Euler–Poisson systems as action-minimizing paths in the Wasserstein space

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Abstract

This paper uses a variational approach to establish existence of solutions (σ_t, v_t) for the 1–d Euler-Poisson system by minimizing an action. We assume that the initial and terminal points σ_0, σ_T are prescribed in $\mathcal{P}_2(I\!R)$, the set of Borel probability measures on the real line, of finite second moment. We show existence of a unique minimizer of the action when the time interval $[0,T]$ satisfies $T < \pi$. These solutions conserve the Hamiltonian and they yield a path $t \to \sigma_t$ in $\mathcal{P}_2(I\!R)$. This path turns out to be the characteristics of an infinite dimensional Hamilton-Jacobi equation in $\mathcal{P}_2(I\!R)$. When $\sigma_t = \delta_{y(t)}$ is a Dirac mass, the Euler-Poisson system reduces to $\ddot{y} + y = 0$ and the Hamilton-Jacobi equation is merely finite dimensional and is given by $\partial_t u + 1/2(\partial_y u)^2 + 1/2y^2 + 1/24 = 0$. The kinetic version of the Euler-Poisson, i.e. the Vlaso v-Poisson system was studied in [1] as a Hamiltonian system.

1 Introduction

Several works are concerned with the Euler-Poisson system and its many variants [11], [19], [23] and [8] (the pressureless case). Some of them have considered the so-called entropy solutions [13], [17]. Here we analyze a different class of solutions for the one-dimensional repulsive Euler-Poisson system with constant background. We anticipate that this new class of solutions is relevant to adapting the weak KAM theory to our infinite dimensional setting, yet that remains to be proved. The goal of this paper is to study solutions which are action-minimizing paths with respect to a Lagrangian L. This Lagrangian is defined on the tangent bundle to $\mathcal{P}_2(I\!R)$ (as defined in [2]). Here, $\mathcal{P}_2(I\!R)$ is the space of Borel probability measures on $I\!R$ with finite second-order moments. When the initial and terminal points of the paths are prescribed, we refer to this problem as the two-point boundary problem. We establish existence of solutions for the two-point boundary problem, along with uniqueness of solutions that are action-minimizers. Our study is facilitated by a remarkable Eulerian-Lagrangian duality property in the space of L^2-absolutely continuous curves. This property is a direct consequence of lemma 2.2. It could also be obtained as a consequence of the more subtle purely analytic result obtained in (33). These results allow to pass from Eulerian to Lagrangian coordinates. Here, we require minimal smoothness property in the time variable and no regularity property in the space variable. The result in (33) is also exploited to obtain conservation of the Hamiltonian along solutions in the action-minimizing class. In the last part of this work, we introduce an infinite-dimensional Hamilton-Jacobi equation. It allows for an interesting interpretation of the Euler-Poisson system as its character istics.

Let us begin by introducing the commonly known form of the pressureless, repulsive Euler-Poisson system with constant background charge

$$\begin{cases} \partial_t \rho_t + \partial_y(\rho_t v_t) = 0 & \text{in } I\!R \times (0,T), \\ \partial_t(\rho_t v_t) + \partial_y(\rho_t v_t^2) = -\rho_t \partial_y \Phi_t & \text{in } I\!R \times (0,T), \\ -\partial^2_{yy}\Phi_t = \rho_t - 1 & \text{in } I\!R. \end{cases} \tag{1}$$

Observe that $y \to y^2/2 - \Phi_t(y)$ is a convex function and so, $\partial_y \Phi_t$ is well-defined except maybe on an at most countable set. Hence, the expression $\rho_t \partial_y \Phi_t$ makes sense since $\partial_y \Phi_t$ is well-defined ρ_t–almost everywhere. If, instead, we try to substitute the density ρ_t by an arbitrary Borel measure σ_t in the expression $\rho_t \partial_y \Phi_t$, we are forced to substitute $\partial_y \Phi_t$ by a function which is defined almost everywhere with respect to σ_t. Thus, to further allow for solutions that are Borel probability measures, we focus on an extension of the momentum equation in (1). Namely we substitute it by

$$\partial_t(\sigma_t v_t) + \partial_y(\sigma_t v_t^2) = \sigma_t[\bar{\gamma}_t - \mathbf{id}] \tag{2}$$

in the distributional sense. Here $\bar{\gamma}_t(y) = \sigma_t(-\infty, y) + 1/2\sigma_t\{y\} - 1/2$. We note that if $d\sigma_t := \rho_t d\mathcal{L}^1$, then this formulation and that from (1) coincide. We are going to justify this and give the precise meaning of $\bar{\gamma}_t$ in the sequel. We arrive to (2) as a natural expression of Newton's second law. To do so, we first need to recall some basic facts from the theory of L^2-absolutely continuous curves in $\mathcal{P}_2(I\!R)$. We shall be quite sketchy, for further details we recommend the comprehensive reference [2]. Let us endow $\mathcal{P}_2(I\!R)$ with the quadratic Wasserstein metric defined by

$$W_2^2(\mu, \nu) := \min_\gamma \int_{I\!R^2} |\bar{y} - y|^2 d\gamma(y, \bar{y}),$$

where the infimum is taken among all probabilities γ on the the product space $I\!R^2$ with marginals μ, ν. The joint distributions γ that realize the minimum are called optimal couplings or optimal plans. Thus, $(\mathcal{P}_2(I\!R), W_2)$ becomes a Polish space on which we define absolutely continuous curves. Suppose in general that $(\mathcal{S}, dist)$ in a complete metric space. We say that $[0, T] \ni t \to \sigma_t \in \mathcal{S}$ lies in $AC^2(0, T; \mathcal{S})$ provided that there exists $f \in L^2(0, T)$ such that $dist(\sigma_t, \sigma_{t+h}) \leq \int_t^{t+h} f(s)ds$ for all $0 < t < t + h < T$. We now take $(\mathcal{S}, dist) = (\mathcal{P}_2(I\!R), W_2)$ and $\sigma \in AC^2(0, T; \mathcal{P}_2(I\!R))$. We arrive to the definition of $\bar{\gamma}_t$ by first considering γ_t as the unique optimal coupling between $\nu_0 := \mathcal{L}^1|_{(-1/2, 1/2)}$ and σ_t. Then we let $\bar{\gamma}_t$ be the barycentric projection of γ_t onto its second marginal σ_t. In general, the barycentric projection $\bar{\gamma}_\mu : I\!R^d \to I\!R^d$ of a plan $\gamma \in \mathcal{P}(I\!R^d \times I\!R^d)$ onto its second marginal $\mu := \pi^2_{\#}\gamma$ is uniquely defined μ–a.e. by

$$\bar{\gamma}_\mu(y) := \int_{I\!R^d} x \; d\gamma_y(x) \quad \text{ for } \mu\text{-a.e. } y \in I\!R^d, \tag{3}$$

where we have disintegrated γ as $\gamma = \int_{I\!R^d} \gamma_y d\mu(y)$. When $d = 1$ one can check that the barycentric projection reduces to $\bar{\gamma}_\mu(y) = \mu(-\infty, y) + 1/2\mu\{y\} - 1/2$. To relate back to (1), we make the following observation: if σ_t vanishes on sets which are at most countable, then $\bar{\gamma}_t$ is nothing but the optimal map $\partial_y\psi_t$ between σ_t and ν_0. Moreover, we have $\partial_y\psi_t$ is differentiable σ_t–almost everywhere and $\partial_{yy}\psi_t = \sigma_t$ in the sense of distributions. In this particular case, ψ_t and the function Φ_t appearing in (1) are related by $\Phi_t(y) = y^2/2 - \psi_t(y)$.

To arrive to our point of view, we shall briefly discuss a system related to (1) that was recently studied from a similar, yet different perspective. The kinetic version of (1) is the Vlasov-Poisson system

$$\begin{cases} \partial_t f(y,v,t) + v\partial_y f(y,v,t) = -\partial_y\phi(y,t)\partial_v f(y,v,t) & \text{in } I\!R \times I\!R \times (0,T), \\ \partial^2_{yy}\phi(\cdot, t) = \rho - 1 & \text{in } I\!R \times (0,T) \\ \rho(y,t) = \int_{I\!R} f(y,v,t)dv. & \end{cases}$$

Indeed, at least at the formal level, if v_t is a velocity for $t \to \rho(\cdot, t) =: \rho_t$ and $f(y, w, t) = \rho_t(y)\delta_{w - v_t(y)}$ (monokinetic solution for Vlasov-Poisson), then (ρ, v) solves (1). Infinite-dimensional Hamiltonian ODE are treated in [1] for Hamiltonians $\mathcal{H} : \mathcal{P}_2(I\!R^d \times I\!R^d) \to I\!R$. Set $\mathcal{H}(\nu) = W_2^2(\nu, \nu_0 \otimes \delta_0)/2$ with ν_0 being the indicator function of the unit cube $X := (-1/2, 1/2)^d$. One can show that the Hamilton ODE in the sense of [1] becomes the Vlasov-Monge-Ampère system introduced in [9]. It appears as an asymptotic approximation for the standard Vlasov-Poisson system describing the evolution of electron clouds in neutralizing uniform media. The two systems are the same if $d = 1$. Let us now explain how the Euler-Poisson system (or Euler-Monge-Ampère in multiple dimensions), can also be regarded as a Lagrangian system in a rela ted context. We use the observation already exploited in prior works, valid for all d: if $\nu \in \mathcal{P}_2(I\!R^{2d})$, then

$$W_2^2(\nu, \nu_0 \otimes \delta_0) = \int_{I\!R^d} |v|^2 d\nu_2(v) + W_2^2(\nu_1, \nu_0). \tag{4}$$

Here ν_1 and ν_2 are, respectively, the first and second marginals of ν. Thus, $\mathcal{H}$ defined above does not "see" the full measure ν, only its marginals. Let us further restrict our attention to measures ν of the form $\nu(E) = \mu(\{x \in I\!R^d |\ (x, \zeta(x)) \in E\})$ with $\mu \in \mathcal{P}_2(I\!R^d)$ and $\zeta : I\!R^d \to I\!R^d$ in $L^2(\mu)$. We write $\nu = (\mathbf{id} \times \zeta)_{\#}\mu$, where $\mathbf{id}$ stands for the identity map on $I\!R^d$. The tangent space $\mathcal{T}_\mu \mathcal{P}_2(I\!R^d)$ is the closure of $\{\nabla\varphi \mid \varphi \in C_c^\infty(I\!R^d)\}$ in $L^2(\mu)$ [2] and, thus, a separable Hilbert space which we choose to identify with its dual. Therefore, we make no distinction between the tangent and cotangent spaces at μ. We can then restrict ourselves to ζ in the cotangent bundle at μ. For ν of the form $(\mathbf{id} \times \zeta)_{\#}\mu$, the right hand side of (4) gives rise t o the Hamiltonian

$$H(\mu, \zeta) := \frac{1}{2}\|\zeta\|^2_{L^2(\mu)} + \frac{1}{2}W_2^2(\mu, \nu_0).$$

The associated Lagrangian considered in this paper is

$$L(\mu, \xi) := \frac{1}{2}\|\xi\|^2_{L^2(\mu)} - \frac{1}{2}W_2^2(\mu, \nu_0) = \sup_{\zeta \in \mathcal{T}_\mu \mathcal{P}_2(I\!R^d)} \{\langle \zeta, \xi \rangle - H(\mu, \zeta)\}.$$

It is defined for $\mu \in \mathcal{P}_2(I\!R^d)$ and $\xi \in \mathcal{T}_\mu \mathcal{P}_2(I\!R^d)$.

We now take $0 < T < \pi$ and consider the action

$$\mathcal{A}_T(\sigma) := \int_0^T L(\sigma_t, \mathbf{v}_t)dt, \qquad \sigma \in \mathcal{C}_T(\mu, \bar{\mu}).$$

Here $\mathcal{C}_T(\mu, \bar{\mu})$ denotes the set all paths in $AC^2(0, T; \mathcal{P}_2(I\!R^d))$ connecting two given probabilities $\mu,\ \bar{\mu} \in \mathcal{P}_2(I\!R^d)$. The Borel map $\mathbf{v} : I\!R^d \times (0, T) \to I\!R^d$ is the velocity of minimal norm associated to σ. By that we mean first that in the sense of distribution, the continuity equation $\partial_t \sigma_t + \nabla_y \cdot (\sigma_t \mathbf{v}_t) = 0$ holds in $I\!R^d \times (0, T)$. Secondly, the norm $\|\mathbf{v}_t\|_{L^2(\sigma_t)}$ is the metric derivative of $|\sigma'|(t)$ for $\mathcal{L}^1$–almost every $t \in (0, T)$. As a consequence, $\mathbf{v}_t \in \mathcal{T}_{\sigma_t}\mathcal{P}_2(I\!R^d)$ for these t (we refer the reader to section 8.3 of [2]). We prove that any critical path for $\mathcal{A}_T$ is a solution for the Euler-Poisson system in the sense of distributions. Using a direct method for proving that $\mathcal{A}_T$ attains a minimizer which is unique in $\mathcal{C}_T(\mu, \bar{\mu})$ seems arduous, mainly because the existence result is complicated by the negative term appearing in L. It prevents us from infering that $\mathcal{A}_T$ satisfies any reasonable lower semicontinuity property useful to our purpose. Also, it is not clear that $\mathcal{A}_T$ is strictly convex in a sense to be specified. When the space dimension $d = 1$, we achieve our goal by switching to the Lagrangian formulation. The key factor here is that for any $\mu,\ \bar{\mu} \in \mathcal{P}_2(\mathbb{R})$, one has $W_2(\mu, \bar{\mu}) = \|M_\mu - M_{\bar{\mu}}\|_{L^2(\nu_0)}$, where M_μ denotes the optimal map such that $M_{\mu\#}\nu_0 = \mu$. We use this property in lemma 2.2 to show that $\sigma \in AC^2(0, T; \mathcal{P}_2(I\!R))$ is equivalent to $M \in H^1(0, T; L^2(\nu_0))$ and $|\sigma'|(t) = |M'|(t)$ for a.e. $t \in (0, T)$. Here we have set $M_t := M_{\sigma_t}$. As a consequence $\|v_t\|_{L^2(\sigma_t)} = |M'|(t)$ for a.e. $t \in (0, T)$. In fact, we manage to prove a stronger property which lead to $|\sigma'|(t) = |M'|(t)$. We establish the identity $M_t' = v_t \circ M_t$ for a.e. $t \in (0,T)$. Here, M' is the functional derivative of M as recalled in (12). Note that here σ_t may be singular and v_t may be completely non-smooth. These facts show that one can switch between the Eulerian and Lagrangian formulations. Before making the latest statement more accurate we point out that the remarkable identity $M_t' = v_t \circ M_t$ yields conservation of the Hamiltonian H along paths minimizing the action $\mathcal{A}_T$. As a consequence of the 1–d setting, one has

$$W_2^2(\mu, \nu_0) = \int_{I\!R} y^2 d\mu(y) - \frac{1}{2}\int_{I\!R^2} |y-\bar y| d\mu(y) d\mu(\bar y) + \frac{1}{12}. \tag{5}$$

The main problem we focus on in this work is a minimization problem in Lagrangian coordinates. By that we mean analyzing a new action $M \to \mathcal{Q}(M) + \mathcal{C}(M)$ on $AC^2(0,T;L^2(\nu_0))$ after the identification of $AC^2(0,T;L^2(\nu_0))$ with the Hilbert space $H^1(0,T;L^2(\nu_0))$ (as in remark 1.1.3 [2]). It consists of a quadratic term $\mathcal{Q}(M)$ and a convex term $\mathcal{C}(M)$ given by

$$2\mathcal{Q}(M) = \int_0^T |M'|^2(t)dt - \int_0^T dt \int_X |M_t|^2 d\nu_0, \quad 4\mathcal{C}(M) = \int_{X^2} |M_t x - M_t \bar x| dx d\bar x.$$

By the previous comments, $\mathcal{A}_T(\sigma) + T/24 = \mathcal{Q}(M) + \mathcal{C}(M)$. Here, as before, we have assumed that $M_t \in L^2(\nu_0)$ is the optimal map that pushes ν_0 forward to σ_t. Let us prescribe the initial and final maps $\bar M_0, \bar M_T \in L^2(\nu_0)$. Let $\mathcal{C}_T(\bar M_0, \bar M_T)$ be the set of $M \in AC^2(0,T;L^2(\nu_0))$ such that $M_0 = \bar M_0$ and $M_T = \bar M_T$. We prove existence and uniqueness of a minimizer of $\mathcal{Q}(M) + \mathcal{C}(M)$ over $\mathcal{C}_T(\bar M_0, \bar M_T)$ if $\bar M_0, \bar M_T \in L^2(X)$. The Euler-Lagrange equation satisfied by the minimizer M^1 is

$$(M^1)_t'' x + M_t^1 x = \frac{1}{2}\int_X W^1(x, \bar x, t) dz. \tag{6}$$

Here $W^1(x,\bar x,t) \in \partial. |\cdot|(M_t^1 x - M_t^1 \bar x)$ is such that $W^1(\bar x, x, t) = -W^1(x,\bar x,t)$. In fact, since $\mathcal{Q}+\mathcal{C}$ is shown to be strictly convex on $\mathcal{C}_T(\bar M_0, \bar M_T)$, (6) characterizes completely its minimizers over $\mathcal{C}(\mu_0, \mu_T)$. We show that if $\bar M_0$ and $\bar M_T$ are monotone nondecreasing, then so are M_t for $\mathcal{L}^1$–almost every $t \in (0,T)$. Note that we are not claiming that $\mathcal{Q}+\mathcal{C}$ is even convex on $H^1(0,T;L^2(\nu_0))$. We have imposed that $T < \pi$ in order to use Poincaré's inequality in remark 3.4 and obtain strict convexity of $\mathcal{Q}$ on the smaller set $\mathcal{C}(\mu_0, \mu_T)$.

