Wasserstein kernels for one-dimensional diffusion problems

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Abstract

We treat the evolution as gradient flow with respect to the Wasserstein distance on a special manifold and construct the weak solution for the initial-value problem by using a time-discretized implicit scheme. The concept of *Wasserstein kernel* associated with one-dimensional diffusion problems with Neumann boundary conditions is introduced. Based on this, features of the initial data are shown to propagate to the weak solution at almost all time levels, whereas, in a case of interest, these features even help obtaining the weak solution. Numerical simulations support our theoretical results.

Keywords: Wasserstein distance; Gradient flow; Wasserstein kernel; One-dimensional diffusion problem; Weak solution; Classical solution
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1 Introduction

1.1 Overview

This work is organized as follows: first we present the general problem and the main tools pertaining to the present approach. We define the Wasserstein kernel, then we use it in two different instances to prove convergence of the time-interpolants (based on the minimizers from the implicit schemes) to the weak solution. Convergence for a nonhomogeneous porous medium equation with exponent $\gamma = 2$ is analyzed first, followed by a much more detailed study of a Stefan problem. The latter includes numerical simulations that confirm the theoretical results.

Let us begin by recalling the setting in [15]. Basically, there we study the N-dimensional generalization of the following nonhomogeneous diffusion problem:

$$u_t - f(u)_{xx} = g(x,t,u) \text{ in } (0,1)\times(0,T) \text{ and } f(u)_x = 0 \text{ on } \{0,1\}\times(0,T), \qquad (P_f)$$

where $f : [0,\infty) \to \mathbb{R},\ g : \mathbb{R}^3 \to \mathbb{R}$ are functions with certain properties (see [15]; the cases studied here are encompassed). Let $0 < T < \infty$. We recall the following:

Definition 1. *A function $u \in L^\infty((0,1)\times(0,T))$ is a weak solution for (P_f) if it satisfies $f(u)_x \in L^1((0,1)\times(0,T))$ and*

$$\int_0^T\int_0^1 \{u\zeta_t - f(u)_x\zeta_x + g(\cdot,\cdot,u)\zeta\}dxdt = -\int_0^1 u_0\zeta(\cdot,0)dx,$$

for all $\zeta \in C_c^\infty([0,1]\times[0,T))$.

We also have a weaker notion, given by:

Definition 2. *A function $u \in L^\infty((0,1)\times(0,T))$ is a generalized solution for (P_f) if it satisfies*

$$\int_0^T\int_0^1 \{u\zeta_t + f(u)\zeta_{xx} + g(\cdot,\cdot,u)\zeta\}dxdt = -\int_0^1 u_0\zeta(\cdot,0)dx,$$

for all $\zeta \in C_c^\infty([0,1]\times[0,T))$ such that $\zeta_x(0,t) = \zeta_x(1,t) = 0$ for all $t > 0$.

Consider now two Lebesgue integrable nonnegative functions u_1 and u_2 of same positive total mass. We recall the definition of the Wasserstein distance of order 2 as found in [7], [14], [12], etc.. For this purpose we introduce

$$P(u_1,u_2) := \left\{ \text{nonnegative Borel measure } \mu \text{ on } [0,1]\times[0,1] \;\middle|\; \int_0^1\int_0^1 \xi(x)d\mu(x,y) = \right.$$

$$\left. = \int_0^1 \xi(x)u_1(x)dx \text{ and } \int_0^1\int_0^1 \xi(y)d\mu(x,y) = \int_0^1 \xi(y)u_2(y)dy \text{ for all } \xi \in C[0,1] \right\}$$

as the set of all admissible transfer plans between the nonnegative finite measures (of the same total mass) u_1dx and u_2dx.

Definition 3. *The (square of the) Wasserstein distance (of order 2) is defined as*

$$d(u_1,u_2)^2 := \inf_{\mu\in P(u_1,u_2)} \int_0^1\int_0^1 |x-y|^2 d\mu(x,y).$$

Properties of the Wasserstein distance will be referenced as they are used throughout this paper.

As Kinderlehrer and Walkington discuss in [10], the functional:

$$S_f(u) := \int_0^1 \Phi_f(u)dx,\ \ u \in \mathcal{M}_{u^*} \text{ (see (1.3) below)},$$

is decreasing along the trajectories of (P_f), where Φ_f satisfies $y\Phi_f'(y) - \Phi_f(y) = f(y)$, but one cannot realize u as the *gradient flow* of the functional S_f in a conventional sense. We demonstrate in [15] that, formally, a solution for (P_f) is a gradient flow of S_f on a certain manifold w.r.t. the Wasserstein distance. This is done by means of the equivalence between the Wasserstein distance and a certain induced distance on $\mathcal{M}_{u^*}$ (proved by Otto in [13]). The approximants for the weak solution are obtained by time-step discretizing the gradient flow. Next, we go briefly over this construction.

1.2 Preliminaries

Let $u^* \in L^1(0,1)$ be nonnegative of positive total mass and let $h > 0$ be fixed. We define the nonlinear functional $F[h,u^*] : \mathcal{M}_{u^*} \to [0,\infty)$ by:

$$F[h,u^*](u) = \frac{1}{2h} d(u,u^*)^2 + S_f(u). \tag{1.1}$$

The gradient flow of S_f on $\mathcal{M}_{u^*}$ w.r.t. the Wasserstein distance admits a time-step discretization of the form (see [13], [12]):

$$\text{Minimize } F[h,u^*](u) \text{ among all } u \in \mathcal{M}_{u^*}, \tag{1.2}$$

where

$$\mathcal{M}_{u^*} := \left\{ u : (0,1) \to [0,\infty) \;\middle|\; u \text{ is measurable and } \int_0^1 u dx = \int_0^1 u^* dx \right\}. \tag{1.3}$$

We will now state the following:

Proposition 1. *The minimization problem (1.2) admits a unique solution.*

For the proof, see [8], [12] and [15]. The proof of the main theorem of the next section uses a construction based on the following corollary (a trivial iterative application of Proposition 1).

Corollary 1. *The following iterative scheme has a unique solution denoted by $\{u_k^h\}_{k\geq 1}$:*

$$\textit{For } k \geq 1,\ u_k^h \textit{ minimizes } F[h, v_{k-1}^h] \textit{ in } \mathcal{M}_{v_{k-1}^h}, \tag{1.4}$$

where

$$v_k^h := u_k^h + \int_{kh}^{(k+1)h} g(\cdot, \tau, u_k^h(\cdot)) d\tau,\ \ k \geq 0. \tag{1.5}$$

The Euler equation of the above variational principle is computed in [10] following an argument due to Otto (see [12]). It is thus proved that:

$$\int_0^1 \int_0^1 (y-x)\xi(y) d\mu(x,y) - h \int_0^1 f(u_k^h)\xi' dx = 0, \tag{1.6}$$

where μ is the (unique) optimal transference plan (see [4]) for v_{k-1}^h and u_k^h and ξ is any smooth and compactly supported function. The proof is based on the so-called *variation of domain* technique involving the *push-forward density* u_ϵ (see the remark below). By letting $\xi = \zeta'$ and taking into account that $(y-x)\zeta'(y) = \zeta(y) - \zeta(x) + (1/2)\zeta''(s)(y-x)^2$ for some s between x and y we obtain:

$$\begin{aligned} \int_0^1 \int_0^1 (y-x)\xi(y) d\mu(x,y) &= \int_0^1 \int_0^1 \left\{ \zeta(y) - \zeta(x) + \frac{1}{2}\zeta''(s)(y-x)^2 \right\} d\mu(x,y) \\ &\leq \int_0^1 (u_k^h - v_{k-1}^h)\zeta dx + \frac{1}{2}\|\zeta''\|_\infty d(v_{k-1}^h, u_k^h)^2. \end{aligned}$$

where we used the marginal property. This, combined with the previous identity, yields (in view of replacing ζ by $-\zeta$ also):

$$\left| \int_0^1 \frac{u_k^h - v_{k-1}^h}{h} \zeta dx - \int_0^1 f(u_k^h)\zeta'' dx \right| \leq \frac{1}{2h}\|\zeta''\|_\infty d(v_{k-1}^h, u_k^h)^2, \tag{1.7}$$

which is the so-called *approximate Euler equation.*

Remark: In the definition of the push-forward density u_ϵ, the authors of [10] use the variation $y = \psi(x, \epsilon)$ defined as the solution of the autonomous

$$\frac{dy}{d\epsilon} = \xi(y), \;\; y|_{\epsilon=0} = x, \;\; \text{for } x \in [0,1].$$

It turns out that $\psi_\epsilon = \psi(\cdot, \epsilon)$ is invertible satisfying $\psi_\epsilon^{-1} = \psi_{-\epsilon}$ and maps $[0,1]$ onto itself if ϵ is small enough and ξ is smooth and compactly supported in $(0,1)$. A short proof is given at the end of this paper (see Appendix). Thus, u_ϵ is defined as

$$u_\epsilon := \frac{u \circ \psi_\epsilon^{-1}}{(\psi_\epsilon)' \circ \psi_\epsilon^{-1}},$$

and it is the push-forward by ϵ of the probability density u. It is easy to see that u_ϵ is also a probability density, being precisely the variation used in (1.4) to give (1.6) (see [10]).

It is known there exists an increasing bijection (the Monge-Kantorovich mass transfer gradient) $\phi : [0,1] \to [0,1]$ such that

$$d(v_{k-1}^h, u_k^h)^2 = \int_0^1 (x - \phi(x))^2 u_k^h(x) dx, \tag{1.8}$$

and, for all $\xi \in C([0,1] \times [0,1])$,

$$\int_0^1 \int_0^1 \xi(x,y) d\mu(x,y) = \int_0^1 u_k^h(y) \xi(\phi(y), y) dy. \tag{1.9}$$

One can also prove that if $0 < \delta \le u_0 \le M < \infty$ a.e. in $(0,1)$, then all u_k^h and v_k^h have the same property (see [12], [11], [15]). From these we deduce (for details, see [11], [15])

$$f(u_k^h)_x - \frac{1}{h}(\phi - \mathrm{id}_{[0,1]}) u_k^h = 0 \text{ a.e. in } [0,1], \tag{1.10}$$

which, by integration against ξ, gives back (1.6) for ξ not necessarily compactly supported (definitions 1 and 2 require broader classes of test functions). Why do we need to approximate u_0 by functions bounded from below away from 0? Because the Wasserstein distance is less degenerate in this case. By this we mean (see [5]) ϕ is given by $\phi = \left(V_{k-1}^h\right)^{-1} \circ U_k^h$, where $V_{k-1}^h(x) = \int_0^x v_{k-1}^h(y) dy$, $U_k^h = \int_0^x u_k^h(y) dy$. Corroborating this and (1.10) we infer that, provided that $0 < \delta \le u_0 \le M < \infty$, $U_k := U_k^h$ satisfies

$$\begin{cases} \left(f(U_k')\right)'(x) = \frac{1}{h} U_k'(x) \left(V_{k-1}^{-1}(U_k(x)) - x\right) \;\; \text{a.e. in } (0,1), \\ \qquad U_k(0) = 0, \\ \qquad U_k(1) = V_{k-1}(1), \end{cases} \tag{1.11}$$

where $V_k := V_k^h$ and $f(U_k') = f(u_k) \in Lip(0,1)$ (see [15]). We now give the following definition:

