so (3.19) yields

$$\lim_{h\to 0} \int_\varepsilon^{T-\varepsilon} \left\| \frac{\tilde{X}_{t+h} - \tilde{X}_t}{h} - \dot{\tilde{X}}_t \right\|_{L^p(I)}^q dt = 0.$$

It follows that

$$\lim_{h\to 0} \int_\varepsilon^{T-\varepsilon} \left\| \frac{\tilde{X}_{t+h} - M_{t+h}\circ g_t}{h} \right\|_{L^p(I)}^q dt = 0. \tag{3.24}$$

Since ρ_t has no atoms, M_t is strictly increasing for all $t \in (0,T)$, thus

$$g_{t+h}(x) \neq g_t(x) \iff M_{t+h}\circ g_{t+h}(x) \neq M_{t+h} \circ g_t(x)$$

for any $t \in (\varepsilon, T-\varepsilon)$ and any h sufficiently small. Consequently, $(g_{t+h}(x) - g_t(x))/h$ is either 0 or

$$\frac{g_{t+h}(x) - g_t(x)}{h} = \frac{F_{t+h} \circ \tilde{X}_{t+h}(x) - F_{t+h}(M_{t+h} \circ g_t(x))}{\tilde{X}_{t+h}(x) - M_{t+h} \circ g_t(x)} \frac{\tilde{X}_{t+h} - M_{t+h} \circ g_t(x)}{h},$$

where we have used (3.22) and the fact that $F_t \circ M_t \equiv \mathrm{Id}$ in I for all $t \in [0,T]$ (this is due to ρ_t being atom-free, i.e. F_t is continuous on $\mathbb{R}$). Thus,

$$\left| \frac{g_{t+h}(x) - g_t(x)}{h} \right| = \left| \fint_{M_{t+h} \circ g_t(x)}^{\tilde{X}_{t+h}(x)} \rho_{t+h}(y)\, dy \right| \left| \frac{\tilde{X}_{t+h}(x) - M_{t+h} \circ g_t(x)}{h} \right|,$$

where we used that $F'_{t+h} = \rho_{t+h}$ and the convention $\fint_a^b f = 0$ if $a = b$.
Since $\rho_t \in L^{1+p^\star}(\mathbb{R})$ for all $t \in [0,T]$, we deduce it has a maximal function $\mathcal{M}\rho_t \in L^{1+p^\star}(\mathbb{R})$ such that $\|\mathcal{M}\rho_t\|_{L^s(\mathbb{R})} \le C_p \|\rho_t\|_{L^s(\mathbb{R})}$ for some finite constant C_p depending only on p.
By the definition of the Hardy-Littlewood maximal function, we get:

$$\left| \frac{g_{t+h}(x) - g_t(x)}{h} \right| \le \mathcal{M}\rho_{t+h}(\tilde{X}_{t+h}(x)) \left| \frac{\tilde{X}_{t+h}(x) - M_{t+h} \circ g_t(x)}{h} \right|. \tag{3.25}$$

Note that $\tilde{X}_{t+h\#}\chi = \tilde{\rho}_{t+h}$, $s = 1 + p^\star$ and $\tilde{\rho}_{t+h} \le \lambda \rho_{t+h}$ imply (if $p > 1$)

$$\begin{aligned}
\int_I (\mathcal{M}\rho_{t+h})^{p^\star}(\tilde{X}_{t+h}(x))\, dx &= \int_{\mathbb{R}} (\mathcal{M}\rho_{t+h})^{p^\star}(y)\, \tilde{\rho}_{t+h}(y)\, dy \\
&\le \lambda \|\rho_{t+h}\|_{L^s(\mathbb{R})} \left\| (\mathcal{M}\rho_{t+h})^{p^\star} \right\|_{L^{s^\star}(\mathbb{R})} \\
&= \lambda \|\rho_{t+h}\|_{L^s(\mathbb{R})} \|\mathcal{M}\rho_{t+h}\|_{L^s(\mathbb{R})}^{p^\star} \\
&\le \lambda C_p \|\rho_{t+h}\|_{L^s(\mathbb{R})}^s .
\end{aligned}$$

Thus,

$$\left\| \mathcal{M}\rho_{t+h} \circ \tilde{X}_{t+h} \right\|_{L^{p^\star}(I)} \le \lambda C_p \|\rho_{t+h}\|_{L^{1+p^\star}(\mathbb{R})}^{1+\frac{1}{p^\star}}, \quad \text{since } s = 1 + p^\star.$$