Due to the lack of differentiability of $|\cdot|$ on the real line, we could not establish (6) by a direct argument. Our strategy was to introduce the function $|\cdot|^s$ which is of class $C^1(I\!R)$ for $s > 1$. We then replace the expression $\mathcal{C}(\sigma)$ by

$$\mathcal{C}^s(\sigma) = 1/4 \int_{X^2} |M_t x - M_t \bar x|^s dx d\bar x.$$

It is easy to derive (40) as the Euler-Lagrange equation satisfied by the minimizer M^s of $\mathcal{Q}(M) + \mathcal{C}^s(M)$ over $\mathcal{C}_T(\bar M_0, \bar M_T)$. We then let s tend to 1 to obtain (6).

The characterization of minimizers in (6) is employed to prove a remarkable result: if the two endpoints are averages of n Dirac masses, then so is the minimizing path. We refer to this as the closedness principle of $\mathcal{P}_{\cdot n}$, the set of averages of n Dirac masses. Unlike this closedness principle, we provide an example showing the following: if the two endpoints are absolutely continuous with respect to the Lebesgue measure, then the minimizing path is not necessarily so. This is in contrast with the case of minimizing paths for the Lagrangian $\|\xi\|^2_{L^2(\mu)}$ (i.e. geodesics in the Wasserstein space), which satisfies the property that if the left endpoint is absolutely continuous with respect to the Lebesgue measure, then so are all measures on the geodesic except possibly the right endpoint [21]. Our example also leads to a situation of nonuniqueness of energy preserving, entropy solutions for the initial-value problem.

For any $x \in X$, let $S(x, \cdot)$ be the solution of $S''(x,t)+S(x,t) = x,\ S(x,0) = M_0 x,\ S(x,T) = M_T x$. Then $S(x,t)$ is given by

$$S(x,t) = (M_0 x - x)\left(\cos t - \frac{\cos T}{\sin T}\sin t\right) + (M_T x - x)\frac{\sin t}{\sin T} + x.$$

Note that depending on M_0 and M_T, the function $x \mapsto S(x,t)$ may fail to be monotone nondecreasing and hence cannot be a solution of (6). However, if $S(\cdot,t)$ is increasing for all $t \in (0,T)$, then the path $t \mapsto S(\cdot,t)_{\#}\nu_0$ is a minimizing path such that $S(\cdot,t)_{\#}\nu_0 << \mathcal{L}^1$ for all $t \in [0,T]$. These measures can be computed explicitly. Conversely, if the minimizing path σ satisfies $\sigma_t << \mathcal{L}^1$ for all $t \in [0,T]$, then we must have $\sigma_t := S(\cdot,t)_{\#}\nu_0$.

A formulation similar to (2) appeared in [13], where the initial value problem for a variant of the Euler-Poisson system (attractive, with zero background charge) was studied for $\sigma_0 << \mathcal{L}^1$ or σ_0 purely atomic. The similarities stop here, as these authors were interested in global solutions satisfying standard entropy conditions. In previous studies such as [8] and [13], one seeks for solutions of the Euler-Poisson system which satisfy the property that when two particles collide, they stick together. Here, we do not impose that condition but obtain that the minimizer of the action $\mathcal{A}_T$ satisfies a similar property. By considering the simpler case of the evolution of two particles one discovers that two particles can collide, stick together for some time and then split ways again. But this kind of interaction can occur only once in any time interval of length at most π (see Remark (4.13)).

The analysis described above, however, takes full advantage of the assumption $d = 1$ through the isometric identification of $(\mathcal{P}_2(\mathbb{R}), W_2)$ with a closed, convex subset of $L^2(\nu_0)$. The isometric identification mentioned above fails in higher dimensions. For dimensions $d > 1$ we have only been able to check that the Euler-Lagrange equation for the action $\mathcal{A}_T$ considered above is the so-called Euler-Monge-Ampère system.

We hope that our results will be preliminary to the search for minimizing invariant measures for the Euler-Poisson system. For finite-dimensional dynamical systems, the Mather-Mañé theory (see [18] [20]) offers an elegant variational approach to the problem of existence of minimizing invariant measures. This theory is closely related to the weak KAM theory in which the associated Hamilton-Jacobi equation plays a decisive role [14]. In the finite-dimensional setting, Hamilton's ODE's describe the characteristics of the associated Hamilton-Jacobi equation. In our setting, the Euler-Poisson system describes, in some sense, the characteristics of the following infinite-dimensional Hamilton-Jacobi equation

$$\frac{\partial U}{\partial t}(\mu,t) + \frac{1}{2}\left\|\nabla_\mu U(\mu,t)\right\|^2_{L^2(\mu)} + \frac{1}{2}W_2^2(\mu,\nu_0) = 0.$$

Here, the initial data $U(\mu,0) = U_0(\mu)$ is prescribed. We have denoted by $\nabla_\mu U(\mu,t)$ the gradient of U with respect to the Wasserstein distance. Its precise definition appears in the next section. We have proved that Bellman's variational formula provides a viscosity solution (extension of the finite dimensional concept). To obtain a principle analogous to the closedness principle, we fixf $0 < t < \pi/2$ and $\mu = (1/n)\sum_{i=1}^n \delta_{x_{i,t}}$. We prove that the optimal path provided by Bellman's formula satisfies $\sigma_s = (1/n)\sum_{i=1}^n \delta_{x_i(s)}$ with $x_i(t) = x_{i,t}$. Furthermore, the $H^2(0,t;I\!R^n)$ path $\mathbf{x}(s) = (x_1(s), ..., x_n(s))$ is optimal for the classical, finite-dimensional Hamilton-Jacobi equation $\partial_t U^n + H^n\big(\mathbf{x}, \nabla U^n\big) = 0$. Here $H^n : I\!R^n \times I\!R^n \to I\!R$ is the restriction of H to $I\!R^n$. In other words, $H^n(\mathbf{x}, \mathbf{p}) = H(\mu_{\mathbf{x}}, \zeta_{\mathbf{p}})$, where $\mu_{\mathbf{x}} = (1/n)\sum_{i=1}^n \delta_{x_i}$ and $\zeta_{\mathbf{p}} : \{x_1, ..., x_2\} \to \{p_1, ..., p_n\}$ is given by $\zeta_{\mathbf{p}}(x_i) = p_i$, $i = 1, ..., n$. Uniqueness of viscosity solutions remains open even in the finite-dimensional case, mainly due to the fact that H (as well as H^n) is not jointly C^1.

The plan of the paper is as follows: the next section contains some notation and useful preliminaries. This includes general compactness results and a Poincaré-Wirtinger inequality for $AC^2(0,T;\mathcal{S})$ for general complete metric spaces $\mathcal{S}$. It also covers properties of optimal maps in the mass transport problem. In section 3 we prove that Euler-Lagrange equation for the action of L is the Euler-Monge-Ampère system in arbitrary dimensions. Section 4 is restricted to the one-dimensional case and contains proofs of many of the main results announced above. The infinite-dimensional Hamilton-Jacobi equation with its connections to the standard finite-dimensional setting is the subject of Section 5.

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2 Preliminaries

In this section we record some notation and definitions used throughout the manuscript. Here $T \in (0,\infty)$ is fixed. We recall well-known facts about the one-dimensional Monge-Kantorovich mass transport theory. A special attention will be devoted to the monotone maps $M : X = (-1/2, 1/2) \to I\!R$ which are square integrable. We recall the concept of metric derivative of an absolutely continuous path in a complete metric space. One of the spaces we consider is $\mathcal{P}_2(I\!R)$, the set of Borel probability measures on $I\!R$ with finite second moments, endowed with the Wasserstein distance. Another space which naturally appears is $L^2(X,\nu_0)$, the set of square integrable functions on X. Here, ν_0 is the restriction to X of the one-dimensional Lebesgue measure. We often denote it by $L^2(\nu_0)$. A specific subset of it is known to be isometric to $\mathcal{P}_2(I\!R)$. We elaborate on that fact in remark 2.1 (ii) as it will be significantly exploited in this work. Suppose next that ν_0 is a Borel probability measure on $I\!R^d$ and let $L^2(\nu_0)$ be the set of $M : I\!R^d \to I\!R^d$ that square integrable with respect to ν_0. Let $\langle\cdot,\cdot\rangle_{\nu_0}$ be the standard inner product on $L^2(\nu_0)$. We will recall the well-known identification of $L^2(0,T;L^2(\nu_0))$ and $L^2(\nu_0 \times \mathcal{L}^1|_{(0,T)})$. We consider the space $H^1(0,T;L^2(\nu_0))$ which consists of the $M \in L^2(0,T;L^2(\nu_0))$ such that the functional derivative M' exists in $L^2(0,T;L^2(\nu_0))$. This $H^1(0,T;L^2(\nu_0))$ is a Hilbert space when endowed with the inner product

$$\langle M, N\rangle = \int_0^T (\langle M_t, N_t\rangle_{\nu_0} + \langle M_t', N_t'\rangle_{\nu_0})dt.$$

2.1 Notation and Definitions

-We suppose that $T>0$ is a constant. We sometimes give it a specific value such as $T=1$.
- $|\cdot|$ is the euclidean norm on $\mathbb{R}^d$ and $\langle\cdot,\cdot\rangle$ is the standard inner product.
- $C_c^\infty(\mathbb{R}^d)$ is the set of functions on $\mathbb{R}^d$ which are infinitely differentiable and of compact support.
- If $\psi:\mathbb{R}^d\to\mathbb{R}\cup\{\pm\infty\}$, ψ^* is its Legendre transform.
- **id** denotes the identity map on $\mathbb{R}^d$ for $d\geq 1$.
- As usual, we denote by $\mathcal{L}^d$ the Lebesgue measure on $\mathbb{R}^d$.
- X denotes the unit cube in $\mathbb{R}^d$, centered at the origin. In particular if $d=1$ then $X=(-1/2,1/2)$. We set $X_T:=X\times(0,T)$. Similarly, $X\times X\times(0,T)=X_T^2$. The measure ν_0 is the restriction of $\mathcal{L}^d$ to X and so it is a Borel probability measure. We write $\nu_0=\mathcal{L}^d|_X$. The product measure of ν_0 by $\mathcal{L}^1|_{(0,T)}$ is the measure on X_T denoted ν. We do not display explicitly its dependence on T since this does not create any confusion in this manuscript.
- $\mathcal{P}_2(\mathbb{R}^d)$ stands for the set of Borel probability measures μ on $\mathbb{R}^d$ with finite second moments:

$$\int_{\mathbb{R}^d}|y|^2d\mu(y)<\infty. \tag{7}$$

- Given $\mu,\nu\in\mathcal{P}_2(\mathbb{R}^d)$, $\Gamma(\mu,\nu)$ is the set of Borel probability measures on $\mathbb{R}^d\times\mathbb{R}^d$ which have μ and ν as their marginals. The Wasserstein distance W_2 between μ and ν is defined by

$$W_2^2(\mu,\nu)=\min_{\gamma\in\Gamma(\mu,\nu)}\int_{\mathbb{R}^{2d}}|x-y|^2d\gamma(x,y).$$

The set of γ where the minimum is achieved is nonempty and is denoted by $\Gamma_o(\mu,\nu)$. We refer the reader to [2] chapter 7 for the properties of W_2 and $\Gamma_o(\mu,\nu)$. $(\mathcal{P}_2(\mathbb{R}^d),W_2)$ is a metric space, complete and separable. We set $\mathcal{M}:=\mathcal{P}_2(\mathbb{R})$.
- If $\mu\in\mathcal{P}_2(\mathbb{R}^d)$, $L^2(\mu)$ is the set of function $\xi:\mathbb{R}^d\to\mathbb{R}^d$ which are μ measurable and such that $\int_{\mathbb{R}^d}|\xi|^2d\mu$ is finite. This is a separable Hilbert space for the inner product $\langle\xi,\bar\xi\rangle_\mu=\int_{\mathbb{R}^d}\langle\xi,\bar\xi\rangle d\mu$. We denote the associated norm by $\|\cdot\|_\mu$. When $m=\mathcal{L}^1|_{(0,T)}$ to distinguish between the space and time variables, we write $\|\cdot\|_{L^2(0,T)}$ for $\|\cdot\|_m$.
- If $\mu\in\mathcal{P}_2(\mathbb{R}^d)$, we denote by $\mathcal{T}_\mu\mathcal{P}_2(\mathbb{R}^d)$ the closure of $\{\nabla\varphi:\varphi\in C_c^\infty(\mathbb{R}^d)\}$ in $L^2(\mu)$. We refer to $\mathcal{T}_\mu\mathcal{P}_2(\mathbb{R}^d)$ as the tangent space to $\mathcal{P}_2(\mathbb{R}^d)$ at μ (see section 8.5 of [2]). When $d=1$ it is easy to check that $\mathcal{T}_\mu\mathcal{P}_2(\mathbb{R})=L^2(\mu)$.
- If $(Z,|\cdot|)$ is a norm space, $L^2(0,T;Z)$ is the set of Borel functions $M:(0,T)\to Z$ such that $\int_0^T|M_t|_Z^2dt<\infty$. Here and throughout this work, we write M_t in place of $M(t)$. When μ is a Borel probability measure on $\mathbb{R}^d$ and $Z=L^2(\mu)$, we identify $L^2(0,T;L^2(\mu))$ with $L^2(\mu\times\mathcal{L}^1|_{(0,T)})$.
- If $U:\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}\cup\{\pm\infty\}$, we denote by $\partial.U(\mu)$ the subdifferential of U at μ, as introduced in [1] (see definition 5.3). As shown in [1], for λ-convex functionals, this definition coincides with the one in (10.3.12) of [2]. Since $\|\cdot\|_\mu$ is uniform and $\partial.U(\mu)$ is a closed convex subset of $L^2(\mu)$, it admits a unique element element of minimal norm. As it is customary in subdifferential analysis, we denote that element by $\nabla_\mu H(\mu)$. We refer to it as the gradient of U with respect to the Wasserstein distance W_2. The super differential of U at μ is denoted by $\partial^\cdot(\mu)$ and consists of the ξ such that $-\xi$ belongs to $\partial.(-U)(\mu)$. We say that U is differentiable at μ if $\partial.U(\mu)$ and $\partial^\cdot U(\mu)$ are nonempty. In that case, both sets coincide.

- We also recall that if $M : I\!R^d \to I\!R^d$ is a Borel map and $\mu \in \mathcal{P}_2(I\!R^d)$ then $M_\# \mu$ is the Borel measure defined by

$$M_\# \mu[C] = \mu[M^{-1}(C)] \quad \text{for all Borel sets} \quad C \subset I\!R^d. \tag{8}$$

- If $\mu,\ \nu$ are Borel probability measures on the real line and μ is atom-free, then it is known that there exists a unique (up to a set of μ–zero measure) optimal map pushing forward μ to ν. It is called *the monotone rearrangement* and is obtained as $G^{-1} \circ F$, where $F,\ G$ are the cumulative distribution functions of μ and ν. We have

$$G(y) = \nu(-\infty, y] \text{ and } G^{-1}(x) = \inf\{y \in I\!R : G(y) \geq x\}.$$

Note that G^{-1} is the left-continuous *generalized inverse* of G (in [24] the right-continuous one is considered). In this work, optimal map on the real line always means *left continuous* optimal map.
- We denote by $\mathcal{M}on$ the set of monotone nondecreasing functions $M : (-1/2, 1/2) \to I\!R$ which are in $L^2(\nu_0)$.
- Suppose $(\mathcal{S}, dist)$ is a complete metric space and $\sigma : (0,T) \to \mathcal{S}$. We write σ_t to denote the value of σ at $t : \sigma_t := \sigma(t)$. If there exists $\beta \in L^2(0,T)$ such that

$$dist(\sigma_t, \sigma_s) \leq \int_s^t \beta(u)du \tag{9}$$

for every $s < t$ in $(0,T)$, we say that σ is absolutely continuous. We denote by $AC^2(0,T;\mathcal{S})$ the set of $\sigma : (0,T) \to \mathcal{S}$ that are absolutely continuous.
- Suppose $\sigma \in AC^2(0,T;\mathcal{S})$. Since $\mathcal{S}$ is complete, $\lim_{t\to 0^+} \sigma_t$ exists and will be denoted σ_0. Similarly, σ_T is well-defined. For $\mathcal{L}^1$–almost every $t \in (0,T)$

$$|\sigma'|(t) := \lim_{h\to 0} \frac{dist(\sigma_{t+h}, \sigma_t)}{|h|} \tag{10}$$

exists. If the above limit exists at t, we say that $|\sigma'|$ exists at t. We have $|\sigma'| \leq \beta$ for every β satisfying (9) and

$$dist(\sigma_t, \sigma_s) \leq \int_s^t |\sigma'|(u)du. \tag{11}$$

The function $|\sigma'|$ is refered to as the metric derivative of σ. For more details, we refer the reader to section 1.1 of [2]. We denote the L^2–norm of $|\sigma'|$ on $(0,T)$ by $\|\sigma'\|_{metric,T}$. In case there is no confusion about the time interval on which we integrate, we simply write $\|\sigma'\|_{metric}$.
- Suppose $s, \bar{s} \in \mathcal{S}$. We denote by $\mathcal{C}_T(s,\bar{s})$ the set of curves $\sigma \in AC^2(0,T;\mathcal{S})$ such that $\sigma_0 = s$ and $\sigma_T = \bar{s}$. Similarly, $\mathcal{C}_T(\cdot,\bar{s})$ denotes the set of curves $\sigma \in AC^2(0,T;\mathcal{S})$ such that $\sigma_T = \bar{s}$.
- If n is a integer, $\mathcal{P}._n$ is the set of n averages of n Dirac masses in $I\!R$. When $d = 1$, we divide $X = (-1/2, 1/2)$ into n intervals of equal length. Recall that ν_0 is the restriction to X of the one-dimensional Lebesgue measure. Suppose $N, \bar{N} \in L^2(\nu_0) =: \mathcal{S}$ and are constant on each of these subintervals. We denote by $\mathcal{C}_T^n(N,\bar{N})$ the set of M in $\mathcal{C}_T(N,\bar{N})$ such that for each $t \in (0,T)$, M_t is constant on each of these subintervals.