Definition 4. *The boundary-value problem (1.11) is called the k^{th} Wasserstein kernel associated with the problem (P_f).*

In the remaining sections of the paper we will put this concept to use. Two main cases are discussed; a one-phase Stefan problem and a nonhomogeneous porous medium equation. Following [8] we define, for $h>0$, the interpolation $u^h:[0,1]\times(0,\infty)\to[0,\infty)$ by:

$$u^h(x,t)=u_k^h(x) \text{ for } kh\le t<(k+1)h \text{ and } x\in[0,1],\ k\ge 0 \text{ integer} . \tag{1.12}$$

The following is true:

Proposition 2. *As $h\downarrow 0$, we have (up to a subsequence):*

$$u^h \rightharpoonup u \text{ weakly} \star \text{ in } L^\infty((0,1)\times(0,T)), \tag{1.13}$$

where $u\in L^\infty((0,1)\times(0,T))$ satisfies:

$$\lim_{h\downarrow 0}\int_0^T\int_0^1 \big\{f(u^h)_x\zeta_x-g(\cdot,\cdot,u^h)\zeta\big\}dxdt=\int_0^1 u_0\zeta(\cdot,0)dx+\int_0^T\int_0^1 u\zeta_t dxdt, \tag{1.14}$$

for all $\zeta\in C_c^\infty([0,1]\times[0,T))$. Consequently,

$$\lim_{h\downarrow 0}\int_0^T\int_0^1 \big\{f(u^h)\zeta_{xx}+g(\cdot,\cdot,u^h)\zeta\big\}dxdt=-\int_0^1 u_0\zeta(\cdot,0)dx-\int_0^T\int_0^1 u\zeta_t dxdt, \tag{1.15}$$

for all $\zeta\in C_c^\infty([0,1]\times[0,T))$ that satisfy $\zeta_x(0,\cdot)\equiv\zeta_x(1,\cdot)\equiv 0$.

For the proof we refer the reader to [10], [15]. It is based on integrating (1.7) over $[kh,(k+1)h]$ and then summing up for $k=1,..,\lfloor T/h\rfloor-1$. Note that, as f is, generally, nonlinear, we cannot simply use (1.13) to identify the l.h.s. in (1.15) as the desired $\int_0^T\int_0^1 f(u)\zeta_{xx}dxdt$ (the problem in [8] is linear, so the identification works there). Based on a technique due to Otto ([12]), we prove the identification in [15] by showing the precompactness of $\{u^h\}_h$ or $\{f(u^h)\}_h$ under some additional hypotheses on f. In what follows we employ the Wasserstein kernel ODE's to identify these limits in some special cases.

2 Heat flow and porous media

This section explores the case $f(s)=s^\gamma$ for some $\gamma\ge 1$. Fix $h>0$ for the moment, use the simplified notation $u_k:=u_k^h$ and let $V_{k-1}(x):=\int_0^x v_{k-1}(y)dy$, $U_k(x):=\int_0^x u_k(y)dy$ and $I:=(0,1)$. As we observed in [15], u_k may be assumed to be Hölder continuous (exponent $1/\gamma$) and, since u_k^γ is Lipschitz, it follows that u_k is differentiable a.e. and

$$u_k'=\frac{u_k^{2-\gamma}}{\gamma h}\big(V_{k-1}^{-1}\circ U_k-\mathrm{id}_I\big) \text{ a.e. in } (0,1). \tag{2.1}$$

2.1 Homogeneous problems

If $g\equiv 0$, we get $V_k=U_k$ and we can write (2.1) as

$$u_k'=cu_k^{2-\gamma}\big(U_{k-1}^{-1}\circ U_k-\mathrm{id}_I\big) \text{ a.e. in } (0,1), \tag{2.2}$$

where $c := 1/(\gamma h)$. Therefore, U_k is the solution for

$$\begin{cases} U''(x) = c\big(U'(x)\big)^{2-\gamma}\big(U_{k-1}^{-1}(U(x)) - x\big) & \text{for a.e. } x \text{ in } (0,1), \\ U(0) = 0, \\ U(1) = 1. \end{cases} \tag{2.3}$$

We have already seen in [15] that $0 < \delta \le u_0 \le M < \infty$ implies $0 < \delta \le u_k^h \le M < \infty$ for all $k,\ h$ (valid in N dimensions). As an immediate consequence, the weak solution for

$$u_t - \big(u^\gamma\big)_{xx} = 0 \text{ in } (0,1)\times(0,\infty) \text{ and } \big(u^\gamma\big)_x = 0 \text{ on } \{0,1\}\times(0,\infty) \tag{2.4}$$

with initial $u(\cdot,0) = u_0$ satisfies $\delta \le u \le M$ a.e. in $(0,1)\times(0,\infty)$. We now show that the weak solution generated by monotone initial data preserves the same monotonicity at almost all time levels. Later, we will prove a similar result for the Stefan problem (in that case, we even employ this feature to obtain the generalized solution).

Lemma 1. *If u_{k-1} is nondecreasing (nonincreasing) and bounded away from zero and infinity, then so is u_k.*

Proof: Observe first that both U_{k-1} and U_k are strictly increasing and lie in $C^{1,1/\gamma}(0,1)$. Assume u_{k-1} is nondecreasing. We deduce that U_{k-1} is convex on $(0,1)$. Suppose now that there exists a small subinterval $(a,b) \subset (0,1)$ on which $U_k < U_{k-1}$. Since (2.2) implies

$$u_k' = c u_k^{2-\gamma}\big(U_{k-1}^{-1}(U_k) - U_{k-1}^{-1}(U_{k-1})\big) \text{ a.e. in } (0,1), \tag{2.5}$$

we deduce $u_k' < 0$ a.e. in (a,b). As $\delta \le u_k \le M$ a.e. in $(0,1)$ and $u_k^\gamma \in Lip(0,1)$, it follows $u_k \in Lip(0,1)$. Thus, (2.5) shows u_k is decreasing on (a,b). Consequently, U_k is strictly concave on any subinterval on which it lies below U_{k-1}. Since U_{k-1} is convex, we infer that the graphs of U_{k-1} and U_k cannot share the same endpoints (as required) unless, of course, $U_{k-1} \le U_k$ on $[0,1]$. Also, U_k must be convex on $[0,1]$, otherwise (2.5) would again be contradicted. Indeed, $U_k \ge U_{k-1}$ shows that u_k is nondecreasing (due to (2.5)) on $(0,1)$. The other case follows likewise. $\square$

Next we prove that the convergence stated in Proposition 2 has one important property.

Lemma 2. *Within the hypotheses of the previous theorem we have*

$$u^h(\cdot,t) \rightharpoonup u(\cdot,t) \ \textit{weakly in } L^1(0,1) \ \textit{ for a.e. } t > 0. \tag{2.6}$$

Proof: First, observe that u_{k-1}^h is a competitor in (1.4). It follows

$$\frac{1}{2h} d(u_k^h, u_{k-1}^h)^2 + S_f(u_k^h) \le S_f(u_{k-1}^h),\ k \ge 1.$$

By summing up these inequalities for $k = 1..\infty$ we obtain

$$\sum_{k=1}^{\infty} d(u_k^h, u_{k-1}^h)^2 \le 2h S_f(u_0) = Ch. \tag{2.7}$$

Fix $\xi \in C_c^\infty(0,1)$. For $T < \infty$ there exists $C > 0$ such that for all positive integers $m,\ n$ with $\max\{m,n\}h \le T$, we have, as a consequence of (2.7), the triangle inequality for d and the Schwartz inequality:

$$d(u_m^h, u_n^h)^2 \le C|m-n|h.$$

We now claim that
$$\left|\int_0^1 \xi u dx - \int_0^1 \xi u^* dx\right| \leq \|\xi'\|_\infty d(u,u^*).$$
Indeed, by (1.9) we have
$$\int_0^1 \xi u dx - \int_0^1 \xi u^* dx = \int_0^1 (\xi(x) - \xi(\phi(x)))u^*(x)dx$$
and the claim is thus proved by using a first-order Taylor estimate, then Hölder's inequality for $p=2$ applied to $\big(u^*\big)^{1/2}$ and $\mathrm{id} - \phi$ and then taking (1.8) into account. Therefore,
$$\left|\int_0^1 \xi u^h(t) dx - \int_0^1 \xi u^h(t') dx\right| \leq C\|\xi'\|_\infty (|t-t'|+h)^{1/2},\ (\forall) t,\ t' \in (0,T).$$
Now let $t \in (0,T)$ and note that for any $\delta > 0$, we have
$$\left|\int_0^1 \xi u^h(t)dx - \int_0^1 \xi u(t)dx\right| \leq \left|\int_0^1 \xi u^h(t)dx - \frac{1}{2\delta}\int_{t-\delta}^{t+\delta}\int_0^1 \xi u^h dxds\right|$$
$$+\left|\frac{1}{2\delta}\int_{t-\delta}^{t+\delta}\int_0^1 \xi u^h dxds - \frac{1}{2\delta}\int_{t-\delta}^{t+\delta}\int_0^1 \xi u dxds\right| + \left|\frac{1}{2\delta}\int_{t-\delta}^{t+\delta}\int_0^1 \xi u dxds - \int_0^1 \xi u(t) dx\right|.$$
The first and second terms of the right hand side converge to zero due to the previous inequality and (1.13). As for the third term, it is no longer the case to use the smoothness of u as the authors of [8] do to prove that it converges to zero for every $t > 0$. This does, however, hold for a.e. $t > 0$. Indeed, this is just the Lebesgue theorem applied to $f(t) := \int_0^1 \xi(x)u(x,t)dx$. □

Due to (1.12) and the fact that the set of all essentially nondecreasing (nonincreasing) functions in $\mathcal{M}_{u_0}$ is closed under weak L^1 convergence, we infer

Proposition 3. *If u_0 is nondecreasing (nonincreasing) and bounded away from zero and infinity, then so is the weak solution u for (2.4) at almost every time level.*