This inequality is also obvious if $p = 1$. Consequently, we can apply Hölder's inequality to (3.25) to get

$$\left\| \frac{g_{t+h} - g_t}{h} \right\|_{L^1(I)} \le \lambda C_p \|\rho_{t+h}\|_{L^{1+p^\star}(\mathbb{R})}^{1+\frac{1}{p^\star}} \left\| \frac{\tilde{X}_{t+h} - M_{t+h} \circ g_t}{h} \right\|_{L^p(I)}.$$

If $1 < q < \infty$, Hölder's inequality for the time-integral now gives

$$\begin{aligned}
&\int_\varepsilon^{T-\varepsilon} \left\| \frac{g_{t+h} - g_t}{h} \right\|_{L^1(I)} dt \\
&\le \lambda C_p \left(\int_\varepsilon^{T-\varepsilon} \|\rho_{t+h}\|_{L^s(\mathbb{R})}^{q^\star(1+\frac{1}{p^\star})} dt \right)^{1/q^\star} \left(\int_\varepsilon^{T-\varepsilon} \left\| \frac{\tilde{X}_{t+h} - M_{t+h} \circ g_t}{h} \right\|_{L^p(I)}^q dt \right)^{1/q}.
\end{aligned} \tag{3.26}$$

Note that $\rho \in L^r_{loc}(0,T;L^s(\mathbb{R}))$ implies $\displaystyle\int_{\varepsilon/2}^{T-\varepsilon/2} \|\rho_t\|^r_{L^s(\mathbb{R})}\,dt < \infty$, so if we introduce $f(t) := \|\rho_t\|^r_{L^s(\mathbb{R})}$, we have $f \in L^1(\varepsilon/2, T-\varepsilon/2)$. This gives

$$\lim_{h\to 0}\int_{\varepsilon}^{T-\varepsilon} |f(t+h)-f(t)|\,dt = 0,$$

which yields

$$\lim_{h\to 0}\int_{\varepsilon}^{T-\varepsilon} \|\rho_{t+h}\|^r_{L^s(\mathbb{R})}\,dt = \int_{\varepsilon}^{T-\varepsilon} \|\rho_t\|^r_{L^s(\mathbb{R})}\,dt < \infty.$$

Together with (3.24) and (3.26) and the arbitrariness of ε, this implies

$$\lim_{h\to 0}\int_{\varepsilon}^{T-\varepsilon} \left\|\frac{g_{t+h}-g_t}{h}\right\|_{L^1(I)} dt = 0 \quad \text{for all} \quad \varepsilon > 0 \quad \text{sufficiently small.}$$

We are trivially led to

$$\int_0^T\int_I \varphi(t,x)\frac{g_{t+h}(x)-g_t(x)}{h}\,dx\,dt \xrightarrow[h\to 0]{} 0 \text{ for any } \varphi\in C^1_c((0,T)\times I),$$

i.e.

$$\int_0^T\int_I \dot\varphi(t,x)g(t,x)\,dx\,dt = 0 \quad \text{for all} \quad \varphi\in C^1_c((0,T)\times I),$$

which implies $g\in W^{1,\infty}(0,T;L^\infty(I))$ with functional derivative $\dot g\equiv 0$. □

There is another case, which is not covered by the statement of Theorem 3.12, but follows easily from it.

Corollary 3.13. *Same conclusions as in Theorem 3.12 hold if $r = s = \infty$.*

Proof: Since for all $1\le p<\infty$ and all $1<q<\infty$ we have $\mathcal{P}_p(\mathbb{R})\subset\mathcal{P}_1(\mathbb{R})$ (with the inequality $W_1\le W_p$ between the metrics) and $L^\infty_{loc}(0,T;L^\infty(\mathbb{R}))\subset L^r_{loc}(0,T;L^\infty(\mathbb{R}))$ (where $r = q^\star(1+1/p^\star)$, as above) we see that Theorem 3.12 applies to the case

$$\rho\in AC^q(0,T;\mathcal{P}_p(\mathbb{R}))\cap L^\infty_{loc}(0,T;L^\infty(\mathbb{R}))$$

as well, simply as a result of the inclusion $AC^q(0,T;\mathcal{P}_p(\mathbb{R}))\subset AC^q(0,T;\mathcal{P}_1(\mathbb{R}))$. □