2.2 Optimal maps

In this subsection, we recall well-known facts about optimal mass transportation theory. We refer the reader to [2] and [24] for more details.

If $\mu, \bar\mu \in \mathcal{P}_2(I\!R)$ and μ vanishes on $(d-1)$–rectifiable sets, then $\Gamma_o(\mu,\bar\mu)$ reduces to a single element $\{\gamma_0\}$. In that case, $\gamma_0 = (\mathbf{id} \times \nabla\phi)_\# \mu$ for some $\phi : I\!R^d \to I\!R \cup \{+\infty\}$ convex lower semicontinuous (see [2] chapter 6 and [15]). The map $\nabla\phi$ is the unique (up to a set of μ–zero measure) optimal map that such that $\nabla\phi_\# \mu = \bar\mu$. What we mean is that $\nabla\phi$ is the unique map that minimizes $M \to \int_{I\!R^d} |\mathbf{id} - M|^2 d\mu$ over the set of Borel maps M satisfying $M_\# \mu = \bar\mu$. In fact, the definition of M matters only on the support of μ. When $d = 1$ and $X = (-1/2, 1/2) \subset I\!R$, ν_0 are as above and $\mu = \nu_0$ then there is a monotone nondecreasing function $M : X \to I\!R$ such that $M_\# \nu_0 = \bar\mu$. It is uniquely defined up to a set of ν_0–measure and is the optimal map that pushes ν_0 forward to $\bar\mu$. Since $\int_{I\!R} y^2 d\bar\mu(y) = \|M\|^2_{\nu_0} < \infty$ and M monotone, it achieves only finite values in X. Hence, the set of discontinuity of M is at most countable. The monotone nondecreasing map can be described explicitly. We next write the expression of the one which is left continuous and which will be used throughout this work. For $y \in I\!R$ and $x \in X$, set

$$N_{\bar\mu}(y) = \bar\mu(-\infty, y] - 1/2, \quad M_{\bar\mu}(x) = \inf_{z \in I\!R} \{z : N_{\bar\mu}(z) \geq x\}.$$

Then $M_{\bar\mu \#} \nu_0 = \bar\mu$ and $W_2(\mu,\bar\mu) = \|M_\mu - M_{\bar\mu}\|_{\nu_0}$. In the next remark we comment on how this well-known identity may be established.

Remark 2.1. *(i) If $M_0, M_T : X \to I\!R^d$ are Borel maps and $M_{t\#}\nu_0 = \sigma_t$ for $t = 0, T$ then $(M_0 \times M_T)_\# \nu_0$ has σ_0 and σ_T as its marginals and so, $W_2^2(\sigma_0, \sigma_T) \leq \int_{I\!R^2} |y - \bar y|^2 d\gamma(y, \bar y) = \|M_0 - M_T\|^2_{\nu_0}$. Hence, if $M \in AC^2(0,T; L^2(\nu_0))$ and $\sigma_t := M_{t\,\#}\nu_0$ then $\sigma \in AC^2(0,T; \mathcal{P}_2(I\!R^d))$. We then exploit the expression of the metric derivative given in (10) to conclude that $|\sigma'|(t) \leq |M'|(t)$ for $\mathcal{L}^1$-almost every $t \in (0,T)$. As a consequence, $\|M'\|_{metric} \geq \|\sigma'\|_{metric}$.*
(ii) Suppose in addition that $d = 1$, M_0 and M_T are monotone nondecreasing. For each n integer, we choose $\sigma_0^n \in \mathcal{P}_2(I\!R)$ absolutely continuous with respect to $\mathcal{L}^1$, of positive density, such that $W_2(\sigma_0^n, \sigma_0) \leq 1/n$. Let $M_0^n : X \to I\!R$ be monotone increasing satisfying $M^n_{0\#}\nu_0 = \sigma_0^n$. The map M_0^n admits an inverse $N^n : I\!R \to X$ which is monotone increasing. Since $M_T \circ N^n$ is monotone nondecreasing and pushes σ_0^n forward to σ_T, we conclude that $W_2^2(\sigma_0^n, \sigma_T) = \|\mathbf{id} - M_T \circ N^n\|^2_{\sigma_0^n}$. The last term is checked to be $\|M_0^n - M_T\|^2_{\nu_0}$. Letting n tends to ∞, we conclude that $W_2^2(\sigma_0, \sigma_T) = \|M_0 - M_T\|^2_{\nu_0}$.

A direct consequence of remark 2.1 is the following lemma.

Lemma 2.2. *Suppose $d = 1$. Suppose $\sigma \in L^2(0,T; \mathcal{P}_2(I\!R))$, $M \in L^2(0,T; L^2(\nu_0))$ are such that M_t is monotone nondecreasing and $M_{t\#}\nu_0 = \sigma_t$. Then $\sigma \in AC^2(0,T; \mathcal{P}_2(I\!R))$ if and only if $M \in AC^2(0,T; L^2(\nu_0))$. In that case $|M'|(t)$ exists if and only if $|\sigma'|(t)$ exists. Both functions coincide where they exist.*

Proof: The proof is a direct consequence of remark 2.1 and (10). QED.

2.3 The spaces $AC^2(0,T;L^2(\nu_0))$ and $AC^2(0,T;\mathcal{P}_2(I\!R^d))$

Recall that

$$X = (-1/2,1/2)^d, \quad X_T = X \times (0,T), \quad \nu_0 = \mathcal{L}^d|_X, \quad \nu := \nu_0 \times \mathcal{L}^1|_{(0,T)}.$$

One can notice that the next two lemmas are still valid if we replace X by an open subset of $I\!R^d$. The first lemma of this section recalls the standard identification of $L^2(0,T;L^2(\nu_0))$ and $L^2(\nu)$. It allows not to distinguish between these two spaces. Given N as in the lemma below, replacing N by $\tilde{N}$ if necessary, we shall always use the convention that $N \equiv \tilde{N}$. Since the proof of the lemma is standard, it will be skipped.

Lemma 2.3. *If $N \in L^2(0,T;L^2(\nu_0))$, then there exists $\tilde{N} \in L^2(\nu)$ such that*

$$\int_0^T dt \int_X N_t(x)\psi(t,x)\, dx = \int_{X_T} \tilde{N}(x,t)\psi(x,t)dxdt \quad \textit{for all functions } \psi \in L^2(\nu).$$

Furthermore, $t \to \tilde{N}(\cdot,t)$ belongs to $L^2(0,T;L^2(\nu_0))$ and for $\mathcal{L}^1$-a.e. t, $\tilde{N}(x,t) = N_t(x)$ for ν_0–almost every $x \in X$.

The next lemma is also elementary and so, its proof will not be given.

Lemma 2.4. *Suppose that $\{M\} \cup \{M^n\}_{n=1}^\infty \subset AC^2(0,T;L^2(\nu_0))$ satisfies*

$$\|M^n\|_{AC^2(0,T;L^2(\nu_0))},\ \|M\|_{AC^2(0,T;L^2(\nu_0))} \leq C$$

for a constant $C > 0$. Suppose that for each $t \in (0,T)$, $\{M_t^n\}_{n=1}^\infty$ converges weakly to M_t in $L^2(\nu_0)$. Then $\{M^n\}_{n=1}^\infty$ converges weakly to M in $L^2(\nu)$ and $\{(M^n)'\}_{n=1}^\infty$ converges weakly to $\{M'\}$ in $L^2(\nu)$.

In the remainder of this subsection $d = 1$, so that $X = (-1/2, 1/2)$. The purpose of the next two lemmas is to show that if $M \in AC^2(0,T;L^2(\nu_0))$ and M_t is monotone nondecreasing and left continuous for each t, then $(t,x) \to M_t x$ is a Borel map. The point is that we do not need to modify $M_t x$ on a set of $\mathcal{L}^2$–zero measure to obtain a Borel map.

Lemma 2.5. *Let $a < b$ be two real numbers and let $M \in AC^2(0,T;L^2(a,b))$. Suppose that for each t, the function $M_t : (a,b) \to I\!R$ is monotone, nondecreasing and continuous. Then $(t,x) \to M_t x$ is continuous on $(a,b) \times (0,T)$.*

Proof: We skip the proof of this lemma since it is an elementary exercise. We give a hint which is based on the following fact on the class of $C^1(a,b)$–convex functions. Suppose $\{f_n\}_{n=1}^\infty \subset C^1(a,b)$ are convex, $f \in C^1(a,b)$ is convex and $\|f_n\|_{L^1(a,b)}$ is bounded. Then $\{f_n\}_{n=1}^\infty$ converges weakly in $L^1_{loc}(a,b)$ to f if and only if it converges pointwise in (a,b) to f. This is also equivalent to $\{f_n\}_{n=1}^\infty$ converges in $C^0_{loc}(a,b)$ to f and $\{f_n'\}_{n=1}^\infty$ converges pointwise in (a,b) to f'. Since monotone maps are derivatives of convex functions, one establishes the lemma. QED.

Lemma 2.6. *Let $R \in C_c^1(I\!R)$ be a nonnegative function such that $\int_{I\!R} R(y)\,dy = 1$. Suppose $N : X \to I\!R$ is a locally bounded function and $N_-(x) := \lim_{y\to x^-} N(y)$, $N_+(x) := \lim_{y\to x^+} N(y)$ exist for all x in X. Set*

$$N^\epsilon = R_\epsilon * N,$$

where $R_\epsilon(z) = \frac{1}{\epsilon}R(\frac{z}{\epsilon})$. Then, for any $x \in (-\frac{1}{2}, \frac{1}{2})$, we have

$$\lim_{\epsilon\to 0} N^\epsilon(x) = \lambda N_+(x) + (1-\lambda)N_-(x),$$

where $\lambda = \int_{-\infty}^0 R(y)\,dy$. As a consequence, the pointwise limit exists everywhere.

Proof: We have

$$N^\epsilon(x) = \int_{-\infty}^0 R(z)N(x-\epsilon z)\,dz + \int_0^\infty R(z)N(x-\epsilon z)\,dz \qquad \forall x \in X.$$

Since N is locally bounded in X, the dominated convergence theorem yields the conclusion. QED.

Proposition 2.7. *Suppose that $M \in AC^2(0,T;L^2(\nu_0))$ and for each t, the function $M_t : X \to I\!R$ is monotone, nondecreasing and left continuous. Then $(x,t) \to M_t x$ is Borel on X_T as a function of two variables.*

Proof: Let R be as in lemma 2.6 such that $\int_{-\infty}^0 R(y)\,dy = 0$. Set $M_t^n = R_{\frac{1}{n}} * M_t$. For $0 < \delta < 1/2$, set $X^\delta = (-\frac{1}{2}+\delta, \frac{1}{2}-\delta)$. Then

$$\|M_t^n - M_s^n\|_{L^2(X^\delta)} \leq \|M_t - M_s\|_{L^2(\nu_0)} \quad \text{for} \quad t,s \in X^\delta.$$

This proves that $M^n \in AC^2(0,T;L^2(X^\delta)$. By Lemma 2.5 we obtain the map $(x,t) \mapsto M_t^n(x)$ is continuous on X_T^δ. By lemma 2.6 $\lim_{n\to\infty} M_t^n(x) = M_t x$ for each $(x,t) \in X_T$. Thus, M is Borel measurable on X_T as a pointwise limit of Borel map. QED.

Observe that the spaces $AC^2(0,T;L^2(\nu_0))$ and $H^1(0,T;L^2(\nu_0))$ coincide (see Remark 1.1.3 of [2]). If $M \in H^1(0,T;L^2(\nu_0))$ we denote by $M' \in L^2(0,T;L^2(\nu_0))$ its functional derivative. It is defined by

$$\lim_{h\to 0} \left\| \frac{M_{t+h} - M_t}{h} - M_t' \right\|_{\nu_0} = 0 \quad \text{for} \quad \mathcal{L}^1-\text{a.e.} \quad t \in (0,T). \tag{12}$$

It is straightforward to check that $|M'|(t) = \|M_t'\|_{\nu_0}$ for $\mathcal{L}^1$–a.e. $t \in (0,T)$. In the next lemma, we shall view M as a map in $AC^2(I\!R;L^2(\nu_0))$ by extending $M_t = M_{0+}$ for $t \leq 0$ and $M_t = M_{T-}$ for $t \geq T$. Recall that M' can be viewed as an element of $L^2(X_T)$. We obtain an extension of M' to $X \times I\!R$ which we identify with an element of $L^2(\nu_0 \times \mathcal{L}^1)$.

Lemma 2.8. *Let $M \in AC^2(0,T;L^2(\nu_0))$ and M' be its functional derivative. Then*

$$\lim_{h\to 0} \int_{X_T} \left| \frac{M_{t+h}x - M_t x}{h} - M_t' x \right|^2 dxdt = 0. \tag{13}$$

As a consequence, there exist sequences $h_k^+ \to 0^+$, $h_k^- \to 0^-$ *and a measurable subset* A *of* $X \times I\!R$ *such that* $\mathcal{L}^2((X \times I\!R) \setminus A) = 0$ *and*

$$\lim_{k\to\infty} \frac{M_{t+h_k^+}x - M_t x}{h_k^+} = \lim_{k\to\infty} \frac{M_{t+h_k^-}x - M_t x}{h_k^-} = M_t' x \tag{14}$$

for all $(x,t) \in A$.

Proof: Set $g(t) = \|M_t'\|_{\nu_0}$ and let $g^*(t)$ be the Hardy-Littlewood maximal function given by

$$g^*(t) = \sup_h \left| \frac{1}{2h} \int_{t-h}^{t+h} g(s)\, ds \right|, \quad t \in I\!R.$$

Note that $g \in L^2(I\!R)$ and so, $g^* \in L^2(I\!R)$. Clearly, $\|(M_{t+h} - M_t)/h - M_t'\|_{\nu_0} \le 2g^*(t) + g(t)$. This, together with (12) and the Lebesgue dominated convergence theorem yields

$$\lim_{h\to 0} \int_0^T \left\| \frac{M_{t+h} - M_t}{h} - M_t' \right\|_{\nu_0}^2 dt = 0.$$

Fubini's theorem implies (13). QED.

Lemma 2.9. *Let* A *be as in the previous lemma. Suppose* $(x,t), (\bar{x},t) \in A$ *and* $M_t x = M_t \bar{x}$. *Then* $M_t' x = M_t' \bar{x}$.

Proof: Without loss of generality, we can assume $x < \bar{x}$. Then

$$\frac{M_{t+h_k^-}\bar{x} - M_{t+h_k^-}x}{h_k^-} \le 0 \le \frac{M_{t+h_k^+}\bar{x} - M_{t+h_k^+}x}{h_k^+}.$$

By (14), this yields $M_t' x = M_t' \bar{x}$. QED.

Remark 2.10. *Let* $\partial_t M$ *denote the distributional derivative in time of* $(x,t) \to M_t x = M(x,t)$. *Then,* $M_t' x = \partial_t M(x,t)$ *for* $\mathcal{L}^2$*-almost every* $(x,t) \in X_T$.

Proof: Let $\xi \in C_c^\infty(X_T)$ be arbitrary. We have

$$\begin{aligned}
&- \int_{X_T} \partial_t M(x,t)\xi(x,t) dx dt = \int_{X_T} M(x,t)\partial_t \xi(x,t) dx dt = \int_0^T dt \int_X M(x,t) \partial_t \xi(x,t) dx \\
&= \lim_{h\to 0} \int_0^T dt \int_X M(x,t) \frac{\xi(x,t+h) - \xi(x,t)}{h} dx = \lim_{h\to 0} \int_X dx \int_0^T M(x,t) \frac{\xi(x,t+h) - \xi(x,t)}{h} dt \\
&= \lim_{h\to 0} \int_X dx \int_0^T \frac{M(x,t-h) - M(x,t)}{h} \xi(x,t) dt = \int_X dx \int_0^T M_t' x \xi(x,t) dt \\
&= \int_{X_T} M_t' x \xi(x,t) dx dt.
\end{aligned}$$

This concludes the proof. QED.