2.2 A nonhomogeneous problem for $\gamma = 2$

Let us consider the following problem:
$$\begin{cases} u_t - \frac{1}{2}(u^2)_{xx} = u \text{ a.e. in } (0,1)\times(0,T), \\ (u^2)_x = 0 \text{ on } \{0,1\}\times(0,T), \\ u(\cdot,0) = u_0 \text{ in } (0,1). \end{cases} \tag{2.8}$$
Due to (2.1), at each step k, U_k is the unique solution for the boundary value problem:
$$\begin{cases} hU''(x) = V_{k-1}^{-1}(U(x)) - x \text{ a.e. in } (0,1), \\ U(0) = 0, \\ U(1) = V_{k-1}(1). \end{cases} \tag{2.9}$$
The uniqueness follows by the maximum principle (see [1]) applied to $-U'' + a(x)U = 0$ with $U(0) = U(1) = 0$ where, as usual, $U := U_1 - U_2$, $V := V_{k-1}$ and $a(x) := 1/\big(hv\big(V^{-1}(y_x)\big)\big)$ where
$$\frac{1}{h}\Big(V^{-1}\big(U_1(x)\big) - V^{-1}\big(U_2(x)\big)\Big) = \frac{1}{hv\big(V^{-1}(y_x)\big)}\big(U_1(x) - U_2(x)\big)$$

for some y_x between $U_1(x)$ and $U_2(x)$. Note that $v_{k-1} := V'$ is positive and bounded away from 0 and infinity. For simplicity, let us assume that $0 < \delta \le u_0 \le M < \infty$ satisfies $\int_0^1 u_0 dx = 1$, i.e. $U_0(1) = 1$. Now, (1.5) implies

$$V_k(x) := U_k(x) + h \int_0^x u_k(y)dy = (1+h)U_k(x),$$

for all $x \in I$ and all $0 \le k \le n-2$. In particular, (2.9) implies $U_k(1) = (1+h)^k$. We now introduce the new, rescaled functions:

$$S_k := U_k/(1+h)^k \tag{2.10}$$

which obviously satisfy $S_k(0) = 0$ and $S_k(1) = 1$. One can also check that S_k is, in fact, the unique solution of:

$$\begin{cases} h(1+h)^k S''(x) = S_{k-1}^{-1}(S(x)) - x \quad \text{a.e. in } (0,1), \\ S(0) = 0, \\ S(1) = 1. \end{cases} \tag{2.11}$$

(Of course, $S_0 = U_0$.) Consequently, $s_k := S_k'$ is the (unique up to a set of measure 0) solution of the minimization scheme:

s_k minimizes:

$$\frac{1}{h(1+h)^k} d(s, s_{k-1})^2 + \int_0^1 s^2 dx \tag{2.12}$$

among all $s \in \mathcal{M}$.

(Here $\mathcal{M} := \mathcal{M}_{u_0}$ with $u_0 \equiv 1$.) Our plan is to show that the set of interpolants $\left\{s^h\right\}_{h\downarrow 0}$, constructed in the same way as $\left\{u^h\right\}_{h\downarrow 0}$ but based on s_k^h rather than u_k^h, is precompact in $L^1(0,1)$. It is easy to show how this leads to the precompactness of $\left\{u^h\right\}_{h\downarrow 0}$ in $L^1(0,1)$. Indeed, note that (2.10) implies $s_k^h := (1+h)^{-k} u_k^h$ so the definition of u^h gives $u^h(t) = (1+h)^{[t/h]} s^h(t)$ for all $t \in (0,T)$. But then $(1+h)^{[t/h]} \to e^t$ as $h \to 0$ and thus our claim is proved. Therefore, if we can find s such that $s^h \to s$ both in $L^1(Q)$ and a.e. in Q, then $u^h \to e^t s := u$ in L^1 and a.e. which renders u the desired weak solution.
The existence of s_k as the unique minimizer for (2.12) is again insured by Proposition 1.

Proposition 4. *$\left\{s^h\right\}_{h\downarrow 0}$ is precompact in $L^1((0,1)\times(0,T))$.*

Proof: The proof follows closely the proof of the precompactness of $\left\{s^h\right\}_{h\downarrow 0}$ in [12]. We will only justify the main ingredients; the way they are glued together is precisely as referred. The precompactness in space is once again not an issue since all it takes is the fact that $I = (0,1)$ is convex and the uniform boundedness of $\left\{\left(s^h\right)_x^2\right\}_{h\downarrow 0}$ in L^2. To justify this, we need to go back to the proof in [12] and observe that the following remains true

$$\|(s_k^2)_x\|_{L^2(I)} \le \frac{C}{h}(\|s_k\|_\infty)^{1/2} d(s_{k-1}, s_k). \tag{2.13}$$

Note that the constant $C > 0$ is independent of $k,\ h$ and it comes from the fact that $1 \le (1+h)^k \le e^T$. The uniform boundedness of the s_k's is also obvious.
Precompactness in time will be a consequence of an inequality of type (2.14). As all the s_k's have now the same mass (unit mass), we can apply Otto's argument from [12] to obtain

$$\int_0^1 (s_{k+j}^h - s_k^h)\left((s_{k+j}^h)^2 - (s_k^h)^2\right)dx \le \frac{C}{h} d(s_k^h, s_{k+j}^h)\left[d(s_k^h, s_{k-1}^h) + d(s_{k+j}^h, s_{k+j-1}^h)\right], \tag{2.14}$$

for some $C > 0$ independent of $k,\ j,\ h$. Finally, we need to verify that there exists some $C > 0$ such that

$$\sum_{k=1}^{n-1} d(s^h_{k-1}, s^h_k)^2 \leq Ch.$$

Since s^h_k minimizes the functional in (2.12) we have

$$\frac{1}{h(1+h)^k} d(s^h_k, s^h_{k-1})^2 + \int_0^1 (s^h_k)^2 dx \leq \int_0^1 (s^h_{k-1})^2 dx,$$

for all $k \geq 1$. Summing from 1 to $n-1$ and taking into account that $1 \leq (1+h)^k \leq e^T$ we obtain the desired estimate. □

Remark: The proposition remains valid even if $f(u) = u^2/2$ is replaced by any function f satisfying

$$cx \leq f'(x) \leq Cx, \text{ for all } x \geq 0$$

for some $c,\ C > 0$. $g(x,t,u) = u$ can also be replaced by $g(x,t,u) = \alpha u$ for some $\alpha \geq 0$.

Next we will try to understand to what extent the ideas above can be applied in the case of a general r.h.s.. It is easy to see that, for a general g in one-dimension, S_k solves:

$$\begin{cases} hV_{k-1}(1)S''(x) = T^{-1}_{k-1}(S(x)) - x \ \text{ a.e. in } (0,1), \\ S(0) = 0, \\ S(1) = 1, \end{cases} \tag{2.15}$$

where

$$T_{k-1}(x) := \frac{V_{k-2}(1)}{V_{k-1}(1)} S_{k-1}(x) + \frac{1}{V_{k-1}(1)} \int_0^x \int_{(k-1)h}^{kh} g(y, \tau, V_{k-2}(1) S'_{k-1}(y)) d\tau dy.$$

Therefore,

$$s^h_k := S'_k \text{ minimizes } \frac{1}{hV_{k-1}(1)} d(s, t^h_{k-1})^2 + \int_0^1 s^2 dx,$$

where $t^h_{k-1} := T'_{k-1}$. Again $s^h(x,t) := u^h(x,t)/V^h(t)$ where $V^h(t) := \int_0^1 u^h(x,t)dx = V^h_{k-1}(1)$ for $k = [t/h]$. As before, we would like to know whether the convergence of $\{s^h\}$ in L^1 implies the convergence of $\{u^h\}$. It is easy to see that the convergence of $\{V^h\}$ a.e. in $(0,T)$ as well as its boundedness from above and from below away from zero would suffice. As $V^h_{k-1}(1) = \int_0^1 u^h_k dx$, the desired boundedness follows trivially. For convergence (up to a subsequence) we have the following lemma (valid for Ω in arbitrary dimension):

Lemma 3. *$\left\{ \int_\Omega u^h(x,t)dx \right\}_{h\downarrow 0}$ is precompact in $L^1(0,T)$.*

Proof: Choose an arbitrary $\xi \in C^\infty_c(0,T)$ and mote that

$$\int_0^T \left(\int_\Omega u^h(x,t)dx \right) \xi_t dt = \xi(h) \sum_{k=1}^{n-1} \xi(kh) \int_\Omega (u^h_{k-1} - u^h_k) dx.$$

As

$$\int_\Omega (u^h_{k-1} - u^h_k) dx = \int_\Omega (u^h_{k-1} - v^h_{k-1}) dx = O(h),$$

we obtain

$$\left| \int_0^T \left(\int_\Omega u^h(x,t)dx \right) \xi_t dt \right| \leq CT \|\xi\|_{L^\infty(0,T)},$$

i.e. $\left\{ \int_\Omega u^h(x,t)dx \right\}_{h\downarrow 0}$ is bounded in $BV(0,T)$. □

Remark: It is obvious, due to (1.13), that (up to a subsequence)

$$\int_\Omega u^h(x,t)dx) \to \int_\Omega u(x,t)dx \text{ in } L^1(0,T) \text{ and a.e. in } (0,T).$$

As all the s_k^h's and the t_k^h's are of unit mass, we can readily write a variant of (2.14):

$$\int_0^1 (s_{k+j}^h - s_k^h)\big((s_{k+j}^h)^2 - (s_k^h)^2\big)dx \le \frac{C}{h} d(s_k^h, s_{k+j}^h)\big[d(s_k^h, t_{k-1}^h) + d(s_{k+j}^h, t_{k+j-1}^h)\big], \tag{2.16}$$

for some $C > 0$ independent of $k,\ j,\ h$. We can also show, as before,

$$\sum_{k=1}^{n-1} d(s_k^h, t_{k-1}^h)^2 \le Ch.$$

In order to complete the proof of the precompactness of $\{s^h\}$ we need a bound of $d(s_k^h, s_{k+j}^h)$ in terms of quantities of the type $d(s_i^h, t_{i-1}^h)$. Unfortunately, we have not been able to obtain such a bound.