Remark 3.14. *A comparison among all solutions of (CE) lying in $AC^1(0,T;\mathcal{P}^{ac}_1(\mathbb{R}))$ may be defined in the spirit of (3.20) above, i.e. ρ, $\tilde\rho\in AC^1(0,T;\mathcal{P}^{ac}_1(\mathbb{R}))$ are said to be* comparable *if there exists $\lambda\in\mathbb{R}$ such that for all $t\in[0,T]$ we have either $\rho(t,\cdot)\le\lambda\tilde\rho(t,\cdot)$ or $\tilde\rho(t,\cdot)\le\lambda\rho(t,\cdot)$, $\mathcal{L}^1$–a.e. in $\mathbb{R}$. This is reminiscent of comparison principles used in PDE to establish uniqueness, yet there are major differences. Our λ (in classical studies $\lambda = 1$) accounts for the constraint that ρ is a probability density at all times and should therefore be at least 1 ($\lambda = 1$ implies trivially $\rho = \tilde\rho$). Also, we do not prove a comparison principle for (CE) with $AC^1(0,T;\mathcal{P}^{ac}_1(\mathbb{R}))$ solutions. Instead, the above theorem gives us uniqueness of solutions in the same comparison class to the initial-value problem (for any $\rho_0\in\mathcal{P}_1(\mathbb{R})$) associated with (CE).*

Remark 3.15. *For any $\rho \in AC^1(0,T;\mathcal{P}_1^{ac}(\mathbb{R}))$ we can define an equivalence relation on $AC^1(0,T;\mathcal{P}_1^{ac}(\mathbb{R}))$ by $\tilde{\rho} \sim \rho$ if $\rho(0,\cdot) \equiv \tilde{\rho}(0,\cdot)$ Lebesgue a.e. in $\mathbb{R}$ and there exists $1 \le \lambda < \infty$ such that $\lambda^{-1}\rho \le \tilde{\rho} \le \lambda\rho$, $\mathcal{L}^2$–a.e. in $[0,T]\times\mathbb{R}$. Then Theorem 3.12 and Corollary 3.13 apply to conclude that, under the given conditions, no two distinct curves in the same equivalence class share the same velocity.*

4 Open Problems and future work

Here we would briefly like to remind the reader that it is unclear how much the conditions on ρ can be relaxed in order for the uniqueness results to remain true. It is possible, for example, that it is sufficient for ρ to be absolutely continuous with respect to the Lebesgue measure at all times for the corresponding Lagrangian flow to be unique (or maybe one can require even less: that ρ has no atoms at all times).

The present study is fundamentally one-dimensional spatially and a generalization to higher spatial dimensions will require some extra-regularity on the velocity and its flow. The article [13] shows that $(Flow)$ will generally not be satisfied by the monotone (rather, cyclically monotone in higher dimensions) rearrangements regardless of the choice of velocity (as the velocity associated with an absolutely continuous curve of probabilities is not unique, in general, in multi-d). However, said reference shows that if the curve consists of absolutely continuous measures, there is one velocity whose Lagrangian flow is the family of optimal maps. Let $M_t := \nabla P_t$ denote the optimal map (for quadratic cost, see [4]) pushing the Lebesgue measure restricted to the unit cube in $\mathbb{R}^d$ to the measure ρ_t on the curve. The c.d.f. of ρ_t is replaced by the gradient of the Legendre transform P_t^* of the convex function P_t, i.e. $F_t := \nabla P_t^*$. Formally, the fact that $\nabla P_t^* \circ \nabla P_t = \mathrm{Id}$ in the unit cube leads to, in light of $\dot{M}(t,x) = \mathbf{v}(t, M(t,x))$,

$$\partial_t F(t,y) + \nabla F(t,y)\mathbf{v}(t,y) = 0.$$

A computation of the type (3.5), with F as above and X a flow of $\mathbf{v}$ reveals the same formal result as in the one-dimensional case, i.e. that the Lagrangian flow of $\mathbf{v}$ is unique (the one consisting of the optimal maps). It will certainly be interesting to explore under what conditions and to what extent the above findings can be made rigorous, possibly based on recent regularity results on P^* in space [6] and time [2].

In future work, the authors would like to address some interesting applications of the theory developed here to spatially monotone solutions of Hamilton-Jacobi equations with non-standard Hamiltonians (sublinear, concave etc) in bounded domains. For example, the equation

$$\partial_t u + H(\partial_x u) = 0$$

with $H(p) = -1/p$ is closely related to Burger's equation

$$\partial_t \rho + \partial_x(\rho^2) = 0 \text{ in } (0,\infty)\times\mathbb{R}.$$

If ρ_0 is a probability density, then the latter equation has a unique entropy solution ρ that conserves mass and stays nonnegative at all times (this follows from the closed formula for the viscous approximation, see, e.g., [8]). The generalized inverse of its cumulative distribution function solves the Hamilton-Jacobi equation above in some precise sense. To explain: if ρ_0 is, say, essentially bounded, then so is ρ at all later times and we can therefore use Corollary 3.13 to infer existence and uniqueness of the Lagrangian flow associated to ρ. Since $v = \rho$ in this case and the Monge-Ampère equation associated with $M(t, \cdot)_{\#}\chi = \rho(t, \cdot)$ is $\rho(t, M(t, x))\partial_x M(t, x) = 1$, we obtain that M satisfies the Hamilton-Jacobi equation above in some precise sense (see Corollary 2.12). Then properties of ρ as the entropy solution for Burger's equation transfer to M as the solution of the Hamilton-Jacobi equation via our theory.