Recall that (see [2] theorem 8.3.1) if $\sigma \in AC^2(0,T;\mathcal{P}_2(I\!R^d))$, then there exists a Borel map $(y,t) \to \mathbf{v}_t(y) \in I\!R^d$ such that

$$\partial_t \sigma_t + \nabla_y \cdot (\sigma_t \mathbf{v}_t) = 0 \qquad \text{in} \quad I\!R^d \times (0,T)$$

in the sense of distributions. We call $\mathbf{v}$ a velocity associated to the path σ. One can choose a velocity associated to the path σ so that it is uniquely determined by the following properties: if $\mathbf{w}$ is another velocity associated to the path, then for $\mathcal{L}^1$-almost every $t \in (0,T)$ $\|\mathbf{v}_t\|_{L^2(\sigma_t)} \leq \|\mathbf{w}_t\|_{L^2(\sigma_t)}$ and $\mathbf{v}_t \in \mathcal{T}_{\sigma_t}\mathcal{P}_2(I\!R^d)$. We refer to $\mathbf{v}_t$ as the *tangent velocity field* at σ_t, or the *velocity of minimal norm* associated to σ.

Remark 2.11. *It is known that (see [2] theorem 8.3.1) for $\mathcal{L}^1$–almost every $t \in (0,T)$, we have $\|\mathbf{v}_t\|_{L^2(\sigma_t)} = |\sigma'|(t)$.*

2.4 Analysis on $AC^2(0,T;\mathcal{S})$

Throughout this subsection $(\mathcal{S}, dist)$ is a complete metric space. We assume the existence of a Hausdorff topology τ on $\mathcal{S}$, weaker that the metric topology. Also, suppose there exists a distance $dist_\tau$ such that on bounded subsets of $(\mathcal{S}, dist)$, the topology τ coincides with the distance topology $dist_\tau$. We assume that closed balls of $(\mathcal{S}, dist)$ are compact for τ and that $dist$ is τ–sequentially lower semicontinuous on $B \times B$ whenever B is a closed ball of $(\mathcal{S}, dist)$. For instance, when $(\mathcal{S}, dist) = (L^2(\nu_0), \|\cdot\|_{\nu_0})$, we choose τ to be the weak topology. When $\mathcal{S} = \mathcal{P}_2(I\!R^d)$, $dist$ is the Wasserstein distance, we choose τ to be the narrow convergence topology (see [2] remark 5.1.1). We extend the Poincaré–Wirtinger inequality from $I\!R^d$ to $\mathcal{S}$.

Proposition 2.12. *Suppose that σ belongs to $AC^2(0,T;\mathcal{S})$ and $s_0 \in \mathcal{S}$. Then*

$$dist(\sigma_t, s_0) \leq dist(\sigma_0, s_0) + \sqrt{t}\|\sigma'\|_{metric} \tag{15}$$

and

$$\pi\|dist(\sigma(\cdot), s_0)\|_{L^2(0,T)} \leq T\|\sigma'\|_{metric} + \sqrt{T}\mathcal{W}_\mathcal{S}(\sigma_0, \sigma_T; s_0) \tag{16}$$

where $\mathcal{W}_\mathcal{S}(\sigma_0, \sigma_T; \nu_o)$ is defined below. We also have

$$\pi\|dist(\sigma(\cdot), s_0)\|_{L^2(0,T)} \leq 2T\|\sigma'\|_{metric} + \pi\sqrt{T}dist(\sigma_T, s_0)). \tag{17}$$

Here, $\mathcal{W}_\mathcal{S}(\sigma_0, \sigma_T; s_0) = \pi\Big(dist(\sigma_0, s_0)\, dist(\sigma_T, s_0) + 1/3 dist^2(\sigma_0, \sigma_T)\Big)^{1/2} + dist(\sigma_0, \sigma_T)$.

Proof: Set $u(t) = dist(\sigma_t, s_0)$. Then for all $h > 0$ and $t \in (0,T)$

$$(|u(t+h) - u(t)|)/h \leq dist(\sigma(t+h), \sigma_t)/h.$$

This proves that $u \in H^1(0,T)$ and for $\mathcal{L}^1$–almost every $t \in (0,T)$, $|u'(t)| \leq |\sigma'|(t)$. We exploit this and the identity $u(t) = u(0) + \int_0^t u'(s)ds$ to obtain (15). Set

$$w(t) := u(t) - u(0) + (t/T)(u(0) - u(T)), \qquad \tilde{w}(t) := |\sigma'|(t) + 1/T\, dist(\sigma_0, \sigma_t).$$

Then $w \in H^1_0(0,T)$ and so, by the standard Poincaré inequality, since $|w'| \leq \tilde{w}$,

$$\pi/T\|w\|_{L^2(0,T)} \leq \|w'\|_{L^2(0,T)} \leq \|\tilde{w}\|_{L^2(0,T)} \leq \|\sigma'\|_{metric} + 1/\sqrt{T}\, dist(\sigma_0, \sigma_T).$$

This, together with the fact that

$$\int_0^T ((1-t/T)a + (t/T)b)^2 dt = (T/3)(b-a)^2 + Tab,$$

yields (16). We use that $\pi\|u - u(T)\|_{L^2(0,T)} \leq 2T\|u'\|_{L^2(0,T)}$ to obtain (17). QED.

Proposition 2.13. *Suppose that* $\sigma : [0,T] \to \mathcal{S}$*,* $\{\sigma^n\}_{n=1}^\infty \subset AC^2(0,T;\mathcal{S})$ *and* $\{\sigma_t^n\}_{n=1}^\infty$ *converges to* σ_t *in* $(\mathcal{S}, dist_\tau)$ *for every* $t \in (0,T)$*. If* $\{\sigma_0^n\}_{n=1}^\infty$ *is bounded in* $(\mathcal{S}, dist)$*, then* $\sigma \in AC^2(0,T;\mathcal{S})$ *and*

$$\liminf_{n\to\infty} \|(\sigma^n)'\|_{metric} \geq \|\sigma'\|_{metric}. \tag{18}$$

Proof: Without loss of generality, we assume the left hand side of (18) is finite and let $\{n_k\}_{k=1}^\infty \subset$ **N** be such that

$$C_0 = \lim_{k\to+\infty} \|(\sigma^{n_k})'\|_{metric} = \liminf_{n\to\infty} \|(\sigma^n)'\|_{metric}.$$

The sequence $\{|(\sigma^{n_k})'|\}_{k=1}^\infty$ is bounded in $L^2(0,T)$ and so, admits a subsequence (not relabelled) which converges weakly to some α in $L^2(0,T)$. Since $\{\sigma_0^n\}_{n=1}^\infty$ is bounded in $(\mathcal{S}, dist)$, (11) yields that $\{\sigma_t^n\}_{n=1}^\infty$ is bounded in $(\mathcal{S}, dist)$ for each $t \in (0,T)$. We use (11) again and the fact that *dist* is $dist_\tau$–lower semicontinuous on *dist*-bounded sets to obtain

$$dist(\sigma_t, \sigma_s) \leq \liminf_{k\to+\infty} dist(\sigma_t^{n_k}, \sigma_s^{n_k}) \leq \liminf_{k\to+\infty} \int_s^t |(\sigma^{n_k})'|(u)du = \int_s^t \alpha(u)du \tag{19}$$

for every $0 < s \leq t < T$. This proves that $\sigma \in AC^2(0,T;\mathcal{S})$. By the minimality property of $|\sigma'|$, (19) yields $|\sigma'| \leq \alpha$. This, together with the lower semicontinuity of $\|\cdot\|_{L^2(0,T)}$ gives that

$$\liminf_{k\to+\infty} \|(\sigma^{n_k})'\|_{metric} \geq \|\alpha\|_{L^2(0,T)} \geq \|\sigma'\|_{metric}.$$

This concludes the proof. QED.

The following proposition will be used often in this work. It appears as proposition 3.3.1 in [2].

Proposition 2.14. *Suppose* $\{\sigma_n\}_{n=1}^\infty$ *is a sequence in* $AC^2(a,b;\mathcal{S})$ *satisfying*

$$M := \sup_n \|(\sigma_n)'\|_{metric}, \quad \bar{M} := \sup_n dist(\sigma_n(0), \nu_0) < \infty$$

for some $\nu_0 \in \mathcal{S}$*. Then, there exist* $\sigma \in AC^2(a,b;\mathcal{S})$ *and a sequence* $\{n_k\}_{k=1}^\infty \subset$ **N** *(independent of* t*) such that* $\{\sigma_{n_k}(t)\}_{k=1}^\infty$ *converges to* σ_t *in* $(\mathcal{S}, dist_\tau)$ *for every* $t \in (a,b)$*.*

3 Actions on $AC^2(0,T;\mathcal{P}_2(\mathbb{R}^d))$ and $AC^2(0,T;L^2(\nu_0))$

In this section we assume that $T \in (0,\pi)$. This condition is not needed in subsection 3.1 but is useful in the remaining subsections.

3.1 Euler-Monge-Ampère systems as minimizers of an action

On $AC^2(0,T;\mathcal{P}_2(I\!R^d))$ we consider the action

$$\mathcal{A}_T(\sigma) := \frac{1}{2}\Big(\|\sigma'\|^2_{metric} - \|W_2^2(\sigma(\cdot),\nu_0)\|^2_{L^2(0,T)} \Big). \tag{20}$$

In the remainder of this subsection, we fix $\sigma \in AC^2(0,T;\mathcal{P}_2(I\!R^d))$ and $\xi \in C_c^\infty((0,T)\times I\!R^d; I\!R^d)$. We define $\sigma_t^s = (\mathbf{id}+s\xi(t,\cdot))_\#\sigma_t$, where we recall that $\mathbf{id}(y)\equiv y$. Let $M \in AC^2(0,T;L^2(\nu_0))$ be such that $M_{t\#}\nu_0 = \sigma_t$ and M_t is monotone nondecreasing.

Lemma 3.1. *For $s \in I\!R$, $\sigma^s \in AC^2(0,T;\mathcal{P}_2(I\!R^d))$ and, if s is small enough,*

$$|(\sigma^s)'|^2(t) - |\sigma'|^2(t) \le C_t^2 s^2 + 2s\int_{I\!R^d} \langle \mathbf{v}_t, \partial_t\xi_t + \nabla_y \xi_t \mathbf{v}_t\rangle d\sigma_t,$$

where $C_t = \|\partial_t\xi + \nabla\xi_t\cdot\mathbf{v}_t\|_{L^2(\sigma_t)}$.

Proof: Set $M_t = \mathbf{id} + s\xi_t$. For arbitrary $F \in C_c^\infty(I\!R^d)$, it is easy to see that the derivative of $t \to \int_{I\!R^d} F d\sigma_t^s = \int_{I\!R^d} F(M_t)d\sigma_t$ with respect to t is $\int_{I\!R^d}\langle \nabla F, \mathbf{v}_t^s\rangle d\sigma_t^s$, where

$$\mathbf{v}_t^s = \big[\mathbf{v}_t + s\big(\partial_t\xi + \nabla\xi_t\cdot\mathbf{v}_t\big)\big]\circ M_t^{-1}.$$

Hence, $\mathbf{v}^s$ is a velocity for the σ^s (we also refer the reader to theorem 2.1 of [3] for more details). By the fact that $|(\sigma^s)'|(t)$ is the minimal norm of admissible velocities of σ^s, we conclude that for $\mathcal{L}^1$-almost every $t \in (0,T)$

$$|(\sigma^s)'|(t) \le \|\mathbf{v}_t^s\|_{L^2(\sigma_t^s)}.$$

From that, it is apparent that the lemma holds. QED.

Lemma 3.2. *Let $\mu \in \mathcal{P}_2(I\!R^d)$, $\eta \in C_c^\infty(I\!R^d)$ and set $\mu^s = (\mathbf{id}+s\eta)_\#\mu$. Let φ be a lower semicontinuous convex function such that $(\nabla\varphi)_\#\nu_0 = \mu$. Then,*

$$\lim_{s\to 0}\frac{W_2^2(\mu^s,\nu_0) - W_2^2(\mu,\nu_0)}{s} = 2\int_{I\!R^d}\langle \nabla\varphi - \mathbf{id}, \eta\circ\nabla\varphi\rangle\, d\nu_0 = 2\int_{I\!R^d}\langle \mathbf{id} - \bar\gamma, \eta\rangle\, d\mu. \tag{21}$$

Here we have set $\gamma = (\mathbf{id}\times\nabla\varphi)_\#\nu_0$. We have denoted by $\bar\gamma \in \mathcal{T}_\mu\mathcal{P}_2(I\!R^d)$ the barycentric projection of γ onto μ (see (3) in the introduction).

Proof: The second equality in (21) is a direct consequence of the assumption $\gamma \in \Gamma_o(\nu_0,\mu)$ and the definition of barycentric projection. Thus, only the first equality in (21) needs to be proved. We have $W_2(\mu^s,\mu) \le s\|\eta\|_\mu$ and so, by the triangle inequality, $W_2(\mu^s,\nu_0)$ tends to $W_2(\mu,\nu_0)$ as s tends to 0. Set $T_s = \mathbf{id}+s\eta$ and let φ_s be the convex function satisfying $(\nabla\varphi_s)_{\#\nu_0} = \mu^s$. Observe that $\Gamma_o(\nu_0,\mu)$ has only one element. Hence, it is obvious that $\{\nabla\varphi_s\}$ converges weakly in $L^2(\nu_0)$ to $\nabla\varphi$ as s tends to 0 (see for instance [1] lemma 3.3). Since $W_2(\mu^s,\nu_0)^2 = \|\nabla\varphi_s - \mathbf{id}\|^2_{\nu_0}$ and

$W_2^2(\mu,\nu_0)^2 = \|\nabla\varphi - \mathbf{id}\|_{\nu_0}^2$ we conclude that $\{\nabla\varphi_s\}$ converges strongly in $L^2(\nu_0)$ to $\nabla\varphi$ as s tends to 0. Note that $(T_s \circ \nabla\varphi)_{\#\nu_0} = \mu^s$ and so,

$$W_2^2(\mu^s,\nu_0) - W_2^2((\mu,\nu_0) \leq \int_X |\mathbf{id} - T_s \circ \nabla\varphi|^2 \, d\nu_0 - \int_X |\mathbf{id} - \nabla\varphi|^2 \, d\nu_0$$
$$= 2s \int_X \langle \nabla\varphi - \mathbf{id}, \eta \circ \nabla\varphi \rangle \, d\nu_0 + s^2 \int_X |\xi \circ \nabla\varphi|^2 \, d\nu_0. \tag{22}$$

On the other hand, for any small enough s, T_s^{-1} exists and $(T_s^{-1} \circ \nabla\varphi_s)_{\#\nu_0} = \mu^s$. Thus,

$$W_2^2(\mu,\nu_0) - W_2^2(\mu^s,\nu_0) \leq \int_X |\mathbf{id} - T_s^{-1} \circ \nabla\varphi_s|^2 \, d\nu_0 - \int_X |\mathbf{id} - \nabla\varphi_s|^2 \, d\nu_0$$
$$= 2s \int_X \langle \frac{(\mathbf{id} - T_s^{-1}) \circ \nabla\varphi_s}{s}, \mathbf{id} - \frac{(\mathbf{id} + T_s^{-1}) \circ \nabla\varphi_s}{2} \rangle \, d\nu_0 =: 2sA_s. \tag{23}$$

It is easy to check that $|T_s^{-1}(y) - y + s\xi(y,t)| \leq Cs^2$ for a constant C independent of s and y and so,

$$A_s = \frac{s^2}{2}\|\eta\|_{\nu_0}^2 + \int_X \langle \mathbf{id} - \nabla\varphi_s, \eta \circ \nabla\varphi_s \rangle \, d\nu_0 + 0(s^2) \tag{24}$$

We combine (22), (23), (24) and use the fact that $\{\nabla\varphi_s\}$ converges strongly in $L^2(\nu_0)$ to $\nabla\varphi$ as s tends to 0 to conclude the proof. We have used the Lebesgue dominated convergence theorem and the fact that $\{\eta \circ \nabla\varphi_s\}$ is bounded in $L^\infty(\nu_0)$. QED.

Theorem 3.3 (Euler-Monge-Ampère system). *If σ minimizes $\mathcal{A}_T$ over $\mathcal{C}_T(\bar\sigma_0, \bar\sigma_T)$, then*

$$\partial_t(\sigma_t \mathbf{v}_t) + \nabla_y \cdot (\sigma_t \mathbf{v}_t \otimes \mathbf{v}_t) = \sigma_t[\bar\gamma^t - \mathbf{id}] \tag{25}$$

in the distributional sense, where $\gamma^t \in \Gamma_o(\nu_0, \sigma_t)$.