3 A Stefan problem

3.1 The classical solution

Let θ_0 be a nonnegative, measurable function on $[0,\infty)$ supported in $[0,1)$ and normalized to satisfy

$$|\{\theta_0 > 0\}| + \int_0^1 \theta_0 dx = 1. \tag{3.1}$$

By a classical solution for the *one-phase*, one-dimensional Stefan problem on $[0,\infty)$ with initial data $\theta(\cdot,0) \equiv \theta_0$ and natural boundary conditions $\theta_x(0,\cdot) \equiv 0$, we understand a pair $(\theta(x,t), r(t))$ with $\theta(\cdot,t) \in C^1[0,r(t)) \cap C^2(0,r(t))$ for $t > 0$, $\theta(x,\cdot) \in C^1(r^{-1}(x),\infty)$ and $x = r(t)$ a differentiable function such that

$$\begin{cases} \theta_t - \theta_{xx} = 0 \text{ for } t > 0,\ x \in (0, r(t)), \\ \theta(r(t),t) = 0 \text{ for } t > 0, \\ D_x^- \theta(r(t),t) = -r'(t) \text{ for } t > 0, \\ \theta_x(0,t) = 0 \text{ for } t > 0, \\ \theta(x,0) = \theta_0(x) \text{ for } x \in [0,\infty), \\ \theta(x,t) = 0 \text{ for } t > 0 \text{ and } x > r(t). \end{cases} \tag{3.2}$$

A dual formulation (see [6], [9]) involves the inverse $t = s(x)$, the equation thus becoming

$$\begin{cases} \theta_t - \theta_{xx} = 0 \text{ for } x \in [0,a),\ t > s(x), \\ \theta(x, s(x)) = 0 \text{ for } x \in [0,a), \\ D_x^- \theta(x,s(x)) s'(x) = -1 \text{ for } x \in [s^{-1}(0), a), \\ \theta_x(0,t) = 0 \text{ for } t > 0, \\ \theta(x,0) = \theta_0(x) \text{ for } x \in [0,\infty), \\ \theta(x,t) = 0 \text{ for } x \in [s^{-1}(0), a) \text{ and } s(x) > t, \end{cases} \tag{3.3}$$

where $0 < a \leq \infty$ such that $s(x) \uparrow \infty$ as $x \uparrow a$. Therefore, for the existence of a classical solution we require that r be invertible with $r^{-1} = s$ and $r((0,\infty)) = (r(0), a)$. The region occupied by the liquid at time $t > 0$ is therefore $(0, r(t))$ while the ice occupies the region $(r(t), 1)$. The function θ represents the temperature and $x = r(t)$ stands for the transition layer between the two states (liquid/solid).
Assuming that (3.2) has a classical solution (which must be positive in the liquid region), it is not hard to see that this solution satisfies, for every $t > 0$, the conservation

$$r(t) + \int_0^{r(t)} \theta(x,t)dx = r(t) + \int_0^1 \theta(x,t)dx = 1. \tag{3.4}$$

This is obtained by simply differentiating the first term w.r.t. t and using $\theta_t = \theta_{xx}$ on $(0, r(t))$ and the side x-derivative condition at $(r(t), t)$. The curve $x = r(t)$ can be proved to be monotone increasing. Indeed, as $\theta > 0$ in the liquid region (to the left of $(r(t), t)$) and $\theta(r(t), t) = 0$, we infer that $\theta_x(r(t), t) \leq 0$, i.e. $r'(t) \geq 0$. We have seen that r is also invertible as a function from $(0, \infty)$ onto $(r(0), a)$ and (3.3) ensures $r' \neq 0$ everywhere. Therefore, $r'(t) > 0$ for every $t \in (0, \infty)$.

In what follows, we make the extra-assumption

$$\theta_0 \text{ is decreasing on } [0,1]. \tag{3.5}$$

We specify that throughout this section f is said to be *decreasing* if $(x-y)(f(x)-f(y)) \leq 0$ for all $x,\ y \in \mathrm{Dom}(f)$. We say that f is *strictly decreasing* if it is decreasing and one-to-one. (*Decreasing* is replaced by *increasing* if $\geq$ replaces $\leq$.)
Note that (3.5) is in accordance with our intuition of the monotonicity of the temperature at the moment the heat source is switched off and right before the system starts evolving under no boundary heat exchange regime. Also, let us further assume (for the moment) that $x = r(t)$ is continuously differentiable and:

$$\theta_0 \in C[0,1] \cap C^1[0, r(0)) \text{ and } D_x^- \theta_0(r(0)) = -r'(0) < 0. \tag{3.6}$$

Proposition 5. *The function $\theta(\cdot, t)$ is monotone decreasing on $[0,1]$ for every $t \geq 0$.*

Proof: Suppose $\theta(\cdot, t_0)$ is decreasing for some $t_0 \geq 0$ (we know this happens for $t_0 = 0$). As $r'(\tau) \geq \varepsilon > 0$ in some small one-sided neighborhood V of $x_0 = r(t_0)$ on the curve $x = r(t)$ (say, $x = r(t)$ for $t_0 < t < t_1$) we find $\theta_x < -\varepsilon < 0$ in V. Due to the smoothness of θ inside $\cup_{t>0}\{t\} \times (0, r(t))$ we find that there exists $c > 0$ such that $\theta_x < -\varepsilon/2$ in the region W bounded by V, its translated by c to the left (denoted by V_c) and the lines $t = t_0$ and $t = t_2$ for some $t_2 \in (t_0, t_1)$. The vertical through x_0 intersects ∂W again at the point (x_0, t') for some $t' > t_0$. The maximum principle applied to θ_x (which also satisfies the heat equation) in the rectangle $R_c = (0, x_0) \times (t_0, t')$ ensures that $\theta_x \leq 0$ in R_c. To the right of R_c, θ_x is again nonpositive (by the construction of W). Therefore, we have found a $t' > t_0$ such that $\theta(\cdot, t)$ is decreasing for all $t \in [t_0, t')$. This leads to the fact that the set $\mathbf{T}$ of all $t > 0$ for which $\theta(\cdot, t)$ is decreasing is open in $[t_0, \infty)$. Due to the smoothness of θ inside the liquid region we also infer that $\mathbf{T}$ is closed in $[t_0, \infty)$. Since we may pick $t_0 = 0$, the proof is concluded. $\square$

3.2 $K^{\downarrow}$: a manifold of monotone decreasing functions

Recall that the *enthalpy* U of θ is defined as the multivalued application $U := \theta + H(\theta)$ (of course, we assume latent heat $L = 1$) where H is the Heaviside graph. A special element is given by $u := 1 + \theta$

if $\theta > 0$ and $u := 0$ if $\theta = 0$. By abuse of language, we can refer to any $u \in U$ as *an enthalpy* (see [15]). Thus, (3.4) becomes $\int_0^1 u(x,t)dx = 1$ for all $t > 0$. Also, as (3.4) is satisfied, we deduce $0 < r(t) < 1$ for all $t > 0$. Therefore, we envision to fashion a nonnegative generalized solution (see Definition 2) in the interval $(0,1)$ that satisfies $\int_0^1 u(x,t)dx = 1$ for a.e. $t > 0$. Indeed, since $\theta \equiv 0$ to the right of $x = r(t)$, we can simply extend it by 0 to the right of $x = 1$ obtaining the solution for our problem. We will also see that the artificial assumption (3.6) (which only was adopted to prove the previous proposition) may be dropped. Thus, under (3.4) and (3.5), we will construct a generalized solution for (3.2) that turns out to be "almost classical" (and weak). This means that the jump of the spatial derivative of θ across the interface $x = r(t)$ will only satisfy $D_x^- \theta(r(t),t) = -r'(t)$ for a.e. $t > 0$ and θ will be continuous except possibly on a set of zero one-dimensional Haussdorf measure. The interface itself is only continuous and strictly increasing; therefore, only a.e. differentiable.
Let us consider the problem:

$$\begin{cases} u_t - \theta_{xx} = 0 & \text{in } (0,1) \times (0,\infty), \\ u \in \theta + H(\theta) & \text{in } (0,1) \times (0,\infty), \\ \theta_x = 0 & \text{on } \{0,1\} \times (0,\infty), \\ \theta(\cdot,0) = \theta_0 & \text{in } (0,1). \end{cases} \tag{P}$$

We will define the function $\alpha : [0,\infty) \to [0,\infty)$ by

$$\alpha(z) := 0 \text{ if } z \in [0,1] \text{ and } \alpha(z) := z - 1 \text{ if } z \in (1,\infty). \tag{3.7}$$

Note that the temperature is retrieved by $\theta := \alpha(u)$, i.e. $u \in \theta + H(\theta)$ is equivalent to $\theta = \alpha(u)$.
Assume (θ, r) is a classical solution for (3.2). Then it is easy to prove that a special $u \in \theta + H(\theta)$ (given by $u := 1 + \theta$ if $\theta > 0$ and $u := 0$ if $\theta = 0$) satisfies $\int_0^1 u(x,t)dx = 1$ (due to (3.4)) for all t and is, in fact, a weak solution for (P) (with $u_0 := 1 + \theta_0$ if $\theta_0 > 0$ and $u_0 := 0$ if $\theta_0 = 0$) in the sense of Definition 1 (and, consequently, a generalized solution in the sense of Definition 2).

An adaptation of an important uniqueness result due to Brezis and Crandall (see [3], [15]) successfully applies here to show that, if θ_0 (and so u_0) is essentially bounded, then the temperatures $\theta^\epsilon := \alpha(u^\epsilon)$ converge strongly in $L^2_{loc}([0,1] \times (0,\infty))$ to $\theta := \alpha(u)$, where u^ϵ, u are the generalized solutions for (P) for initial data u_0^ϵ and u_0 respectively such that $u_0^\epsilon \to u_0$ strongly in $L^2(0,1)$ (see [15] for proof). Therefore, from now on we can prove our results for $u_0^\epsilon \in \theta_0 + H(\theta_0)$ given by $u_0^\epsilon := 1 + \theta_0$ if $\theta_0 > 0$ and $u_0^\epsilon := \epsilon$ if $\theta_0 = 0$ (for some $0 < \epsilon < 1$). Indeed, it will become clear that the existence and certain properties of the solution u^ϵ (due to the above mentioned convergence) apply to u, i.e. the solution for (P) with initial data $u_0 \in \theta_0 + H(\theta_0)$ given by $u_0 := 1 + \theta_0$ if $\theta_0 > 0$ and $u_0 := 0$ if $\theta_0 = 0$.
In this case, (1.11) reads

$$\begin{cases} \theta_k'(x) = \frac{1}{h} u_k(x) \big(U_{k-1}^{-1}(U_k(x)) - x \big) & \text{a.e. in } (0,1), \\ U_k(0) = 0, \\ U_k(1) = 1, \end{cases} \tag{3.8}$$

where U_k is the antiderivative of u_k that vanishes at 0 and $\theta_k := \alpha(u_k) \in Lip(0,1)$ (see [15]).
We now define

$$K^\downarrow := \{u \in K \mid u \text{ is essentially bounded and decreasing on } (0,1)\}, \tag{3.9}$$

and based on (3.8) we will prove the following

Lemma 4. *If $u_{k-1} \in K^\downarrow$ and is bounded away from zero and infinity, then so is u_k and $u_k \in K^\downarrow$.*

Proof: Observe first that both U_{k-1} and U_k are strictly increasing and absolutely continuous. As $u_{k-1} \in K^{\downarrow}$ we deduce that U_{k-1} is concave on $(0,1)$. Suppose now that there exists a small subinterval $(a,b) \subset (0,1)$ on which $U_k > U_{k-1}$. From (3.8) we write

$$h\theta_k' = u_k\big(U_{k-1}^{-1}(U_k) - U_{k-1}^{-1}(U_{k-1})\big) \text{ a.e. in } (0,1), \tag{3.10}$$

which implies θ_k is strictly increasing on (a,b) (as it is Lipschitz continuous). It follows (due to its nonnegativity) that $\theta_k > 0$ in (a,b), i.e. $u_k > 1$ and $\theta_k = u_k - 1$ in (a,b). We infer $u_k|_{(a,b)} \in Lip(a,b)$ and (3.10) now reads

$$hu_k' = u_k\big(U_{k-1}^{-1}(U_k) - U_{k-1}^{-1}(U_{k-1})\big) > 0 \text{ a.e. in } (a,b).$$