Similarly, we know there is a unique bounded solution ρ for the heat equation

$$\partial_t \rho - \partial_{xx}\rho = 0 \text{ in } (0, \infty) \times \mathbb{R}$$

if $\rho(0, \cdot) \equiv \rho_0$ is an essentially bounded probability density on $\mathbb{R}$ (see, e.g., [8]). This solution is also smooth and everywhere positive for all $t > 0$, and one can show that $1/\rho$ admits a classical flow (in space) over $[0, 1]$, i.e. M solves $\partial_x M(t, x) = 1/\rho(t, M(t, x))$ with the properties $M(t, 0+) = -\infty$, $M(t, 1-) = \infty$; $M(t, \cdot)$ is precisely the optimal map pushing χ forward to $\rho(t, \cdot)$. Just as discussed above in the case of Burger's equation, we infer via Corollary 2.12 that M solves the second-order non-standard Hamilton-Jacobi equation

$$\partial_t M + \frac{\partial_{xx} M}{(\partial_x M)^2} = 0.$$

We expect that an elegant theory of monotone viscosity solutions for such Hamilton-Jacobi equations in bounded domains can be built around the connection between such equations (of the general form $u_t = F(t, x, u_x, u_{xx})$) and the corresponding scalar conservation laws or diffusion equations on the real line. In work in progress we are analyzing this connection in much more generality than presented above. We would like to emphasize here that the method of characteristics fails even in these particular cases, and even for "nice", monotone initial M_0.

Acknowledments

Mohamed Amsaad would like to thank the Department of Mathematics of West Virginia University for its support. Adrian Tudorascu acknowledges the partial support provided by the Simons Foundation award #246063. The second author wishes to express his gratitude to Wilfrid Gangbo and Truyen Nguyen for some stimulating discussions that led to the inception of this project. We also thank Alberto Bressan for his suggestion leading to Example 1.3. Last but not least, we thank the anonymous referee for indicating ways to clarify the presentation.

References

[1] L. Ambrosio, *Transport equation and Cauchy problem for BV vector fields*, Invent. Math. **158** (2004), 227–260.

[2] L. Ambrosio, M. Colombo, G. De Philippis, A. Figalli, *Existence of Eulerian solutions to the semigeostrophic equations in physical space: the 2-dimensional periodic case*, Comm. Part. Diff. Eq. **37**, No. 12 (2012), 2209–2227.

[3] L. Ambrosio, N. Gigli and G. Savaré, *Gradient flows in metric spaces and the Wasserstein spaces of probability measures*, Lectures in Mathematics, ETH Zurich, Birkhäuser, 2005.

[4] Y. Brenier, *Polar factorization and monotone rearrangement of vector-valued functions*, Comm. Pure Appl. Math. **44**, No. 4 (1991), 375–417.

[5] G. Crippa, *The flow associated to weakly differentiable vector fields*, PhD Thesis, 2007.

[6] G. De Philippis, A. Figalli. $W^{2,1}$ *regularity for solutions of the Monge-Ampère equation*, Invent. Math. (2012) DOI: 10.1007/s00222-012-0405-4.

[7] R. J. DiPerna, P. L. Lions, *Ordinary differential equations, transport theory and Sobolev spaces*, Invent. Math. **98** (1989), 511–547.

[8] L. C. Evans, *Partial Differential Equations*, AMS Graduate Studies in Mathematics (1998).

[9] M. Fréchet, *Sur la distance de deux lois de probabilite*, Comptes Rendus Acad. Sci. **244** (1957), 689.

[10] W. Gangbo, T. Nguyen and A. Tudorascu, *Euler-Poisson systems as action minimizing paths in the Wasserstein space*, Arch. Rational Mech. Anal. **192**, No. 3 (2009), 419–452.

[11] G. Loeper, *On the regularity of the polar factorization of time dependent maps*, Calc. Var. PDE, **22**, Issue 3 (2004), 343–374.

[12] T. Nguyen, and A. Tudorascu, *Pressureless Euler/Euler-Poisson systems via adhesion dynamics and scalar conservation laws*, SIAM J. Math. Anal. **40**, No. 2 (2008), 754–775.

[13] A. Tudorascu, *On the velocities of flows consisting of cyclically monotone maps*, Indiana U. Math. J. **59**, No. 3 (2010), 929–956.

[14] C. Villani, *Topics in optimal transportation*, Graduate Studies in Mathematics **58**, American Mathematical Society, 2003.