Proof: We use lemmas 3.1, 3.2 and the fact that σ minimizes $\mathcal{A}_T$ to obtain

$$0 \leq \liminf_{s \to 0^+} \frac{\mathcal{A}_T(\sigma^s) - \mathcal{A}_T(\sigma)}{s} \leq \int_0^T dt \Big(\int_{I\!R^d} \langle \mathbf{v}_t, \partial_t \xi_t + \nabla_y \xi_t \mathbf{v}_t \rangle \Big) d\sigma_t$$
$$- \int_0^T \Big(\int_{I\!R^d} \langle \mathbf{id} - \bar\gamma^t, \xi_t \rangle \Big) d\sigma_t dt.$$

Since we can substitute ξ by $-\xi$, the conclusion of the theorem follows. QED.

3.2 Another action as a quadratic form in $AC^2(0,T;L^2(\nu_0))$

Throughout this subsection, $d = 1$ and $\bar\sigma_0, \bar\sigma_T \in \mathcal{P}_2(I\!R)$. We will not directly study minimizers of the action $\mathcal{A}_T$ over $\mathcal{C}_T(\bar\sigma_0, \bar\sigma_T)$ for two reasons. Indeed, unlike the second order moment, the second term $\mu \to \int_{I\!R^2} |y - \bar y| d\mu(y) d\mu(\bar y)$ which appears in the Wasserstein distance (5) is not differentiable. Note that, so far, we only know that (25) is a necessary condition satisfied by the minimizer of $\mathcal{A}_T$. To obtain an Euler-Lagrange equation which characterizes completely the minimizers of $\mathcal{A}_T$, we regularize W_2 to obtain a differentiable function

$$\mathcal{W}^s(\mu) := \int_{I\!R} |y|^2 d\mu(y) - \frac{1}{2s} \int_{I\!R^2} |y - \bar{y}|^s d\mu(y) d\mu(\bar{y}) + \frac{1}{12}.$$

We introduce the corresponding action

$$\mathcal{A}^s_T(\sigma) := \frac{1}{2} \int_0^T \left[|\sigma'|^2(t) - \mathcal{W}^s(\sigma_t) \right] \, dt \tag{26}$$

and later study its minimizers of as s tends to 0. Observe that the term $-1/2W_2^2(\nu_0, \cdot)$ which appears in $\mathcal{A}_T$ is not lower semicontinuous for the narrow convergence on $\mathcal{P}_2(I\!R)$. This is a source of further difficulty we encounter while trying to directly minimize $\mathcal{A}_T$.

Let $M \in AC^2(0, T; L^2(\nu_0))$ be such that $M_t : X \to I\!R$ is the unique (ν_0–almost everywhere) monotone nondecreasing map satisfying $M_{t\#}\nu_0 = \sigma_t$. We decompose $\mathcal{A}_T$ into two terms which will satisfies some lower semicontinuity properties in a sense to be made precise. These are

$$1/2\|\sigma'\|^2_{metric} - 1/2 \int_0^T \int_X y^2 d\sigma_t(y) \quad \text{and} \quad 1/4 \int_0^T dt \int_{I\!R^2} |y - \bar{y}| d\sigma_t(y) d\sigma_t(\bar{y}).$$

The second term can be expressed as a function of M:

$$\mathcal{C}(M) = 1/4 \int_{X_T^2} |M_t x - M_t \bar{x}| dx d\bar{x} dt.$$

It is a convex functional on the Hilbert space $H^1(0, T, L^2(\nu_0))$ with the standard inner product. Lemma 2.2 yields that the first expression is the quadratic form $\mathcal{Q}(M) = \mathcal{B}(M, M)$ where $\mathcal{B}$ is the bilinear form defined on $H^1(0, T, L^2(\nu_0))$ defined by

$$\mathcal{B}(M, N) = 1/2 \int_0^T (\langle M', N' \rangle_{\nu_0} - \langle M, N \rangle_{\nu_0}) dt, \qquad \mathcal{Q}(M) = \mathcal{B}(M, M). \tag{27}$$

We regularize $\mathcal{C}$ to obtain a differentiable convex functional

$$\mathcal{C}^s(M) = 1/4s \int_{X_T^2} |M_t x - M_t \bar{x}|^s dt dx d\bar{x}.$$

We shall study the action $\mathcal{Q}(M) + \mathcal{C}^s(M)$ on $AC^2(0, T; L^2(\nu_0))$.

Remark 3.4. *Let $\bar{M}_0, \bar{M}_T \in L^2(\nu_0)$ and let $M, \tilde{M} \in \mathcal{C}_T(\bar{M}_0, \bar{M}_T)$. (a) If and $\alpha \in [0, 1]$, then*

$$\mathcal{Q}((1 - \alpha)M + \alpha\tilde{M}) = \mathcal{Q}(M) + \alpha^2 \mathcal{Q}(\tilde{M} - M) + 2\alpha\mathcal{B}(M, \tilde{M} - M). \tag{28}$$

We apply Poincare's inequality (with $s_0 = 0$) in proposition 2.12 to $\mathcal{S} = L^2(\nu_0)$ and obtain

$$2\mathcal{Q}(\tilde{M} - M) \geq (1 - T^2/\pi^2) \|(\tilde{M} - M)'\|^2_{metric}. \tag{29}$$

We combine (28) and (29) to obtain: if $M \neq \tilde{M}$, then

$$\alpha \to \mathcal{Q}((1 - \alpha)M + \alpha\tilde{M}) \quad \text{ is strictly convex.} \tag{30}$$

(b) In particular, if we set $M_t = (1 - t/T)\bar{M}_0 + (t/T)\bar{M}_T$, then (28) ($\alpha = 1$) and (29) imply

$$\mathcal{Q}(\tilde{M}) \geq \mathcal{Q}(M) + 1/2(1 - T^2/\pi^2)\Big(\|\tilde{M}'\|^2_{metric} - 1/T\|\bar{M}_T - \bar{M}_0\|^2_{L^2(\nu_0)}\Big) + 2\mathcal{B}(M, \tilde{M} - M). \quad (31)$$

We have used that $\|(\tilde{M} - M)'\|^2_{metric} = \|\tilde{M}'\|^2_{metric} - 1/T\|\bar{M}_T - \bar{M}_0\|^2_{\nu_0}$.

Proposition 3.5 ($\mathcal{Q}$ and $\mathcal{Q} + \mathcal{C}^s$ are sequentially weakly lower semicontinuous). *Let $s \geq 1$ and let $\{M\} \cup \{M^n\}_{n=1}^\infty \subset \mathcal{C}_T(\bar{M}_0, \bar{M}_T)$. Suppose that for each $t \in (0, T)$, $\{M^n_t\}_{n=1}^\infty$ converges weakly to M_t in $L^2(\nu_0)$. Then,*

$$\liminf_{n\to+\infty} \mathcal{Q}(M^n) \geq \mathcal{Q}(M) \quad \text{and} \quad \liminf_{n\to+\infty} \mathcal{Q}(M^n) + \mathcal{C}^s(M^n) \geq \mathcal{Q}(M) + \mathcal{C}^s(M).$$

Proof: We assume without loss of generality that the sequence $\{\mathcal{Q}(M^n)\}_{n=1}^\infty$ is bounded independently of n. Poincaré's inequality (16) ensures that

$$\sup_{n\in\mathbf{N}} \|(M^n)'\|_{metric}, \quad \sup_{n\in\mathbf{N}} \|M^n\|_{L^2(\nu)} < \infty.$$

By lemma 2.4, $\{M^n\}_{n=1}^\infty$ converges weakly to M in $L^2(\nu)$ and $\{(M^n)'\}_{n=1}^\infty$ converges weakly to M' in $L^2(\nu)$. Note that, by (28),

$$\mathcal{Q}(M^n) \geq \mathcal{Q}(M) + 2\mathcal{B}(M, M^n - M), \quad (32)$$

where we have used (29) to ensure that $\mathcal{Q}(M^n - M) \geq 0$. Because $\mathcal{B}(M, \cdot)$ is linear and continuous on $H^1(0, T, L^2(\nu_0))$, (32) gives that $\liminf_{n\to+\infty} \mathcal{Q}(M^n) \geq \mathcal{Q}(M)$. Setting $E^n(x, y, t) := M^n_t(x) - M^n_t(y)$ we obtain that $\{E^n\}_{n=1}^\infty$ converges weakly to E in $L^2(\nu \times \nu_0)$, where $E(x, y, t) := M_t x - M_t y$. This, together with the lower semicontinuity of the $L^s(\nu \times \nu_0)$-norm, yields

$$\liminf_{n\to+\infty} \mathcal{C}^s(M^n) \geq \mathcal{C}^s(M).$$

This concludes the proof of the proposition. QED.

4 Lagrangian minimizing paths in $\mathcal{P}_2(\mathbb{R})$

Throughout this section, $d = 1$ and so, $X = (-1/2, 1/2)$, $\nu_0 = \mathcal{L}^1|_X$ and $\nu = \mathcal{L}^2|_{X_T}$. We suppose that $T \in (0, \pi)$. It is convenient to introduce $\mathcal{M}on$, the set of monotone nondecreasing functions $M \in L^2(\nu_0)$.

4.1 More on properties of paths in $AC^2(0, T; \mathcal{P}_2(\mathbb{R}))$

The method of proof for most of the results in this subsection exploits strongly that $d = 1$. As far as we know, some of them, such as proposition 4.2, are not available in the literature. The main point of that proposition is that (33) holds although the velocity v may fail to be smooth in any sense. We use $\mathcal{Q}$ and $\mathcal{C}^s$ as defined in the previous section.

Suppose that $\bar{M}_0, \bar{M}_T \in \mathcal{M}on$. The purpose of the next remark is to show that the minimizer of $\mathcal{Q} + \mathcal{C}^s$ over $\mathcal{C}(\bar{M}_0, \bar{M}_T)$ coincides its minimizer over $\mathcal{C}(\bar{M}_0, \bar{M}_T) \cap \{M|\ M_t \in \mathcal{M}on\}$.

Remark 4.1. *Suppose $\tilde{M} \in AC^2(0,T;L^2(\nu_0))$ and set $\sigma_t = \tilde{M}_{t\#}\nu_0$ so that by remark 2.1, $\sigma \in AC^2(0,T;\mathcal{P}_2(I\!R))$. Let $M_t \in \mathcal{M}on$ be such that $M_{t\#}\nu_0 = \sigma_t$. By lemma 2.2, $M \in AC^2(0,T;L^2(\nu_0))$. We combine remark 2.1 and lemma 2.2 to conclude that $\|M'\|_{metric} \leq \|\tilde{M}'\|_{metric}$. Since*

$$\|M_t\|_{L^2(\nu_0)} = \int_{I\!R} |z|^2 d\sigma_t(z) = \|\tilde{M}_t\|_{L^2(\nu_0)},$$

it is apparent that $\mathcal{Q}(M) \leq \mathcal{Q}(\tilde{M})$. Similarly,

$$\mathcal{C}^s(M) = 1/4s \int_0^T \int_{I\!R^2} |z-w|^s d\sigma_t(z) d\sigma_t(w) = \mathcal{C}^s(\tilde{M})$$

and so, $\mathcal{Q}(M) + \mathcal{C}^s(M) \leq \mathcal{Q}(\tilde{M}) + \mathcal{C}^s(\tilde{M})$. We use lemma 2.2 to obtain that $\mathcal{A}^s_T(\sigma) = \mathcal{Q}(M) + \mathcal{C}^s(M) - T/24$.

Proposition 4.2. *Suppose $\sigma \in AC^2(0,T;\mathcal{P}_2(I\!R))$. Let v be the Borel velocity of minimal norm associated to σ and let $M_t \in \mathcal{M}on$ be such that $M_{t\#}\nu_0 = \sigma_t$. For each t, modifying M_t on a countable subset of X if necessary, we may assume without loss of generality that M_t is left continuous (see subsection 2.2). Then, we have*

$$M_t' x = v_t(M_t x) \tag{33}$$

for ν-almost every $(x,t) \in X_T$.

Proof: First, recall that $M \in AC^2(0,T;L^2(\nu_0))$. Let A and the sequence $\{h_k^+\}$ be given as in lemma 2.8 and let $\mathcal{K}$ be a countable dense subset of $C_c^1(I\!R)$. We use the definition of v and lemma 2.8 to obtain a Borel $J \subset (0,T)$ of full measure satisfying the following: for each $t \in J$, $\frac{d}{dt}\int_{I\!R} \varphi d\sigma_t = \int_{I\!R} v_t \varphi' d\sigma_t$ for all $\varphi \in \mathcal{K}$. Furthermore, the set

$$D_t := \{x \in X : \ (x,t) \in A \text{ and } x \text{ is a Lebesgue point of } M_t'\} \subset X$$

has full ν_0–measure. For $t \in J$, let us define

$$\bar{M}_t x = \liminf_{n \to \infty} \left(n \int_{x-\frac{1}{n}}^x M_t' \bar{x} d\bar{x} \right), \quad x \in X.$$

Note that $\bar{M}_t$ is a Borel map as a liminf of continuous functions on X. We also define $S_t(z) = -1/2 + \sigma_t(-\infty, z]$, which is, up to an additive constant, the right continuous distribution function of σ_t. Then let us introduce the map $w_t = \bar{M}_t \circ S_t$, which is a Borel map as a composition of two Borel maps. Fix $t \in J$ arbitrary. We have

$$w_t(M_t x) = \bar{M}_t \circ S_t \circ M_t x = \bar{M}_t x_t^r, \tag{34}$$

where x_t^r is the right endpoint of $I_{t,x} := \{\bar{x} \in X : M_t \bar{x} = M_t x\}$. Notice that since M_t is left continuous, $I_{t,x}$ is a right closed interval (possibly degenerate) containing x. If $I_{t,x}$ contains only $x = x_t^r$ then $w_t(M_t x) = M_t' x$ holds as a direct consequence of (34). Suppose next that $I_{t,x}$ is an interval of positive length so that x_t^r is the largest value in the interval. For n large enough

$[x_t^r - 1/n, x_t^r] \subset I_{t,x}$. Because D_t has full ν_0–measure, lemma 2.9 gives that $\int_{x_t^r-\frac{1}{n}}^{x_t^r} M_t'\bar{x}d\bar{x} = 1/nM_t'x$ and so, $\bar{M}_t x_t^r = M_t'x$. We combine this with (34) to obtain

$$w_t(M_t x) = M_t'x \text{ for all } x \in D_t. \tag{35}$$

We combine (14) and (35) to obtain for $t \in J$ and $\varphi \in \mathcal{K}$,

$$\begin{aligned}\int_{I\!R} w_t\varphi' d\sigma_t &= \int_X w_t(M_t x)\varphi'(M_t x)dx = \int_X M_t'x\varphi'(M_t x)dx \\ &= \lim_{k\to\infty}\int_X \frac{\varphi(M_{t+h_k^+}x) - \varphi(M_t x)}{h_k^+}dx \\ &= \lim_{k\to\infty}\int_{I\!R} \varphi\ \frac{d\sigma_{t+h_k^+} - d\sigma_t}{h_k^+} = \int_{I\!R} v_t\varphi' d\sigma_t. \end{aligned} \tag{36}$$

As $\mathcal{K}$ is dense in $C_c^1(I\!R)$, we conclude that the equalities in (36) hold for all $\varphi \in C_c^1(I\!R)$. As $\sigma_t \in \mathcal{P}_2(\mathbb{R})$ is a Borel probability measure on $I\!R$, $\{\varphi' |\ \varphi \in C_c^1(I\!R)\}$ is itself dense in $L^2(\sigma_t)$. Here, we have used that $I\!R$ is of dimension 1. Hence (36) holds if we substitute φ' by any element of $L^2(\sigma_t)$. This proves that $w_t = v_t\ \sigma_t$–almost everywhere. Next, observe that M_t maps ν_0–negligible sets onto σ_t–negligible sets. Hence, for $t \in J$, using (35) we conclude that $M_t'x = w_t(M_t x) = v_t(M_t x)$ for almost every $x \in X$. Consequently, for all $\xi \in C_c^\infty(X_T)$

$$\begin{aligned}\int_{X_T} \xi(x,t)M_t'xdxdt &= \int_0^T dt\int_X \xi(x,t)M_t'xdx = \int_0^T dt\int_X \xi(x,t)w_t(M_t x)dx \\ &= \int_0^T dt\int_X \xi(x,t)v_t(M_t x)dx = \int_{X_T} \xi(x,t)v_t(M_t x)dxdt.\end{aligned}$$

This proves the proposition. QED.

4.2 Minimizing paths on $\mathcal{P}_2(I\!R)$

We recall that $\mathcal{Q}$ and $\mathcal{C}^s$ are defined in section 3.2. Since $T < \pi$, Poincaré inequality yields that sublevel subsets of $\mathcal{Q} + \mathcal{C}^s$ are contained in sublevel subsets of $\|\cdot\|_{AC^2(0,T;L^2(\nu_0))}$. Observe that $\mathcal{Q} + \mathcal{C}^s$ is differentiable only for $s > 1$. We will see that if $\bar{M}_0$, $\bar{M}_T$ are monotone nondecreasing satisfying $M_{0\#}\nu_0 = \bar{\sigma}_0$ and $M_{T\#}\nu_0 = \bar{\sigma}_T$, then minimizing $\mathcal{A}_T$ over $\mathcal{C}_T(\bar{\sigma}_0, \bar{\sigma}_T)$ is equivalent to minimizing $\mathcal{Q} + \mathcal{C}$ over $\mathcal{C}_T(\bar{M}_0, \bar{M}_T)$. Furthermore, there is a unique minimizer in each one of these two problems. We analyze the minimizers of $\mathcal{Q} + \mathcal{C}$ by studying first those of $\mathcal{Q} + \mathcal{C}^s$.