Consequently, U_k is strictly convex on any subinterval on which it exceeds U_{k-1}. Since U_{k-1} is concave, we infer that the graphs of U_{k-1} and U_k cannot share the same endpoints (as required) unless, of course, $U_{k-1} \geq U_k$ on $[0,1]$. Similarly it can be proved that U_k must be concave on $[0,1]$, otherwise (3.10) would again be contradicted. Indeed, $U_k \leq U_{k-1}$ shows that θ_k is decreasing (due to (3.8)) on $(0,1)$. Therefore, there exists $0 < r_k < 1$ (the inequalities are strict due to (3.4)) such that $\{\theta_k > 0\} = [0, r_k)$. This means $u_k = \theta_k + 1$ and is also decreasing on $[0, r_k)$, i.e. U_k is concave on $(0, r_k)$. Also, $\theta_k = 0$ a.e. in $[r_k, 1)$ and, again due to (3.8) (also because $0 < \delta \leq u_k$), it follows that $U_k = U_{k-1}$ a.e. in $[r_k, 1)$. Consequently, U_k is concave on $[0,1]$, i.e. $u_k \in K^{\downarrow}$. $\square$

Obviously, the following now holds:

Corollary 2. *If $u_0 \in K^{\downarrow}$ and is bounded from above and from below away from 0, then so is u_k and $u_k^h \in K^{\downarrow}$ for all $k,\ h$.*

Due to the considerations above and Lemma 2, $u(\cdot,t) \in K^{\downarrow}$ for almost all $t > 0$. Observe that, even with α being convex on $[0,\infty)$ and $u^h(\cdot,t) \rightharpoonup u(\cdot,t)$ in $L^1(0,1)$ for a.e. $t > 0$, it does not necessarilly follow that $\alpha(u^h(\cdot,t)) \rightharpoonup \alpha(u(\cdot,t))$ in $L^1(0,1)$ to give us a weak solution. We can actually go even further and state that, even if the weak L^1 limit of $\alpha(u^h)$ exists as in (1.15), we still cannot infer convergence to $\alpha(u)$ in any reasonable sense. In [15] we approximate α by smooth functions and then prove precompactness of $\{u^h\}_{h\downarrow 0}$ to infer existence for the approximate problem. Then we pass to the limit by using a consequence of a technique inspired by [3].
As stated in the beginning of this paper, here we plan to use monotonicity to compensate and obtain the desired weak convergence. More precisely, our next goal is to prove that if we start with a decreasing u_0, we can, by the same means, look for and actually find a weak solution in $K^{\downarrow}$ (defined in (3.9)).
Let us first state one fundamental lemma.

Lemma 5. *If $\{v_n\}_n \subset K^{\downarrow}$ such that $v_n \rightharpoonup v$ in $L^1(0,1)$ then $v \in K^{\downarrow}$ and $\alpha(v_n) \rightharpoonup \alpha(v)$ in $L^1(0,1)$.*

Proof: Obviously, $K^{\downarrow}$ is closed under L^1 weak convergence. We will concentrate our efforts on demonstrating that

$$v_n \rightharpoonup v \text{ implies } \alpha(v_n) \rightharpoonup \alpha(v) \text{ weakly in } L^1(0,1), \tag{3.11}$$

where v_n for all n and v lie in $K^{\downarrow}$. Fix $\xi \in L^\infty(0,1)$ and denote $a_n := \int_0^1 \alpha(v_n)\xi dx$. We will deduce that $\{a_n\}_n$ converges to $l := \int_0^1 \alpha(v)\xi dx$ by showing that any subsequence of $\{a_n\}_n$ contains a subsequence convergent to l. For every n denote $x_n \in [0,1]$ the level-1 "threshold" $v_n(x_n - 0) \geq 1 \geq v_n(x_n + 0)$ and x^* the corresponding for v. That is, x^* is the smallest satisfying $v(x-0) \geq 1 \geq v(x+0)$. Take now any subsequence of $\{v_n\}_n$ and out of it extract a subsequence (we do not relabel) $\{v_n\}_n$ such that $\{x_n\}_n$ is monotone. We encounter three possibilities:

- $x_n \downarrow 0$!

Here it is easy to show that $x^* = 0$ and it only happens when $v \equiv 1$ a.e.. We want to prove now that

$$\int_{\{v_n > 1\}} (v_n - 1)\xi = \int_0^{x_n} (v_n - 1)\xi \to 0.$$

As

$$\int_0^{x_n} \xi dx \to 0 \text{ and } \left| \int_0^{x_n} v_n \xi dx \right| \leq \|\xi\|_\infty \int_0^{x_n} v_n dx,$$

it suffices to show $\int_0^{x_n} v_n \to 0$. Let m be a positive integer greater than 2. As $v_n \rightharpoonup v \equiv 1$ we deduce that, for sufficiently large n, we have

$$\frac{1}{m-1} > \int_0^{1/m} v_n \text{ since } \int_0^{1/m} v_n \to \frac{1}{m}.$$

As $x_n \downarrow 0$ and so x_n will eventually be smaller than $1/m$, for even larger n we will have

$$\frac{1}{m-1} > \int_0^{x_n} v_n \geq 0,$$

so the claim is proved.

- $x_n \uparrow 1$!

Again, we obtain that $v \equiv 1$ and the rest of the proof follows as before.

- $x_n \uparrow x^* \in (0,1)$ or $x_n \downarrow x^* \in (0,1)$!

W.l.o.g. consider the second situation. As $\int_0^{x_n} \xi \to \int_0^{x^*} \xi$ trivially, we only need to prove that $\int_0^{x_n} v_n \xi \to \int_0^{x^*} v\xi$. We have

$$\int_0^{x_n} v_n \xi - \int_0^{x^*} v\xi = \int_0^{x^*} (v_n - v)\xi + \int_{x^*}^{x_n} v_n \xi.$$

The first integral tends obviously to zero. The second integral does the same if we take into account that the v_n's are uniformly essentially bounded in every small left-sided neighbourhood of x^*. Indeed, fix $\delta > 0$ such that $0 < x^* - \delta$. If that were not the case then, for every $M > 0$, there would exist N such that $v_N \geq M$ on $(0, x^* - \delta)$ (due to the fact that v_N is decreasing). For M large enough, the integral of v_N would then exceed 1, contradicting thus $v_N \in K$. $\square$

Now we are ready for:

Theorem 1. *If $u_0 \in K^\downarrow$ then there exists a generalized solution $u \in L^\infty((0,1) \times (0,\infty))$ satisfying $u(\cdot,t) \in K^\downarrow$ for almost all $t > 0$.*

Proof: We can restrict all work to $K^\downarrow$ and we find the minimizers u_k^h (defined in (1.4)) and, consequently, all $u^h(\cdot,t)$'s (see (1.12)) lying in $K^\downarrow$ (Corollary 2). It follows $u(\cdot,t) \in K^\downarrow$ for a.e. $t > 0$, where u is the one from Proposition 2 (as $K^{\downarrow}$ is closed under weak L^1 convergence). According to the same proposition, all that remains to be proved is

$$\lim_{h\downarrow 0}\int_0^\infty\int_0^1 \alpha(u^h(x,t))\xi_{xx}(x,t)dxdt = \int_0^\infty\int_0^1 \alpha(u(x,t))\xi_{xx}(x,t)dxdt, \tag{3.12}$$

for all the appropriate test functions ξ. For this, it suffices to choose a $t > 0$ for which (2.6) holds and then use Lemma 5 to deduce $\alpha(u^h(\cdot,t)) \rightharpoonup \alpha(u(\cdot,t))$ in $L^1(0,1)$. Then consider the x-integrals over $(0,1)$ against $\xi_{xx}(\cdot,t)$ (which converge), integrate w.r.t. t and take into account that

$$f^h(t) := \int_0^1 \alpha(u^h(x,t))\xi_{xx}(x,t)dx = \int_{\{u^h(\cdot,t)>1\}} \{u^h(x,t)-1\}\xi_{xx}(x,t)dx$$

is, in fact, uniformly bounded since $u^h(\cdot,t) \in K$. Therefore, as the time integral is actually with finite limits due to ξ, we use dominated convergence to infer (3.12). $\square$

Remark: It is interesting to investigate whether we can, in fact, drop the monotonicity of θ_0, i.e. relax (3.5) to read

$$0 < \theta_0 \le M < \infty \text{ on } [0, r(0)) \text{ and } \theta_0 = 0 \text{ on } [r(0), 1]. \tag{3.13}$$

We can easily make slight changes to the argument from the proof of Lemma 5 and apply it to the set (instead of $K^{\downarrow}$)

$$K^M_{1,\epsilon} = \{u \in K \bigm| M \ge u \ge 1 \text{ a.e. on } [0,\gamma),\ u = \epsilon \le 1 \text{ a.e. on } [\gamma, 1] \text{ for some } \gamma \in [0,1]\},$$

where $1 \le M < \infty$. This is in view of considering $u_0 := 1 + \theta_0$ if $\theta_0 > 0$ and $u_0 := \epsilon < 1$ if $\theta_0 = 0$. It is not, however, that obvious how to prove an alternative to Lemma 4 with $K^M_{1,\epsilon}$ instead of $K^{\downarrow}$. This is exactly what we do below.

Lemma 6. *If $u_{k-1} \in K^M_{1,\epsilon}$, then $u_k \in K^M_{1,\epsilon}$ for all integers $k \ge 1$.*

Proof: First we note that only a negligible set of zeros of θ_k lies in $(0, r_{k-1}) := \{u_{k-1} > 1\}$. Otherwise, denote by $O_k := \{\theta_k = 0\} \cap (0, r_{k-1})$ and assume $meas(O_k) > 0$. As $\theta_k \in W^{1,\infty}(0,1)$ it follows that both θ_k and θ_k' vanish a.e. in O_k. This implies two things: first, $u_k \le 1$ a.e. in O_k and secondly, by (3.8) we obtain $U_{k-1} = U_k$ a.e. in O_k. The latter implies $u_{k-1} = u_k$ a.e. in O_k and, as $u_{k-1} > 1$ a.e. in $(0, r_{k-1})$, it contradicts the former. Therefore, $\theta_k > 0$ and so $u_k > 1$ a.e. in $\{u_{k-1} > 1\}$. Since $u_{k-1} \in K^M_{1,\epsilon}$, we deduce the graph of $U_{k-1}\big|_{[r_{k-1},1]}$ is a straight line of slope ϵ connecting $(r_{k-1}, U_{k-1}(r_{k-1}))$ and $(1,1)$. Observe now that $U_k(r_{k-1}) \le U_{k-1}(r_{k-1})$. If not, we proceed as in the proof of Lemma 4 and deduce that the graph of U_k would miss the endpoint $(1,1)$. Now, if $U_k(x) = U_{k-1}(x)$ for some $x \in [r_{k-1}, 1]$, then $U_{k-1} = U_k$ on $[x,1]$, i.e. $u_k = \epsilon$ a.e. in $[x,1]$. Indeed, if the two graphs part again to the right of x, then U_k becomes strictly convex or strictly concave (see the proof of Lemma 4) and misses the prescribed endpoint at (1,1). Let $r_k := \min\{x \in (0,1) \mid u_k = \epsilon \text{ a.e. in } [x,1]\}$. All we need to show is $u_k > 1$ a.e. in $(0, r_k)$. We have seen that this is true in $(0, r_{k-1})$. As noted, on (r_{k-1}, r_k) the graph of U_k is below that of U_{k-1}. Therefore, $\theta_k' < 0$ a.e. in (r_{k-1}, r_k) which means $\theta_k \ne 0$ (i.e. $\theta_k > 0$) a.e. in (r_{k-1}, r_k). This concludes our proof. $\square$

We can now state the more general result:

Theorem 2. *If $u_0 \in K^M_{1,\epsilon}$ then there exists a generalized solution $u \in L^\infty((0,1)\times(0,\infty))$ satisfying $u(\cdot,t) \in K^M_{1,\epsilon}$ for almost all $t > 0$.*

Therefore, it suffices for the initial data θ_0 to be bounded from above, strictly positive a.e. in $(0, x_0)$ and identically zero in (x_0, ∞) to obtain existence of the weak solution for (P).