Theorem 4.3. *Suppose $\bar{M}_0, \bar{M}_T \in L^2(\nu_0)$. Then $\mathcal{Q} + \mathcal{C}^s$ admits a unique minimizer M^s on $\mathcal{C}_T(\bar{M}_0, \bar{M}_T)$. If furthermore, $\bar{M}_0, \bar{M}_T \in \mathcal{M}on$ then for $\mathcal{L}^1$-almost every $t \in (0,T)$, $M_t^s \in \mathcal{M}on$.*

Proof: We choose $\mathcal{S} = L^2(\nu_0)$ with the metric $dist = \|\cdot\|_{\nu_0}$. We choose τ to be the weak topology and we apply proposition 2.13. Set $M_t = (1-t/T)\bar{M}_0 + (t/T)\bar{M}_T$ and let $\{M^n\}_{n=1}^\infty$ be a minimizing sequence for $\mathcal{Q} + \mathcal{C}^s$ over $\mathcal{C}_T(\bar{M}_0, \bar{M}_T)$. We may assume that $1 + \mathcal{Q}(M) + \mathcal{C}^s(M) \geq \mathcal{Q}(M^n) + \mathcal{C}^s(M^n)$. We use the Poincare's inequalities (15)–(16) to obtain that $\{M^n\}_{n=1}^\infty$ is bounded in $H^1(0,T;L^2(\nu_0))$. In particular, we obtain a constant C independent of n such that

$$\|\,|(M^n - M)'|\,\|_{L^2(0,T)} \leq C. \tag{37}$$

Thanks to (37) we can apply proposition 2.14. We obtain a subsequence of $\{M^n\}_{n=1}^{\infty}$ (we do not relabel) such that for each $t \in [0,T]$, $\{M_t^n\}_{n=1}^{\infty}$ converges weakly to some M_t^s in $L^2(\nu_0)$. We use proposition 2.13 to obtain that $M^s \in AC^2(0,T;L^2(\nu_0))$. One can readily conclude that $M^s \in \mathcal{C}_T(\bar{M}_0, \bar{M}_T)$. By lemma 2.4, $\{M^n\}_{n=1}^{\infty}$ converges weakly to M^s in $L^2(\nu)$ and $\{(M^n)'\}_{n=1}^{\infty}$ converges weakly to $(M^s)'$ in $L^2(\nu_0)$. By proposition 3.5, $\mathcal{Q} + \mathcal{C}^s$ is sequentially weakly lower semicontinuous on $\mathcal{C}_T(\bar{M}_0, \bar{M}_T)$. We use these to conclude that M^s minimizes $\mathcal{Q} + \mathcal{C}^s$ over $\mathcal{C}_T(\bar{M}_0, \bar{M}_T)$. By remark 3.4, $\mathcal{Q}$ is strictly convex on $\mathcal{C}_T(\bar{M}_0, \bar{M}_T)$. Hence $\mathcal{Q} + \mathcal{C}^s$ is strictly convex as the sum of a strictly convex function and a convex function. This proves that the minimizer M^s is unique.

Suppose, in addition, that $\bar{M}_0, \bar{M}_T \in \mathcal{M}on$. Let $\tilde{M}^s$ be such that $\tilde{M}_t^s$ is monotone nondecreasing and $\tilde{M}^s_{t\#}\nu_0 = M^s_{t\#}\nu_0$. By remark 4.1, $\tilde{M} \in AC^2(0,T;L^2(\nu_0))$ and $\mathcal{Q}(\tilde{M}) + \mathcal{C}^s(\tilde{M}) \leq \mathcal{Q}(M) + \mathcal{C}^s(M)$. Thus $\tilde{M}$ is another minimizer of $\mathcal{Q} + \mathcal{C}^s$ over $\mathcal{C}_T(\bar{M}_0, \bar{M}_T)$. By uniqueness, $M^s = \tilde{M}^s$. This proves that $\mathcal{L}^1$-almost every $t \in (0,T)$, M_t^s is monotone nondecreasing. QED.

Remark 4.4. *Suppose* $M \in AC^2(0,T;L^2(\nu_0))$ *so that* $M \in L^2(\nu)$. *If, in addition,* $M_t \in \mathcal{M}on$ *for each* $t \in [0,T]$ *then that for* $0 < r < 1/2$, *we have* $r|M_t x| \leq \int_X |M_t y| dy$ *on* $[-1/2+r, 1/2-r] =: X^r$. *Hence,*

$$r\int_{-1/2+r}^{1/2-r} |\partial_x M_t x| \leq r(M_t(1/2-r) - M_t(-1/2+r)) \leq 2\int_X |M_t y| dy. \tag{38}$$

We have used that $\partial_x M_t$ *is a nonnegative measure on* X. *Thus,* $r\|\partial_x M\|_{X_T^r} \leq 2\sqrt{T}\|M\|_\nu$. *Let* $\partial_t M$ *be the distributional derivative of* M *which coincides with* M' *by remark 2.10. Then* $\|\partial_t M\|_\nu = \|M'\|_{metric}$. *This proves the existence of a constant* $\bar{C}(r)$ *dependent on* T *but independent of* M *such that*

$$\|M\|_{BV(X_T^r)} \leq \bar{C}(r)\|M\|_{AC^2(0,T;L^2(\nu_0))}. \tag{39}$$

4.3 Euler-Poisson system in 1–d in terms of its associated flow

Suppose $s > 1$, so that $\mathcal{Q} + \mathcal{C}^s$ not only is it strictly convex, but it is also Gâteaux differentiable. If $\bar{M}_0, \bar{M}_T \in L^2(\nu_0)$, standard arguments give that $M^s \in \mathcal{C}_T(\bar{M}_0, \bar{M}_T)$ minimizes $\mathcal{Q} + \mathcal{C}^s$ over $\mathcal{C}_T(\bar{M}_0, \bar{M}_T)$ if and only if

$$\left(M^s\right)_t'' x + M_t^s x = 1/2\int_X (M_t^s x - M_t^s y)|M_t^s x - M_t^s y|^{s-2} dy \tag{40}$$

in the sense of distributions on X_T. An analogous characterization of M^1 can be stated although $\mathcal{Q} + \mathcal{C}^1$ is not Gâteaux differentiable everywhere. As in the previous section set $X^r = [-1/2 + r, 1/2 - r]$.

Proposition 4.5. *Suppose* $M \in \mathcal{C}_T(\bar{M}_0, \bar{M}_T)$, W *is a Borel function on* X_T^2 *such that* $W(x,y,t) \in \partial.|\cdot|(M_t x - M_t y)$ *and* $W(x,y,t) = -W(y,x,t)$ *for* $\nu \times \nu_0$- *almost every* $(x,y,t) \in X_T^2$. *If*

$$M_t'' x + M_t x = \frac{1}{2}\int_X W(x,\bar{x},t) d\bar{x} \tag{41}$$

in the sense of distributions on X_T, *then* M *is the unique minimizer of* $\mathcal{Q}+\mathcal{C}$ *over* $\mathcal{C}_T(\bar{M}_0, \bar{M}_T)$. *If we further assume that* $\bar{M}_0, \bar{M}_T$ *are monotone nondecreasing, then, for* $\mathcal{L}^1$*–almost every* $t \in (0,T)$, M_t *is monotone nondecreasing.*

Proof: Let $\tilde{M} \in \mathcal{C}_T(\bar{M}_0, \bar{M}_T)$ be arbitrary. Since $W(x,y,t) = -W(y,x,t)$, we have

$$\int_{X_T^2} W(x,\bar{x},t)(\tilde{M}_t x - M_t x) dx d\bar{x} dt = -\int_{X_T^2} W(x,\bar{x},t)(\tilde{M}_t \bar{x} - M_t \bar{x}) dx d\bar{x} dt.$$

By the definition of $\mathcal{B}$ in (27) and (41),

$$-4\bar{\mathcal{B}}(M, \tilde{M} - M) = \frac{1}{2}\int_{X_T^2} W(x,\bar{x},t)\Big((\tilde{M}_t x - \tilde{M}_t \bar{x}) - (M_t x - M_t \bar{x})\Big) dx d\bar{x} dt \tag{42}$$

We exploit (42) and the fact that $W(x,y,t) \in \partial.|\cdot|(M_t x - M_t y)$ to conclude that

$$\frac{1}{4}\int_{X_T^2} (|\tilde{M}_t x - \tilde{M}_t \bar{x}| - |M_t x - M_t \bar{x}|) dx d\bar{x} dt \geq -2\mathcal{B}(M, \tilde{M} - M).$$

This, in view of (28) (set $\alpha = 1$) and the fact that $\mathcal{Q}(\tilde{M} - M) \geq 0$, yields $\mathcal{Q}(\tilde{M}) + \mathcal{C}(\tilde{M}) \geq \mathcal{Q}(M) + \mathcal{C}(M)$. Hence, M minimizes $\mathcal{Q} + \mathcal{C}$ over $\mathcal{C}_T(\bar{M}_0, \bar{M}_T)$. For each $t \in [0,T]$ define M_t^o to be the unique monotone nondecreasing map such that $M^o_{t\,\#}\nu_0 = M_{t\,\#}\nu_0 =: \sigma_t$. By remark 2.1, the fact that $M \in AC^2(0,T; L^2(\nu_0))$ yields $\sigma \in AC^2(0,T; \mathcal{P}_2(I\!R))$. Lemma 2.2 ensures that $\mathcal{C}_T(\bar{M}_0, \bar{M}_T)$ and

$$\|M'\|_{metric} \geq \|\sigma'\|_{metric} = \|(M^o)'\|_{metric}.$$

This, along with the fact that $\mathcal{C}(M) = \mathcal{C}(M^o)$, yields that M^o is also a minimizer of $\mathcal{Q} + \mathcal{C}$ over $\mathcal{C}_T(\bar{M}_0, \bar{M}_T)$. The strict convexity of $\mathcal{Q} + \mathcal{C}$ obtained in (30) ensures uniqueness of its minimizer over $\mathcal{C}_T(\bar{M}_0, \bar{M}_T)$. Thus, $M = M^o$. QED.

We are next going to show the converse of proposition 4.5. For that, let $\bar{M} = (1 - t/T)\bar{M}_0 + (t/T)\bar{M}_T$. In this work we are interested in what happens when s tends to 1. Instead of imposing that s is bounded above by an arbitrary constant, we simply impose that $s \leq 2$. Observe that for $s \in [1,2]$ we have

$$\sup_{s' \in [1,2]} \mathcal{Q}(\bar{M}) + \mathcal{C}^{s'}(\bar{M}) \geq \mathcal{Q}(M^s) + \mathcal{C}^s(M^s) \geq \mathcal{Q}(M^s).$$

Hence Poincare's inequality (16) yields

$$\sup_{s \in \in [1,2]} \|(M^s)'\|_{metric} + \sup_{t \in [0,T]} \|M_t^s\|_{\nu_0}^2 < \infty. \tag{43}$$

The above supremum depends only on T, $\bar{M}_0$ and $\bar{M}_T$.

Theorem 4.6. *Suppose $\bar{M}_0, \bar{M}_T \in L^2(\nu_0) \cap \mathcal{M}on$ and $M^1 \in \mathcal{C}_T(\bar{M}_0, \bar{M}_T)$. Then M^1 minimizes $\mathcal{Q} + \mathcal{C}$ over $\mathcal{C}_T(\bar{M}_0, \bar{M}_T)$ if and only if $\big(M^1\big)' \in AC^2(0,T; L^2(\nu_0))$ and*

$$\big(M^1\big)''_t x + M_t^1 x = \int_X W_{M^1}(x,\bar{x},t) d\bar{x} \tag{44}$$

in the sense of distributions on X_T, for some $W_{M^1} \in L^\infty(X_T^2)$ satisfying $W_{M^1}(x,\bar{x},t) \in \partial.|\cdot|(M_t^1 x - M_t^1 \bar{x})$ and $W_{M^1}(x,\bar{x},t) = -W_{M^1}(\bar{x},x,t)$.

Proof: Let M^s be the minimizer of $\mathcal{Q}+\mathcal{C}^s$ over $\mathcal{C}_T(\bar{M}_0,\bar{M}_T)$. By the previous proposition, $M^s_t \in \mathcal{M}on$. We set $X^r_T=(0,T)\times[-1/2+r,1/2-r]$. We use (16) to obtain existence of a constant $\bar{C}(r)$ which depends on r but is independent of $s\in[1,2]$ such that $\|M^s\|_{BV(X^r_T)}\leq \bar{C}(r)$. We next invoke the compactness of bounded subsets of $BV(X^r_T)$ in $L^1(X^r_T)$. We obtain existence of a sequence $\{s_j\}_{j=1}^\infty \subset [1,2]$ converging to 1 as j tends to $+\infty$ and such that $\{M^{s_j}\}_{j=1}^\infty$ converges to some $\tilde{M}^1$ in $L^1_{loc}(X_T)$. Passing to a subsequence if necessary, we may assume that $\{M^{s_j}\}_{j=1}^\infty$ converges ν-almost everywhere to $\tilde{M}^1$. Passing to another subsequence, it is easy to check that $\tilde{M}^1 \in \mathcal{C}_T(\bar{M}_0,\bar{M}_T)$ and $\{M^{s_j}_t\}_{j=1}^\infty$ converges weakly to $\tilde{M}^1_t$ in $L^2(\nu_0)$ for each $t\in(0,T)$. By Egoroff's theorem, $\{M^{s_j}\}_{j=1}^\infty$ converges uniformly to $\tilde{M}^1$ except on a set of ν-small measure. By (43) $\{|M^{s_j}_t x - M^{s_j}_t y|^{s_j}\}_{j=1}^\infty$ is weakly closed in $L^1(\nu\times\nu_0)$ and so,

$$\lim_{j\to+\infty}\int_{X^2_T}|M^{s_j}_t x - M^{s_j}_t \bar{x}|^{s_j}dtdxd\bar{x}=\int_{X^2_T}|\tilde{M}^1_t x-\tilde{M}^1_t\bar{x}|dtdxd\bar{x}. \tag{45}$$

This, together with the fact that proposition 3.5 provides the lower semicontinuity of $\mathcal{Q}+\mathcal{C}$, yields $\liminf_{j\to+\infty}\mathcal{Q}(M^{s_j})+\mathcal{C}^{s_j}(M^{s_j})\geq \mathcal{Q}(\tilde{M}^1)+\mathcal{C}(\tilde{M}^1)$. Hence, $\tilde{M}^1$ minimizes $\mathcal{B}^1_T$ over $\mathcal{C}_T(\bar{M}_0,\bar{M}_T)$. Uniqueness of minimizers being ensured by theorem 4.3, we conclude that $\tilde{M}^1=M^1$. We use that

$$\liminf_{j\to+\infty}\mathcal{Q}(M^1)+\mathcal{C}^{s_j}(M^1)\geq\liminf_{j\to+\infty}\mathcal{Q}(M^{s_j})+\mathcal{C}^{s_j}(M^{s_j})\geq\mathcal{Q}(M^1)+\mathcal{C}(M^1)$$

and (45) to conclude that $\mathcal{Q}(M^1)=\liminf_{j\to+\infty}\mathcal{Q}(M^{s_j})$. Using (29) we obtain

$$0=\liminf_{j\to+\infty}\mathcal{Q}(M^{s_j}-M^1)\geq\liminf_{j\to+\infty}1/2(1-T^2/\pi^2)\|(M^{s_j}-M^1)'\|_{metric}.$$

This, together with Poincaré's inequality (16) proves that $\{M^{s_j}\}_{j=1}^\infty$ converges strongly to M^1 in $H^1(0,T;L^2(\nu_0))$. Set

$$W_{M^s}(x,\bar{x},t):=(M^s_t x-M^s_t\bar{x})|M^s_t x-M^s_t\bar{x}|^{s-2}.$$

We have that W_{M^s} is uniformly bounded in $L^2(X^2_T)$ and $W_{M^s}(x,\bar{x},t)\in\partial.G^s(M^s_t x-M^s_t\bar{x})$ with $G^s(e):=\frac{|e|^s}{s}$. Hence, up to a subsequence we do not relabel, $\{W_{M^{s_j}}\}_{j=1}^\infty$ converges weakly in $L^2(X^2_T)$ to some W_{M^1} in $L^2(X^2_T)$. The fact that $\{M^{s_j}\}_{j=1}^\infty$ converges strongly to M^1 in $H^1(0,T;L^2(\nu))$ yields $W_{M^1}(x,\bar{x},t)\in\partial.|\cdot|(M^1_t x-M^1_t\bar{x})$. Using (40) we obtain (44). It is apparent that $W_{M^1}(x,\bar{x},t)=-W_{M^1}(\bar{x},x,t)$ for $\nu\times\nu_0$-almost every $(x,\bar{x},t)\in X^2_T$. We use proposition 4.5 to conclude that any solution of (44) is a minimizer of $\mathcal{Q}+\mathcal{C}$. Observe that $W_{M^1}\in L^\infty(X^2_T)$ and

$$\left|\int_X W_{M^1}(x,\bar{x},t)d\bar{x}\right|\leq 1.$$

Since $M^1\in AC^2(0,T;L^2(\nu_0))$, (44) implies $\big(M^1\big)'\in AC^2(0,T;L^2(\nu_0))$. QED.