3.3 The free boundary and the regularity of θ

Let us now assume that $0 \leq \theta_0 \leq M < \infty$ and (3.5) holds. Also assume that (3.1) stands, i.e. $|\{\theta_0 > 0\}| + \int_0^1 \theta_0 dx = 1$. As indicated in the previous section, let us fix $0 < \epsilon < 1$ and let $u_0^\epsilon \in \theta_0 + H(\theta_0)$ be given by $u_0^\epsilon := 1 + \theta_0$ if $\theta_0 > 0$ and $u_0^\epsilon := \epsilon$ if $\theta_0 = 0$. Then let

$$\eta^\epsilon := \int_0^1 u_0^\epsilon dx = 1 + \epsilon|\{\theta_0 = 0\}|.$$

Recall that $u_k^{\epsilon,h}$ satisfies

$$\begin{cases} (\theta_k^{\epsilon,h})'(x) = \frac{1}{h} u_k^{\epsilon,h}(x)\big((U_{k-1}^{\epsilon,h})^{-1}(U_k^{\epsilon,h}(x)) - x\big) & \text{a.e. in } (0,1), \\ U_k^{\epsilon,h}(0) = 0, \\ U_k^{\epsilon,h}(1) = \eta^\epsilon, \end{cases} \tag{3.14}$$

where $U_k^{\epsilon,h}$ is the antiderivative of $u_k^{\epsilon,h}$ that vanishes at 0 and $\theta_k^{\epsilon,h} := \alpha(u_k^{\epsilon,h}) \in Lip(0,1)$ (see [15]). Note that U_0^ϵ is strictly increasing and concave (of supraunitary slope) up to $r_0 := \min\{r \in [0,1],\ \theta_0(r) = 0\}$ ($0 < r_0 < 1$ due to (3.1) and (3.5)) and $U_0^\epsilon(x) = \epsilon(x-1) + \eta^\epsilon$ for $x \in (r_0, 1]$. (In this context, supraunitary slope means, obviously, $u_0^\epsilon > 1$). We will prove by induction that all $U_k^{\epsilon,h}$'s have a similar structure. As proved in [15], all $\theta_k^{\epsilon,h}$'s are Lipschitz continuous functions on $[0,1]$. They are also decreasing due to Corollary 2. We show that the sequence $\{r_k^{\epsilon,h}\}_k$ is increasing, where $r_k^{\epsilon,h} := \min\{r \in [0,1],\ \theta_k^{\epsilon,h}(r) = 0\}$. As anticipated, assume that $U_{k-1}^{\epsilon,h}$ looks like U_0^ϵ, i.e. it is strictly increasing and concave (of supraunitary slope) up to $r_{k-1}^{\epsilon,h}$ and $U_{k-1}^{\epsilon,h}(x) = \epsilon(x-1) + \eta^\epsilon$ for $x \in (r_{k-1}^{\epsilon,h}, 1]$.

Lemma 7. *If $\theta_{k-1}^{\epsilon,h}$ is essentially decreasing on $(0,1)$ then so is $\theta_k^{\epsilon,h}$ and $r_k^{\epsilon,h} \geq r_{k-1}^{\epsilon,h}$. Also $U_{k-1}^{\epsilon,h} \geq U_k^{\epsilon,h}$ a.e. in $[0, r_k^{\epsilon,h})$ and $U_k^{\epsilon,h}(x) = \epsilon(x-1) + \eta^\epsilon$ for a.e. $x \in [r_k^{\epsilon,h}, 1]$.*

Proof: The first statement follows from Lemma 4. The rest is a fairly simple exercise consisting of a more in-depth geometrical exploitation of the proof of the same lemma. First suppose $r_k^{\epsilon,h} < r_{k-1}^{\epsilon,h}$. Then $\big(\theta_k^{\epsilon,h}\big) = \big(\theta_k^{\epsilon,h}\big)' = 0$ a.e. in $(r_k^{\epsilon,h}, 1)$ and (3.14) implies $U_{k-1}^{\epsilon,h} = U_k^{\epsilon,h}$ a.e. in $(r_k^{\epsilon,h}, 1)$. In particular, $U_{k-1}^{\epsilon,h} = U_k^{\epsilon,h}$ a.e. in $(r_k^{\epsilon,h}, r_{k-1}^{\epsilon,h})$ and so $u_{k-1}^{\epsilon,h} = \big(U_{k-1}^{\epsilon,h}\big)' = \big(U_k^{\epsilon,h}\big)' = u_k^{\epsilon,h}$ a.e. in $(r_k^{\epsilon,h}, r_{k-1}^{\epsilon,h})$. Also, since $\theta_k^{\epsilon,h} = 0$ a.e. in $(r_k^{\epsilon,h}, r_{k-1}^{\epsilon,h})$, it follows that $\big(U_k^{\epsilon,h}\big)' = u_k^{\epsilon,h} \leq 1$ a.e. in $(r_k^{\epsilon,h}, r_{k-1}^{\epsilon,h})$. However, $u_{k-1}^{\epsilon,h} > 1$ a.e. in $(r_k^{\epsilon,h}, r_{k-1}^{\epsilon,h})$ (because $\theta_{k-1}^{\epsilon,h} > 0$ there), so we arrive to a contradiction.
The second part follows easily as we observe that $\theta_k^{\epsilon,h} = 0$ and thus $U_{k-1}^{\epsilon,h} = U_k^{\epsilon,h}$ a.e. in $[r_k^{\epsilon,h}, 1]$. □

Remark: As $\theta_k^{\epsilon,h} = u_k^{\epsilon,h} - 1$ on $[0, r_k^{\epsilon,h})$ (i.e. the set where $u_k^{\epsilon,h} > 1$) and $\theta_k^{\epsilon,h}$ is Lipschitz continuous we infer that $u_k^{\epsilon,h}$ is decreasing and Lipschitz continuous on $[0, r_k^{\epsilon,h})$ bounded from below by 1. Then it jumps at $r_k^{\epsilon,h}$ to ϵ where it stays up to 1.

Let $u^{\epsilon,h}$ be the standard time interpolant of the $u_k^{\epsilon,h}$'s (see (1.12)) and u^ϵ be the weak solution for (P). If we let $U^{\epsilon,h}(x,t) := \int_0^x u^{\epsilon,h}(y,t)dy$ and $U^\epsilon(x,t) := \int_0^x u^\epsilon(y,t)dy$ note that, due to Lemma 5, $U^{\epsilon,h}(\cdot,t) \to U^\epsilon(\cdot,t)$ pointwise in $(0,1)$ for a.e. $t > 0$.
Since $u^{\epsilon,h}(x,t) = u_{[t/h]}^{\epsilon,h}(x)$ in $[0,1]$, we infer that for a.e. $t > 0$ we have $U_{[t/h]}^{\epsilon,h} \to U^\epsilon(\cdot,t)$ in $[0,1]$ pointwise. Of course, $U^\epsilon(\cdot,t)$ is also concave and, if we consider a subsequence of $\{h\}$ (still denoted

by $\{h\}$) such that $\{r^{\epsilon,h}_{[t/h]}\}_h$ is monotone, we easily see that $U^\epsilon(\cdot,t)$ has almost the same structure as $U^{\epsilon,h}_{[t/h]}$. That is, there exists $r^\epsilon(t) \in [0,1]$ (the limit of $\{r^{\epsilon,h}_{[t/h]}\}_h$) such that $\big(U^\epsilon\big)'(\cdot,t) = u^\epsilon(\cdot,t)$ is decreasing on $(0,1)$, $u^\epsilon \geq 1$ on $[0, r^\epsilon(t))$ and $u^\epsilon(\cdot,t) = \epsilon$ on $(r^\epsilon(t), 1]$. Also, due to Lemma 7, $t \to r^\epsilon(t)$ is increasing.
Therefore, we have just given a sketch of the proof of the following

Proposition 6. *There exists an increasing function $t \to r^\epsilon(t) \in [0,1]$ defined for a.e. $t > 0$ such that $u^\epsilon(\cdot,t)$ is decreasing for any such t and*

$$u^\epsilon(\cdot,t) \geq 1 \ \text{on} \ [0, r^\epsilon(t)) \ \text{and} \ u^\epsilon(\cdot,t) = \epsilon \ \text{on} \ [r^\epsilon(t), 1]. \tag{3.15}$$

As we have already mentioned in the previous section, there exists a weak solution u for (P) with initial data $u_0 := 1 + \theta_0$ if $\theta_0 > 0$ and $u_0 := 0$ if $\theta_0 = 0$ such that $\theta^\epsilon \to \theta$ (where $\theta = \alpha(u)$ as before) strongly in $L^2((0,1)\times(0,T))$ and a.e. in $(0,1)\times(0,T)$. We now prove the following

Proposition 7. *There exists an increasing function $t \to r(t) \in [0,1]$ defined for a.e. $t > 0$ such that $u(\cdot,t)$ is decreasing for any such t and*

$$u(\cdot,t) \geq 1 \ \text{on} \ [0, r(t)) \ \text{and} \ u(\cdot,t) = 0 \ \text{on} \ [r(t), 1]. \tag{3.16}$$

Proof: We obviously have $\theta^\epsilon(\cdot,t) \to \theta(\cdot,t)$ strongly in $L^2(0,1)$ and for a.e. $t > 0$. Then, as in [15], the maximal monotone operators theory implies $u^\epsilon(\cdot,t) \rightharpoonup u(\cdot,t)$ weakly in $L^2(0,1)$. Consequently, $U^\epsilon(\cdot,t) \to U(\cdot,t)$ pointwise in $(0,1)$, where $U(x,t) := \int_0^x u(y,t)dy$. Therefore, $U(\cdot,t)$ is also concave and consider a subsequence (still indexed by ϵ) such that $\{r^\epsilon(t)\}_\epsilon$ is monotone. Assume this subsequence converges to something that we denote by $r(t)$. Then, $\eta^\epsilon \downarrow 1$ also as $\epsilon \downarrow 0$ and we easily infer (3.16) by (3.15). The previous proposition also ensures that r^ϵ is increasing as a function of t. It is not hard to see that the same property is now passed to r. □

Note that, due to (3.4), $0 < r(t) < 1$ for all t for which it is defined. Let us define $r(t) := \lim_{\tau\uparrow t} r(\tau)$ at those t's where it was not previously defined. Thus, r is defined everywhere on $(0,\infty)$ and is increasing. Therefore, $r = r(t)$ is continuous possibly with the exception of a countable set of t's in $(0,\infty)$. At those t's we redefine it to satisfy $r(t) := \lim_{\tau\uparrow t} r(\tau)$. Thus, r becomes left-continuous.