Remark 4.7. *Let M^1 be as in theorem 4.6. Note that, in particular, if for $\mathcal{L}^2$-almost every (x,t), $M^1_t x\neq M^1_t\bar{x}$ for $\mathcal{L}^1$-almost every $\bar{x}\in X$, (44) reads off*

$$\big(M^1\big)''_t x+M^1_t x=x.$$

Corollary 4.8. *Suppose $\bar{\sigma}_0, \bar{\sigma}_T \in \mathcal{P}_2(I\!R)$. Let $\bar{M}_0, \bar{M}_T$ be monotone nondecreasing maps satisfying $\bar{M}_{0\,\#}\nu_0 = \bar{\sigma}_0$ and $\bar{M}_{T\,\#}\nu_0 = \bar{\sigma}_T$, and let M^s be the unique minimizer of $\mathcal{Q} + \mathcal{C}^s$ over $\mathcal{C}_T(\bar{M}_0, \bar{M}_T)$ given by theorem 4.3. Then $\sigma^s := M^s_{t\,\#}\nu_0$ is the unique minimizer of $\mathcal{A}^s_T$ over $\mathcal{C}_T(\bar{\sigma}_0, \bar{\sigma}_T)$.*

Proof: Let $\sigma \in AC^2(0,T;\mathcal{P}_2(I\!R))$ be arbitrary and let $M_t \in \mathcal{M}on$ such that $M_{t\,\#}\nu_0 = \sigma_t$. By remark 4.1, $M \in AC^2(0,T;L^2(\nu_0))$. We have

$$\mathcal{A}^s_T(\sigma^s) + T/24 = \mathcal{Q}(M^s) + \mathcal{C}^s(M^s) \leq \mathcal{Q}(M) + \mathcal{C}^s(M) = \mathcal{A}^s_T(\sigma) + T/24. \tag{46}$$

The inequality in (46) being strict unless $M^s = M$, we conclude that σ^s is the unique minimizer of $\mathcal{A}^s_T$ over $\mathcal{C}_T(\bar{\sigma}_0, \bar{\sigma}_T)$. QED.

Suppose $\bar{\sigma}_0, \bar{\sigma}_T \in \mathcal{P}_2(I\!R)$ and σ^1 is the unique minimizer of $\mathcal{A}_T$ found in corollary 4.8. Let v be the Borel velocity of minimal norm associated to σ. For each $t \in [0,T]$, let M^1_t be the monotone nondecreasing map such that $M^1_{t\,\#}\nu_0 = \sigma^1_t$. Since the tangent space $\mathcal{T}_{\sigma^1_t}\mathcal{P}_2(I\!R)$ is a separable Hilbert space, we shall identify it with its dual and so, we can view the tangent vector v_t as a cotangent vector. Although the Hamiltonian H is defined on the cotangent bundle of $\mathcal{P}_2(I\!R)$, now it still makes sense to write $H(\sigma^1_t, v_t)$. By remark 2.11 and lemma 2.2, for $\mathcal{L}^1$–almost every $t \in (0,T)$

$$2H(\sigma^1_t, v_t) = \|v_t\|^2_{L^2(0,T)} + W_2^2(\sigma^1_t, \nu_0) = \int_X \big[\big|\big(M^1_t\big)'\big|^2 + |M^1_t - \mathbf{id}|^2\big] d\nu_0.$$

Whereas $t \to H(\sigma^1_t, v_t)$ may not be continuous, because $M^1 \in H^2(0,T;L^2(\nu_0))$, we have that $t \to 2\bar{H}(t) := \int_X [|\big(M^1\big)'_t|^2 d\nu_0 + W_2^2(\sigma^1_t, \nu_0)] d\nu_0$ is continuous. We next state that this map is constant. As a consequence of theorem 4.6 , $\bar{H} \in H^1(0,T)$. To prove that it is constant, it suffices to show that its distributional derivative is null. Let us observe that here, we have obtained conservation of the Hamiltonian without assuming that $\sigma^1_t << \mathcal{L}^1$. In fact, in general, the property $\sigma^1_t << \mathcal{L}^1$ fails. The proof relies on (33) of proposition 4.2 and on the following lemma.

Lemma 4.9. *Suppose that $M : X \to I\!R$ is monotone nondecreasing and $W \in L^\infty(X_T^2)$ satisfying $W(x,\bar{x}) \in \partial.|\cdot|(Mx - M\bar{x})$ and $W(x,\bar{x}) = -W(\bar{x},x)$. If $\xi \in L^1(\sigma)$ is a Borel map for $\sigma := M_\#\nu_0$, then*

$$2\int_X x\xi(Mx)dx = \int_{X^2} \xi(Mx)W(x,\bar{x})dxd\bar{x}.$$

Proof: We set

$$A = \{(x,\bar{x}) |\, Mx > M\bar{x}\}, \quad B = \{(x,\bar{x}) |\, Mx < M\bar{x}\}.$$

We use that M is monotone nondecreasing to conclude that

$$\begin{aligned} 2\int_X x\xi(Mx)dx &= \int_X dx\int_{-1/2}^{x}\xi(Mx)d\bar{x} - \int_X dx\int_x^{1/2}\xi(Mx)d\bar{x} \\ &= \int_A \xi(Mx)dxd\bar{x} + \int_{\{(x,\bar{x})|\, x>\bar{x}, Mx=M\bar{x}\}}\xi(Mx)dxd\bar{x} \\ &- \int_B \xi(Mx)dxd\bar{x} - \int_{\{(x,\bar{x})|\, x<\bar{x}, Mx=M\bar{x}\}}\xi(Mx)dxd\bar{x} \\ &= \int_A \xi(Mx)dxd\bar{x} - \int_B \xi(Mx)dxd\bar{x} \qquad (47) \\ &= \int_A \xi(Mx)W_M(x,\bar{x})dxd\bar{x} + \int_B \xi(Mx)W_M(x,\bar{x})dxd\bar{x}. \qquad (48) \end{aligned}$$

To obtain the equality in (47), we have used that

$$\int_{\{(x,\bar{x})|\, x>\bar{x}, Mx=M\bar{x}\}}\xi(Mx)dxd\bar{x} = \int_{\{(x,\bar{x})|\, x<\bar{x}, Mx=M\bar{x}\}}\xi(Mx)dxd\bar{x}.$$

By Fubini's theorem

$$\begin{aligned} \int_{\{Mx=M\bar{x}\}}\xi(Mx)W(x,\bar{x})dxd\bar{x} &= \int_{\{Mx=M\bar{x}\}}\xi(M\bar{x})W(\bar{x},x)d\bar{x}dx \\ &= -\int_{\{Mx=M\bar{x}\}}\xi(Mx)W(x,\bar{x})d\bar{x}dx \end{aligned}$$

and so, the three previous expressions vanish. This, together with (48), yields the proof. QED.

Theorem 4.10. *Setting* $\bar{H}(t) := 1/2\int_X |\big(M^1\big)'_t|^2 d\nu_0 + 1/2W_2^2(\sigma_t^1,\nu_0)$*, we have that* $\bar{H}(0) = \bar{H}(t)$ *for all* $t\in[0,T]$.

Proof: To simplify the notation, take $M := M^1$. Let $\varphi \in C_c^1(0,T)$ be arbitrary. Since $M,\ M' \in H^1(0,T;L^2(\nu_0))$, it can be shown that

$$\lim_{h\to 0}\int_{X_T}\left|\frac{\big(M_{t+h}x)^2 - \big(M_t x\big)^2}{h} - 2M_t x M'_t x\right| dxdt = 0$$

and

$$\lim_{h\to 0}\int_{X_T}\left|\frac{\big(M'_{t+h}x)^2 - \big(M'_t x\big)^2}{h} - 2M'_t x M''_t x\right| dxdt = 0.$$

By the above limits and theorem 4.6, one has

$$\begin{aligned} -\int_0^T \varphi'(t)\bar{H}(t)dt &= \frac{1}{2}\lim_{h\to 0}\int_{X_T}\frac{\varphi(t-h)-\varphi(t)}{h}\big[|M'_t|^2 + |M_t - \mathbf{id}|^2\big]d\nu_0 dt \\ &= \int_{X_T}\varphi(t)M'_t(M''_t + M_t - \mathbf{id})d\nu_0 dt \\ &= \int_0^T\varphi(t)\int_X M'_t x\Big(\frac{1}{2}\int_X W_M(x,\bar{x},t)d\bar{x} - x\Big)dxdt. \end{aligned}$$

Thus, $\bar H \in H^1(0,T)$ and, by proposition 4.2, we have for $\mathcal{L}^1$–almost every $t \in (0,T)$,

$$\frac{d}{dt}\bar H(t) = \int_X \mathbf{v}_t(M_t x)\Big(\frac{1}{2}\int_X W_M(x,\bar x,t)d\bar x - x\Big)dx.$$

We use lemma 4.9 to conclude that $\frac{d}{dt}\bar H(t) = 0$ for $\mathcal{L}^1$–almost every $t \in (0,T)$. Because $\bar H$ is continuous, $\bar H(t) = \bar H(0)$ for every $t \in (0,T]$. QED.

4.4 Minimizing paths whose endpoints are discrete measures.

We denote by $\mathcal{P}_{\cdot n}$ the set of measures of the form $1/n\sum_{i=1}^n \delta_{x_i}$ where $(x_1,\cdots,x_n) \in I\!R^n$. We show that if $\bar\sigma_0^1, \bar\sigma_T^1 \in \mathcal{P}_{\cdot n}$ and σ^1 minimizes $\mathcal{A}_T$ over $\mathcal{C}_T(\bar\sigma_0^1,\bar\sigma_T^1)$ then at each time t, $\sigma_t^1 \in \mathcal{P}_{\cdot n}$. Let

$$L_{\cdot n} = \{S : X \to I\!R \mid \; S|_{(c_{i-1},c_i)} \text{ is constant}\}, \quad c_i = -1/2 + i/n, \;\; i = 1,\cdots,n\} \tag{49}$$

which is a closed subspace of $L^2(\nu_0)$. For $\bar M_0, \bar M_T \in L_{\cdot n}$, we denote by $\mathcal{C}_T^n(\bar M_0,\bar M_T)$ the set of paths $M \in AC^2(0,T;L_{\cdot n})$ satisfying $M_0 = \bar M_0$ and $M_T = \bar M_T$. To $\mathbf{x} = (x_1,\cdots,x_n) \in I\!R^n$, we associate the measure $\sigma_{\mathbf{x}} = 1/n\sum_{i=1}^n \delta_{x_i}$ and the map $M_{\mathbf{x}} \in L_{\cdot n}$ defined by

$$M_{\mathbf{x}}(x) = x_i \quad \text{if} \quad x \in (c_{i-1}, c_i).$$

For $\mathbf{r} \in H^1(0,T;I\!R^n)$, we define the action

$$\mathcal{A}_{\cdot n}^s(\mathbf{r}) = \frac{1}{2}\int_0^T \sum_{i=1}^n (\frac{|\dot r_i(t)|^2}{n} - \frac{|r_i(t)|^2}{n})dt + \frac{1}{4ns}\int_0^T \sum_{i,j=1}^n |r_i(t) - r_j(t)|^s - \frac{T}{24} = \mathcal{A}_T^s(\sigma_{\mathbf{r}}).$$

Remark 4.11. *It is easy to show that $\mathcal{A}_{\cdot n}^s$ admits a minimizer $\mathbf{r}^s$ over over the set $\{\mathbf{r} \in H^1(0,T)^n \mid \mathbf{r}(0) = \mathbf{x}^0, \mathbf{r}(T) = \mathbf{x}^T\}$. For $s \in (1,2)$, the Euler-Lagrange equations satisfied are*

$$\ddot{\mathbf{r}}_i^s(t) + \mathbf{r}_i^s(t) = \frac{s}{2n}\sum_{j\neq i} w_{ij}^s(t) \quad w_{ij}^s(t) = (\mathbf{r}_i^s(t) - \mathbf{r}_j^s(t))|(\mathbf{r}_i^s(t) - \mathbf{r}_j^s(t)|^{s-2}, \quad i = 1,\cdots,n, \tag{50}$$

in the sense of distributions on $(0,T)$. This proves that $\mathbf{r}^s \in H^2(0,T)^n$. For $\mathbf{u} \in H^1(0,T)^n$ (independent of s) and $s \in [1,2]$, we have

$$\mathcal{A}_{\cdot n}^s(\mathbf{r}^s) \le \sup_{\bar s \in [1,2]} \mathcal{A}_{\cdot n}^{\bar s}(\mathbf{u}) < \infty$$

and so, Poincaré's inequality gives that $\|\mathbf{r}^s\|_{H^1(0,T)^n}$ is bounded by a constant independent of s. This, together with (50) yields that, in fact, $\|\mathbf{r}^s\|_{H^2(0,T)^n}$ is bounded by a constant independent of s.

We show that the set $\mathcal{P}_{\cdot n}(I\!R)$ is closed under the Lagrangian minimizing paths in the following sense.

Theorem 4.12. *Suppose $\bar\sigma_0 = 1/n\sum_{i=1}^n \delta_{x_i^0}$, $\bar\sigma_T = 1/n\sum_{i=1}^n \delta_{x_i^T}$ and let σ^1 be the path found in corollary 4.8, minimizing $\mathcal{A}_T^1$ over $\mathcal{C}_T(\bar\sigma_0,\bar\sigma_T)$. Then $\sigma_t^1 \in \mathcal{P}_{\cdot n}(I\!R)$ for a.e. $t \in (0,T)$.*

Proof: Assume without loss of generality that $x^k_{i-1} \leq x^k_i$ for $k = 0, T$ and $i = 1, \cdots, n$. We set

$$\bar{M}_0 = M_{\mathbf{x}^0}, \quad \bar{M}_T = M_{\mathbf{x}^T}, \quad \mathbf{x}^k = (x^k_1, \cdots, x^k_n) \quad \text{for } k = 0, T.$$

Let $s \in [1, 2]$. We are going to apply proposition 2.13 with $\mathcal{S} = (L_{\cdot n}, \| \cdot \|_{L^2(\nu_0)})$ and τ is the L^2-weak topology which coincides with the $\|\cdot\|_{\nu_0}$-topology when restricted to $L_{\cdot n}$. Since bounded sets of $L_{\cdot n}$ are closed for the τ-topology, we can apply proposition 2.13 and obtain as in proposition 4.3, existence of a path M^s minimizing $\mathcal{Q} + \mathcal{C}^s$ over $AC^2(0, T; L_{\cdot n})$. Let $\sigma^s(t) = M^s_{\#}\nu_0$ and let $\bar{M}^s_t$ be monotone nondecreasing such that $\sigma^s(t) = \bar{M}^s_{\#}\nu_0$. As argued in the proof of theorem 4.3, by remark 4.1,

$$\bar{M}^s \in AC^2(0, T; L_{\cdot n}), \quad \text{and} \quad \mathcal{Q}(\bar{M}^s) + \mathcal{C}^s(\bar{M}^s) \leq \mathcal{Q}(M^s) + \mathcal{C}^s(M^s).$$

The strict convexity of $\mathcal{Q} + \mathcal{C}^s$ (direct consequence of (30)) yields $M^s = \bar{M}^s$. This proves that M^s_t is monotone nondecreasing for each $t \in [0, T]$. If $\mathbf{r} = (r_1, \cdots, r_n) \in H^1(0, T; I\!R^n)$, by abuse of notation we denote the map $t \to M_{\mathbf{r}(t)} \in L_{\cdot n}$ by $M_{\mathbf{r}}$. Set

$$\sigma_{\mathbf{r}(t)} = \frac{1}{n} \sum_{i=1}^n \delta_{r_i(t)}, \qquad r^s_i(t) := M^s_t((c_{i-1} + c_i)/2) \quad (i = 1, \cdots n).$$

If $\mathbf{r}(0) = \mathbf{x}^0$ and $\mathbf{r}(T) = \mathbf{x}^T$ then $M_{\mathbf{r}}(0) = \bar{M}_0$ and $M_{\mathbf{r}}(T) = \bar{M}_T$. By lemma 2.2 and remark 2.1

$$\|(M^s)'\|_{metric} = \|(\sigma^s)'\|_{metric}, \quad \|(M_{\mathbf{r}})'\|_{metric} \geq \|(\sigma_{\mathbf{r}})'\|_{metric}.$$