Theorem 3. *After possibly redefining $\theta := \alpha(u)$ on negligible sets, the following are true:*
(i) $t \to r(t)$ is strictly increasing and continuous;
(ii) θ is continuous in $(0,1)\times(0,\infty)$ possibly with the exception of a set of zero one-dimensional Haussdorf measure;
(iii) θ is strictly positive, smooth and satisfies $\theta_t = \theta_{xx}$ to the left of $x = r(t)$, i.e. in the region $\cup_{t>0}\{t\}\times(0,r(t))$;
(iv) $\theta_x(0,\cdot) \equiv 0$ on $(0,\infty)$;
(v) $\theta \equiv 0$ to the right of $x = r(t)$, i.e. in the region $\cup_{t>0}\{t\}\times(r(t),1)$;
(vi) $D^-_x\theta(r(t),t) = -r'(t)$ for a.e. $t > 0$;
(vii) $\theta(x,0) = \theta_0(x)$ in $(0,1)$ in the limit sense.

Proof: If we let $\theta = \alpha(u)$ we infer, since u is null a.e. to the right of $r(t)$ and is also the weak solution for (P), that

$$\int_0^\infty \int_0^{r(t)} \big\{(1+\theta)\zeta_t + \theta\zeta_{xx}\big\}dxdt = -\int_0^{r(0)} (1+\theta_0)\zeta(x,0)dx \tag{3.17}$$

for all $\zeta \in C_c^\infty([0,1] \times [0,\infty))$ such that $\zeta_x(0,t) = \zeta_x(1,t) = 0$ for all $t > 0$. If we consider any rectangle

$$R_{a,b}^{t_1,t_2} := (a,b) \times (t_1,t_2) \subset \cup_{t>0}\{t\} \times (0,r(t))$$

and any $\zeta \in C_c^\infty(R_{a,b}^{t_1,t_2})$ for $0 < t_1 < t_2 < \infty$ and $0 < a < b < r(t_1)$, we see that (3.17) applied to such functions ζ translates into the fact that θ is a solution in the sense of distributions of the heat equation $\theta_t - \theta_{xx} = 0$ in $R_{a,b}^{t_1,t_2}$. Therefore, θ is smooth in $R_{a,b}^{t_1,t_2}$ and satisfies $\theta_t - \theta_{xx} = 0$ in $R_{a,b}^{t_1,t_2}$ in the classical sense (see, say, [1]). We now claim that $r(t) = \min\{x \in [0,1],\ \theta(x,t) = 0\}$. Indeed, if that were not the case, then we would have $\theta(\cdot,t_0) = 0$ for some $t_0 > 0$ on $(r(t_0) - \delta, 1]$ (δ sufficiently small). It is, obviously, easy to construct a rectangle as the one from above (with $a = 0$) which has a point from $\{t_0\} \times (r(t_0) - \delta, 1]$ in its interior. According to the maximum principle for the classical solutions of the heat equation (see [1]) we infer that $\theta = 0$ at any time level between t_1 and t_0 on the entire interval $(0,1)$ (recall that θ is decreasing at almost all time levels). However, that contradicts the conservation identity (3.4). Therefore, θ is strictly positive and smooth in $\cup_{t>0}\{t\} \times (0,r(t))$ and identically zero in $\cup_{t>0}\{t\} \times [r(t),1]$ (after a possible redefinition in a negligible set). (iii) and (v) are thus proved. Next we would like to prove (ii). In light of all the above, it suffices to prove

$$\lim_{x \uparrow r(t)} \theta(x,t) = 0 \text{ for almost all } t > 0.$$

We now refer the reader again to [15] where we prove that the generalized spatial derivative of θ, i.e. θ_x lies in $L^2((0,1) \times (0,T))$ for all finite $T > 0$. This implies

$$\theta_x(\cdot,t) \in L^2(0,1) \text{ for almost all } t > 0.$$

As all $\theta(\cdot,t)$ are also in $L^2(0,1)$ (being bounded by M, the upper bound of θ_0) we infer that $\theta(\cdot,t) \in H^1(0,1)$ for a.e. $t > 0$. This eliminates the possibility of a jump at $r(t)$ for a.e. $t > 0$ since all $H^1(0,1)$ functions are equal almost everywhere to continuous functions (see [2]).
Now we prove that r is continuous everywhere. For this assume that a (finite, of course) jump occurs at some point $t_0 > 0$. Consequently, there exists $\varepsilon > 0$ such that $\lim_{t \downarrow t_0} r(t) = r(t_0) + \varepsilon$. Note that, for any $t > 0$, the function

$$\zeta(x,s) := \zeta(s) = \begin{cases} 1 \text{ on } [0, t-\delta], \\ \frac{t-s+\delta}{2\delta} \text{ on } (t-\delta, t+\delta), \\ 0 \text{ on } [t+\delta, \infty), \end{cases}$$

for some small $\delta > 0$ can be approximated such that it becomes admissible as a test function in (3.17). A Lebesgue point argument shows that, by letting $\delta \downarrow 0$, we obtain

$$r(t) + \int_0^{r(t)} \theta(x,t)dx = 1 \text{ for almost all } t > 0. \tag{3.18}$$

As r is increasing and θ is nonnegative (and 0 to the right of $x = r(t)$), it follows that there exist two sequences $\{t_n^+\}_n \downarrow t_0$ and $\{t_n^-\}_n \uparrow t_0$ such that

$$r(t_n^-) + \int_0^{r(t_0)} \theta(x,t_n^-)dx = 1 \geq r(t_n^+) + \int_0^{r(t_0)} \theta(x,t_n^+)dx. \tag{3.19}$$

For any $t_n \to t_0$, if we fix $\epsilon > 0$, we can write

$$\left| \int_0^{r(t_0)} \{\theta(x,t_n) - \theta(x,t_0)\}dx \right| \leq \int_0^{r(t_0)-\epsilon/(4M)} |\theta(x,t_n) - \theta(x,t_0)|dx + \epsilon/2,$$

where $M := \|\theta\|_\infty$. Due to (iii), $\int_0^{r(t_0)-\epsilon/(4M)} |\theta(x,t_n)-\theta(x,t_0)|dx \le \epsilon/2$ for sufficiently large n. Consequently,

$$\lim_{n\uparrow\infty}\int_0^{r(t_0)} \theta(x,t_n^-)dx = \int_0^{r(t_0)} \theta(x,t_0)dx = \lim_{n\uparrow\infty}\int_0^{r(t_0)} \theta(x,t_n^+)dx.$$

This and (3.19) contradict the jump by ε at t_0, thus proving the continuity of r.
In order to complete the proof of (i) assume that $r \equiv c$ is constant on (t_1,t_2), where $0 < t_1 < t_2 < \infty$. Consider now test functions $\zeta \in C_c^\infty(R_{0,1}^{t_1,t_2})$. The equality (3.17) easily gives

$$\int_{t_1}^{t_2}\int_0^c \big\{\theta\zeta_t + \theta\zeta_{xx}\big\}dxdt = 0, \text{ for all } \zeta \in C_c^\infty(R_{0,1}^{t_1,t_2}).$$

Due to (v), we can actually write

$$\int_{t_1}^{t_2}\int_0^1 \big\{\theta\zeta_t + \theta\zeta_{xx}\big\}dxdt = 0, \text{ for all } \zeta \in C_c^\infty(R_{0,1}^{t_1,t_2}). \tag{3.20}$$

It follows (as before) that θ is a classical solution of the heat equation $\theta_t = \theta_{xx}$ in the open rectangle $R_{0,1}^{t_1,t_2}$ and, again by the maximum principle, $\theta \equiv 0$ in this whole rectangle. Therefore, we arrive to a contradiction and thus (i) is completed.
To prove (vi) we fix $T > 0$ and note that, for all $\zeta \in C_c^\infty((0,1)\times(0,T))$, (3.17) becomes

$$\int_0^T\int_0^{r(t)} \big\{(1+\theta)\zeta_t - \theta_x\zeta_x\big\}dxdt = 0, \tag{3.21}$$

since $\theta \equiv 0$ along $x = r(t)$ (integration by parts). If we consider the vector field $F := (-\theta_x, 1+\theta)$, then the r.h.s. of (3.21) can be written as $\int_0^T\int_0^{r(t)} F\cdot\nabla\zeta dxdt$. Due to (i) we can apply *Green's* theorem to obtain

$$\int_0^T\int_0^{r(t)} F\cdot\nabla\zeta dxdt = \int_\Gamma (F\cdot\nu)\zeta ds - \int_0^T\int_0^{r(t)} (\nabla\cdot F)\zeta dxdt,$$

where Γ is the portion of $x = r(t)$ for $0 < t < T$ ($\zeta \equiv 0$ on the rest of the boundary). According to (iii) and (3.21), it follows that $\nabla\cdot F \equiv 0$ in $\cup_{0<t<T}\{t\}\times(0,r(t))$ and

$$\int_\Gamma \zeta(F\cdot\nu)ds = 0 \text{ for all } \zeta \in C_c^\infty((0,1)\times(0,T)).$$

Consequently, $F\cdot\nu = -\nu_x\theta_x + \nu_t(1+\theta) = 0$ (the values of θ and θ_x on Γ are limits from the left) a.e. along Γ which, according to (i) and $\theta \equiv 0$ along Γ, yields (vi).
(vii) now follows trivially from (3.17). □

Remark: It is not hard to see that (3.5) may be replaced by the weaker (3.13) and Theorem 3 remains true.

Remark: It is easy to see that our solution is the same as the one found in [9] by entirely different means. In [9] one can also find higher regularity results for the free boundary.

3.4 Asymptotic behavior and numerical simulations

We keep the assumptions on θ_0 from the previous section and let (θ, r) be a classical solution as in (3.2). Recall that Φ_α is a solution of $y\Phi'(y) - \Phi(y) = \alpha(y)$ and we pick

$$\Phi_\alpha(y) = y \text{ if } y \in [0,1] \text{ and } \Phi_\alpha(y) = 1 + y \log y \text{ if } y \in (1, \infty).$$

Next we will show how the decay in the entropy S_α provides the right tool for proving asymptotic decay to zero of the solution for (P). Let us write

$$S_\alpha(u(\cdot,t)) = \int_0^{r(t)} \Phi_\alpha(1+\theta(x,t))dx + \int_{r(t)}^1 \Phi_\alpha(0)dx = \int_0^{r(t)} \Phi_\alpha(1+\theta(x,t))dx,$$

since $\Phi_\alpha(0) = 0$. According to Theorem 3, we may compute, for all $t > 0$,

$$\begin{aligned}\frac{d}{dt}S_\alpha(u(\cdot,t)) &= \Phi_\alpha(1+\theta(r(t),t))r'(t) + \int_0^{r(t)} \theta_t(x,t)\Phi_\alpha'(1+\theta(x,t))dx \\ &= r'(t) + \int_0^{r(t)} \theta_{xx}(x,t)[1+\log(1+\theta(x,t))]dx \\ &= -\int_0^{r(t)} \frac{\theta_x^2(x,t)}{1+\theta(x,t)}dx. \end{aligned} \tag{3.22}$$

As $\theta(r(t),t) = 0$, we easily infer $\|\theta(\cdot,t)\|_\infty^2 \le r(t)\int_0^{r(t)} \theta_x^2(x,t)dx$, leading to

$$\frac{d}{dt}S_\alpha(u(\cdot,t)) \le -\frac{1}{r(t)} \cdot \frac{\|\theta(\cdot,t)\|_\infty^2}{1+\|\theta(\cdot,t)\|_\infty}.$$

Since $0 < r(0) \le r(t) < 1$ and $S_\alpha(u(\cdot,t)) \ge 0$, it follows

$$\int_0^\infty \frac{\|\theta(\cdot,t)\|_\infty^2}{1+\|\theta(\cdot,t)\|_\infty}dt < \infty. \tag{3.23}$$

As the experimental results will show, we seem to be having $\lim_{t\uparrow\infty} \|\theta(\cdot,t)\|_{L^\infty(0,1)} = 0$. In order to prove this we need to recall that θ has the following important property (see [15]):

$$\text{if } \theta_0 \in L^\infty(0,1), \text{ then } \theta \in L^\infty(Q_0) \text{ and } \|\theta\|_{L^\infty(Q_0)} \le \|\theta_0\|_{L^\infty(0,1)},$$

where $Q_{t_0} := (0,1) \times (t_0, \infty)$ for $t_0 \ge 0$. Now, if we fix $0 < t_0 < \infty$, it is a well known fact that $\theta\big|_{Q_{t_0}}$ is the solution for our equation with initial data $\theta(\cdot, t_0)$. It follows, after possibly changing the variable $t \to t - t_0$ to go back and work in Q_0, that

$$\theta \in L^\infty(Q_{t_0}) \text{ and } \|\theta\|_{L^\infty(Q_{t_0})} \le \|\theta(\cdot,t_0)\|_{L^\infty(0,1)}.$$

Therefore, $t \to \|\theta(\cdot,t)\|_{L^\infty(0,1)}$ is essentially decreasing in $(0,\infty)$. The word "essentially" may, in fact, be suppressed due to the smoothness of θ to left of $x = r(t)$. Along with (3.23), this yields

$$\lim_{t\uparrow\infty} \|\theta(\cdot,t)\|_{L^\infty(0,1)} = 0. \tag{3.24}$$

We show tables that predict the kind of behavior we have noted and proved. These were obtained by adapting the C-code (based on the relaxation algorithm) used by Kinderlehrer and Walkington in [10] and generously provided to us by the authors. The code provides a numerical implementation

k	$t = 0.015625$	$t = 0.0625$	$t = 0.25$	$t = 1.0$	$t = 4.0$
00	2.427382	1.949334	1.388693	1.054514	1.010024
01	2.362357	1.924320	1.383408	1.053967	1.010020
02	2.239278	1.874992	1.372722	1.052895	1.010017
03	2.075957	1.802934	1.356408	1.051311	1.010014
04	1.891817	1.710225	1.335211	1.049212	1.010013
05	1.700818	1.598537	1.308621	1.046612	1.010013
06	1.502935	1.471821	1.276945	1.043543	1.010011
07	1.287788	1.340215	1.240152	1.040023	1.010008
08	0.545401	1.225322	1.198408	1.036084	1.010002
09	0.001515	1.155991	1.152050	1.031768	1.009997
10	0.025960	0.001868	1.101533	1.027088	1.009985
11	0.018432	0.025436	1.047650	1.022124	1.009984
12	0.020482	0.018932	0.951794	1.016906	1.009982
13	0.019838	0.019988	0.000872	1.011463	1.009980
14	0.020059	0.020158	0.027815	1.005907	1.009977
15	0.019982	0.019927	0.017717	0.616585	1.009974

Table 1: $n = 16,\ h = 1/1024,\ \epsilon = 0.02$

of time-interpolating the minimizers from Corollary 1.
We consider $\theta_0(x) := 2 - 4x$ on $[0, 1/2]$ and $\theta_0 := 0$ on $[1/2, 1]$. Observe that θ_0 satisfies (3.1). For $\epsilon = 0.02$ and $\epsilon = 0.005$ respectively, we construct the $u_0^\epsilon \in \theta_0 + H(\theta_0)$ as before, i.e. $u_0^\epsilon := 3 - 4x$ on $[0, 1/2]$ and $u_0^\epsilon := \epsilon$ on $(1/2, 1]$. A numerical solution u^ϵ ($u^{\epsilon,h}$ for $h = 1/1024$) is given at time levels $t = 0.015625$, 0.0625, 0.25, 1, and 4 for each ϵ (Table 1 for $\epsilon = 0.02$ and Table 2 for $\epsilon = 0.005$) as a piecewise constant function on a uniform 16-division of the interval $[0, 1]$. It is easy to see that the $u^{\epsilon,h}$'s do stay decreasing in x being greater than 1 first and then jumping to ϵ where they stay up to the endpoint 1. The asymptotic behavior is also visible in each table. The code becomes unstable as $\epsilon \downarrow 0$, therefore we do not give any results for samller ϵ.

Appendix

Proposition 8. *(Variation of domain) Let $\xi \in C_c(0, 1)$ be Lipschitz and consider the autonomous ODE*

$$\frac{d\phi}{d\tau} = \xi(\phi),\ \ \phi|_{\tau=0} = x,\ \ \text{for } x \in [0, 1],\ \ \tau \in \mathbb{R}. \tag{3.25}$$

Then, for every $x \in [0, 1]$ and every τ with $|\tau|$ small enough, the solution $\phi(\tau; x)$ lies in $[0, 1]$ and therefore $\xi(\phi)$ is well-defined. Furthermore, $\phi(\tau; \cdot)$ is invertible and $\phi^{-1}(\tau; \cdot) \equiv \phi(-\tau; \cdot)$ for any real number τ.

Proof: First extend ξ to $\zeta \in C^1(\mathbb{R})$ by letting it be zero both on $(-\infty, 0]$ and $[1, \infty)$. Note that ζ itself is Lipschitz continuous and $Lip(\zeta) = Lip(\xi)$. Now we replace ξ by its extension ζ in (3.25) observing that now $\zeta(\phi)$ makes sense everywhere as ζ is defined on the entire real line.
We have

$$\phi(t + s; x) = \phi(t; x) + \int_t^{t+s} \zeta(\phi(\tau; x))d\tau,\ \ t,\ \ s \in \mathbb{R},$$

k	$t=0.015625$	$t=0.0625$	$t=0.25$	$t=1.0$	$t=4.0$
00	2.428561	1.952677	1.404553	1.069223	1.002524
01	2.358358	1.939759	1.399927	1.068909	1.002520
02	2.239327	1.900519	1.390122	1.068270	1.002518
03	2.083282	1.819204	1.375711	1.067289	1.002516
04	1.909051	1.737442	1.357692	1.065919	1.002514
05	1.728137	1.628803	1.335509	1.064123	1.002512
06	1.544491	1.501350	1.309958	1.061819	1.002510
07	1.366275	1.342408	1.279810	1.058953	1.002507
08	0.351146	1.176068	1.244167	1.055442	1.002501
09	0.000102	1.015961	1.200618	1.051212	1.002495
10	0.006610	0.000026	1.147894	1.046178	1.002485
11	0.004548	0.006272	1.085207	1.040286	1.002484
12	0.005164	0.004762	0.495588	1.033497	1.002482
13	0.004909	0.004127	0.000064	1.025797	1.002480
14	0.005066	0.005956	0.010874	1.017258	1.002478
15	0.004973	0.004664	0.002305	0.245824	1.002474

Table 2: $n = 16,\ h = 1/1024,\ \epsilon = 0.005$

and
$$\phi(s;\phi(t;x)) = \phi(t;x) + \int_0^s \zeta(\phi(\tau;\phi(t;x)))d\tau.$$
Upon changing the variable $\tau \to \tau + t$ in the first integral and subtracting the two expressions we obtain
$$|\phi(t+s;x) - \phi(s;\phi(t,x))| \le \int_0^s |\zeta(\phi(t+\tau;x)) - \zeta(\phi(\tau;\phi(t;x)))|d\tau,\ \ s > 0.$$
We fix x and t. As ζ is Lipschitz we can apply the Gronwall lemma and obtain the continuation property
$$\phi(s;\phi(t;x)) = \phi(t+s;x),\ (\forall)s > 0,\ t \in \mathbb{R}.$$
Therefore, $\phi(t;\cdot)$ is invertible and $\phi^{-1}(t;\cdot) = \phi(-t;\cdot)$.
Apply the integral form $\phi(t;\cdot) = \cdot + \int_0^t \zeta(\phi(\tau;\cdot))d\tau$ to x and y then subtract to obtain
$$|\phi(t;x) - \phi(t;y)| \le |x-y| + \int_0^t |\zeta(\phi(\tau;x)) - \zeta(\phi(\tau;y))|d\tau.$$
The Gronwall lemma gives $|\phi(t;x) - \phi(t;y)| \le e^{tLip(\xi)}|x-y|$, i.e. $\phi(t;\cdot)$ is Lipschitz continuous. We already know that it is invertible, so $\phi(t;\cdot)$ is strictly monotone and continuous thus mapping $[0,1]$ onto the compact interval
$$[\min\{\phi(t;0),\phi(t;1)\},\max\{\phi(t;0),\phi(t;1)\}].$$
Analyzing $\phi(t;1) = 1 + \int_0^t \zeta(\phi(\tau;1))d\tau$ note that, due to the continuity of $t \to \phi(t;1)$ at $t = 0$, $\phi(\tau;1)$ gets close to 1 as τ gets small and so, since ζ vanishes near 1, we obtain $\phi(t;1) = 1$ for all small enough t. Similarly, $\phi(t;0) = 0$ for all small enough t. This finishes our proof. $\square$

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