Thus

$$\mathcal{A}^s_{\cdot n}(\mathbf{r}^s) = \mathcal{A}^s_T(\sigma^s) = \mathcal{Q}(M^s) + \mathcal{C}^s(M^s) - T/24 \leq \mathcal{Q}(M_{\mathbf{r}}) + \mathcal{C}^s(M_{\mathbf{r}}) - T/24 = \mathcal{A}^s_T(\sigma_{\mathbf{r}}) = \mathcal{A}^s_{\cdot n}(\mathbf{r}).$$

This proves that $\mathbf{r}^s$ minimizes $\mathcal{A}^s_{\cdot n}$ over the set $\{\mathbf{r} \in H^1(0, T)^n | \, \mathbf{r}(0) = \mathbf{x}^0, \mathbf{r}(T) = \mathbf{x}^T\}$. We use remark 4.11 to obtain existence of a sequence $\{s_j\}_{j=1}^\infty$ converging to 1 as j tends to ∞, such that $\{\mathbf{r}^{s_j}\}_{j=1}^\infty$ converges to some $\mathbf{r}^1$ in $C^1([0, T])$. Extracting if necessary a subsequence we do not relabel, we obtain that $\{w^{s_j}_{ij}\}_{j=1}^\infty$ converges weakly $*$ in $L^\infty(0, T)$ to some w_{ij}. It is apparent that $w_{ij} = -w_{ji}$ and $w_{ij}(t) \in \partial . | \cdot |(r^1_i(t) - r^1_j(t))$. One uses (50) to obtain that

$$\ddot{\mathbf{r}}^1_i(t) + \mathbf{r}^1_i(t) = \frac{1}{2n} \sum_{j \neq i} w^1_{ij}(t), \quad i = 1, \cdots, n. \tag{51}$$

We set $w^1_{ii}(t) \equiv 0$ and

$$M(x, t) = \mathbf{r}^1_i(t), \qquad W(x, \bar{x}, t) = w^1_{ij}(t)$$

for $x \in (c_{i-1}, c_i)$ and $\bar{x} \in (c_{j-1}, c_j)$ to discover that $\tilde{M}^1_0 = \bar{M}_0$, $M^1_T = \bar{M}_T$ and

$$\left(M^1\right)''_t x + M^1_t x = \frac{1}{2} \int_X W(x, \bar{x}, t) d\bar{x}.$$

This, together with theorem 4.6, shows that M^1 is the unique minimizer of $\mathcal{Q}+\mathcal{C}$ over $\mathcal{C}_T(\bar{M}_0, \bar{M}_T)$. Set $\sigma^1_t = M^1_{t\#}\nu_0 \in \mathcal{P}^n(I\!R)$. We use corollary 4.8 to conclude that σ^1 is the unique minimizer of $\mathcal{A}_T$ over $\mathcal{C}_T(\bar{\sigma}_0, \bar{\sigma}_T)$. QED.

Remark 4.13. *To motivate the statement made in the introduction about the behavior of a simple two-particle system, let us consider* $\mu_0 = 1/2(\delta_{x_0} + \delta_{y_0})$, $\mu_T = 1/2(\delta_{x_T} + \delta_{y_T})$ *and let* σ^1 *be the unique minimizer of* $\mathcal{A}_T$ *over* $\mathcal{C}(\mu_0, \mu_T)$ *with* $T < \pi$. *We learned from the closedness principle of the set* $\mathcal{P}_{\cdot 2}$ *that* σ^1 *must satisfy* $\sigma^1_t = 1/2(\delta_{x_t} + \delta_{y_t})$. *Assume* $t_1 < t_2$, $x(t_1) = y(t_1) =: a_1$ *and* $x(t_2) = y(t_2) =: a_2$. *In other words, the two particles of the system collide at two distinct times. Then, it is obvious that we must have* $x(t) = y(t)$ *for all* $t \in [t_1, t_2]$. *Indeed, let* z *be the unique solution of the ODE* $\ddot{z} + z = 0$ *on* $[t_1, t_2]$ *such that* $z(t_1) = a_1$, *and* $z(t_2) = a_2$. *Then the map* $M_t x = z(t)$ *defined on* $(t_1, t_2) \times X$ *and* $W_M(y, \bar{y}, t) \equiv 0$ *defined on* $X^2 \times (t_1, t_2)$ *satisfies (6). Thus, it minimizes the action*

$$M \to \int_{t_1}^{t_2} dt \int_X (|M'|^2 - |M|^2) dx + \frac{1}{2} \int_{t_1}^{t_2} dt \int_{X^2} |M_t x - M_t \bar{x}| dx d\bar{x}$$

over $\mathcal{C}(a_1, a_2)$. *Define* $\sigma_t := M_{t\#}\nu_0$. *Since* σ *minimizes* $\int_{t_1}^{t_2} L(\sigma_t, v_t) dt$ *over* $\mathcal{C}(\delta_{a_1}, \delta_{a_2})$, *we conclude that* $\sigma_t = \sigma^1_t$ *for* $[t_1, t_2]$.

4.5 Minimizing paths σ^1 such that $\sigma^1(0), \sigma^1(1) << \mathcal{L}^1$.

In this subsection, we show that minimizing paths σ^1 may escape the set of absolutely continuous measures in spite of $\sigma^1(0), \sigma^1(1) << \mathcal{L}^1$. Suppose that $T = 1$ and $\bar{\sigma}_0 << \mathcal{L}^1$ and $\bar{\sigma}_1 << \mathcal{L}^1$ are two Borel probability measures. The main observation in this section is that they may be chosen so that if σ^1 minimizes $\mathcal{A}_1$ over $\mathcal{C}_1(\bar{\sigma}_0, \bar{\sigma}_1)$ then $\sigma^1_t = \delta_0$ for all $t \in [1/2, 3/4]$. In other words, σ^1 escapes the set of measures that are absolutely continuous with respect to *calLone*.

In light of theorem 4.6 and corollary 4.8, to identify appropriate $\bar{\sigma}_0$ and $\bar{\sigma}_1$, it suffices to determine maps $M^1_t : X \to I\!R$ monotone nondecreasing which satisfies the following properties. (i) $M^1 \in \mathcal{C}_1(\bar{M}_0, \bar{M}_1)$, (ii) (44) holds, (iii) $M^1_{0\#}\nu_0, M^1_{1\#}\nu_0 << \mathcal{L}^1$ and (iv) $M^1_t \equiv 0$ for all $t \in [1/2, 3/4]$. Define

$$f(t) = \frac{\cos(t - 1/2)}{\cos 1/2}, \quad \alpha_t = 1 - \frac{f(t)}{f(1/2)}, \quad M_t x = \alpha_t x \ \ t \in [0, 1].$$

Note that $\alpha_t > 0$ unless $t = 1/2$ and $\alpha_{1/2} = 0$. The map M_t is increasing (as a function of x) for $t \in [0, 1]$ provided that $t \neq 1/2$. But $M_{1/2}, M'_{1/2} \equiv 0$. Since $\ddot{f} + f \equiv 0$, we have that $M''_t x + M_t x = x$ for every $(t, x) \in X_1$. Set $W(x, \bar{x}, t) = (x - \bar{x})/|x - \bar{x}|$. We use that $M \in C^\infty(X_1)$ to conclude that

$$M''_t x + M_t x = x = \frac{1}{2} \int_X W(x, \bar{x}, t) d\bar{x}, \tag{52}$$

in the sense of distributions on X_1. Similarly, we define

$$g(t) = \frac{\cos(t - 3/4)}{\cos 1/4}, \quad \beta_t = 1 - \frac{g(t)}{g(3/4)}, \quad N_t x = \beta_t x \quad t \in [1/2, 1], \quad \beta = \beta_1 = 1 - \cos(1/4),$$

As before, the map N_t is increasing (as a function of x) for $t \in [1/2, 1]$ provided that $t \neq 3/4$. Note that $N_{3/4}, M'_{3/4} \equiv 0$. We use that $N \in C^\infty(X_1)$ to obtain

$$N''_t x + N_t x = x = \frac{1}{2} \int_X W(x, \bar{x}, t) d\bar{x}, \tag{53}$$

in the sense of distributions on $(1/2, 1) \times X$.

The monotone maps M^1. For $(x,t) \in X_1$, we define

$$M_t^1 x = \begin{cases} M_t x & \text{if} \quad t \in [0,1/2] \\ 0 & \text{if} \quad t \in [1/2,3/4] \\ N_t x & \text{if} \quad t \in [3/4,1] \end{cases}, \quad W_{M^1}(x,\bar{x},t) = \begin{cases} \frac{x-\bar{x}}{|x-\bar{x}|} & \text{if} \quad t \in [0,1/2] \\ 0 & \text{if} \quad t \in [1/2,3/4] \\ \frac{x-\bar{x}}{|x-\bar{x}|} & \text{if} \quad t \in [3/4,1]. \end{cases}$$

We have that $W_{M^1}(x,\bar{x},t) \in \partial.|\cdot|(M_t^1 x - M_t^1 \bar{x})$ for $\mathcal{L}^3$–almost every $(x,\bar{x},t) \in X^2 \times (0,1)$. Clearly, $M \in C([0,1], L^2(\nu_0))$. Because $\big(M^1\big)'_{(1/2)^-} = \big(M^1\big)'_{(1/2)^+}$ and $\big(M^1\big)'_{(3/4)^-} = \big(M^1\big)'_{(3/4)^+}$ we obtain that $M \in H^2(0,1; L^2(\nu_0))$. One uses (52) , (53) and that

$$\big(M^1\big)''_t x + M_t^1 x = 0 = \frac{1}{2}\int_X W_{M^1}(x,\bar{x},t)d\bar{x},$$

in the sense of distributions on $(1/2, 3/4) \times X$ to obtain that

$$\big(M^1\big)''_t x + M_t^1 x = \frac{1}{2}\int_X W_{M^1}(x,\bar{x},t)d\bar{x}.$$

in the sense of distributions on X_1.

Note that if we set $\sigma_t^1 = M^1_{t\#}\nu_0$ then

$$\sigma_t^1 = \begin{cases} \frac{1}{\alpha_t}\chi_{(-\alpha_t/2,\alpha_t/2)} & \text{if} \quad t \in [0,1/2) \\ \delta_0 & \text{if} \quad t \in [1/2,3/4] \\ \frac{1}{\beta_t}\chi_{(-\beta_t/2,\beta_t/2)} & \text{if} \quad t \in (3/4,1]. \end{cases}$$

The velocity in Lagrangian and Eulerian coordinates is given by

$$\big(M^1\big)'_t x = \begin{cases} -\tan(t-\frac{1}{2})x & \text{if} \quad t \in [0,1/2) \\ 0 & \text{if} \quad t \in [1/2,3/4] \\ -\tan(t-\frac{3}{4})x & \text{if} \quad t \in [3/4,1] \end{cases}, \quad v_t^1(y) = \begin{cases} -\frac{\tan(t-\frac{1}{2})}{\alpha_t}y & \text{if} \quad t \in [0,1/2) \\ 0 & \text{if} \quad t \in (1/2,3/4) \\ -\frac{\tan(t-\frac{3}{4})}{\beta_t}y & \text{if} \quad t \in (3/4,1). \end{cases}$$

We have proven the following theorem.

Theorem 4.14. *Let*

$$\bar{\sigma}_0 = \frac{1}{\alpha_0}\chi_{(-\alpha_0/2,\alpha_0/2)}, \quad \bar{\sigma}_1 = \frac{1}{\beta_1}\chi_{(-\beta_1/2,\beta_1/2)}, \quad v_0(y) = \frac{\tan 1/2}{1-\cos 1/2}y \quad \text{for} \quad |y| \le \alpha_0/2.$$

(i) **Optimal path escaping the set of absolutely continuous measures.** *σ^1, defined above, minimizes $\mathcal{A}_1^1$ over $\mathcal{C}_1(\bar{\sigma}_0,\bar{\sigma}_1)$ and $\sigma_t^1 = \delta_0$ for $t \in [1/2,3/4]$.*
(ii) **Non uniqueness in the initial value problem.** *The path σ defined by $\sigma_t = M_{t\#}\nu_0$, minimizes $\mathcal{A}_1^1$ over $\mathcal{C}_1(\bar{\sigma}_0,\bar{\sigma}_0)$. Thus, σ and σ^1 are two solutions of the Euler-Poisson system, distinct for $t \in (1/2,3/4)$, with the same initial point $\bar{\sigma}_0$ and the same initial velocity $v_0 = v_0^1$.*

5 An infinite-dimensional Hamilton-Jacobi equation

Throughout this section we assume that $0 < T < \pi/2$ and we set

$$X = (-1/2, 1/2), \quad \nu_0 = \mathcal{L}^1|_X, \quad \nu = \nu_0 \times \mathcal{L}^1.$$

Recall that $\mathcal{T}_\mu \mathcal{P}_2(I\!R)$ is a separable Hilbert space and so, it can be identified with its dual. The tangent bundle $\mathcal{T}\mathcal{P}_2(I\!R)$ of $\mathcal{P}_2(I\!R)$ is the union over $\mu \in \mathcal{P}_2(I\!R)$ of $\{\mu\} \times \mathcal{T}_\mu \mathcal{P}_2(I\!R)$. We also identify it with the cotangent bundle. Let us recall the expression of the Lagrangian we have been using in this work:

$$L(\mu, \xi) = \frac{1}{2}\|\xi\|_\mu - \frac{1}{2}W_2^2(\mu, \nu_0), \qquad (\mu, \xi) \in \mathcal{T}\mathcal{P}_2(I\!R).$$

The Hamiltonian associated to L is then

$$H(\mu, \zeta) := \sup \left\{ \langle \zeta, \xi \rangle_\mu - L(\mu, \xi) | \ \xi \in \mathcal{T}_\mu \mathcal{P}_2(I\!R) \right\} = \frac{1}{2}\|\zeta\|_\mu^2 + \frac{1}{2}W_2^2(\mu, \nu_0),$$

which is defined for $(\mu, \zeta) \in \mathcal{T}\mathcal{P}_2(I\!R)$. The aim of this section is to prove existence of solutions (in some sense) for the infinite-dimensional Hamilton-Jacobi equation

$$\frac{\partial U}{\partial t}(\mu, t) + \frac{1}{2}\int_{I\!R} |\nabla_\mu U(\mu, t)(x)|^2 d\mu(x) + \frac{1}{2}W_2^2(\mu, \nu_0) = 0, \quad U(\mu, 0) = U_0(\mu). \tag{54}$$

As in the finite-dimensional case, we show that the value function

$$U(\mu, t) := \min_{\sigma \in \mathcal{C}_t(\cdot, \mu)} \left\{ U_0(\sigma_0) + \int_0^t L(\sigma(\tau), \dot\sigma(\tau)) d\tau \right\} \tag{55}$$

is a viscosity solution to (54). Its characteristics are paths in $AC^2(0, T; \mathcal{P}_2(I\!R))$ satisfying the Euler-Poisson system. Here, $\dot\sigma$ is the velocity of minimal norm associated to σ and

$$\mathcal{C}_t(\cdot, \mu) := \{\sigma \in AC^2(0, t; \mathcal{P}_2(I\!R)) \mid \sigma_t = \mu\}.$$

5.1 Augmented action; the terminal point problem

Throughout this subsection, we fix $U_0 : \mathcal{P}_2(I\!R) \to I\!R$ and define

$$V_0(M) := U_0(M_\# \nu_0). \tag{56}$$

We suppose that V_0 is Frechet differentiable, λ–convex for some real number λ satisfying
(H1) $T\lambda^- < 1 - 4T^2/\pi^2$.
We also assume that $\delta_{L^2} V_0$, the L^2–Frechet differential of V_0, is continuous in the following sense:

(H2) $\{\delta_{L^2} V_0(M_n)\}_{n=1}^\infty$ *converges to* $\delta_{L^2} V_0(M)$
in the sense of distributions on X, whenever $\{M_n\}_{n=1}^\infty \subset L^2(\nu_0)$ converges to M locally in $L^2(\nu_0)$.

We will show that the value function U defined in (55) is a minimum which satisfies the Hamilton-Jacobi equation (54). Our study includes "linear" functions U_0 of the form

$$U_0(\mu) = \int_{I\!R} u_0 d\mu,$$

where $u_0 \in C^1(I\!R)$ is λ–convex as a function defined on $I\!R$ and $|u_0(t)| \leq C(|t|^p + 1)$ for some $C > 0$ and $p \in [1, 2)$. In that case $V_0(M) = \int_X u_0 \circ M d\nu_0$.

For $M \in AC^2(0, T; L^2(\nu_0)$ we define

$$\mathbf{B}^s_T(M) = \mathcal{Q}_T(M) + \mathcal{C}^s_T(M) + V_0(M_0)$$

where, $\mathcal{Q}$, $\mathcal{C}^s$ are defined in subsection 3.2. We write $\mathcal{Q}_T$ in place of $\mathcal{Q}$ and $\mathcal{C}^s_T$ in pace of $\mathcal{C}^s$ to display the dependence in T.. Not to make the notation any heavier, we refrain from displaying the dependence on V_0 in $\mathbf{B}_T$. For $\bar{M}_T \in L^2(\nu_0)$, set

$$\mathcal{C}_T(\cdot, \bar{M}_T) := \{M \in AC^2(0, T; L^2(\nu_0)) \mid M_T = \bar{M}_T\}. \tag{57}$$

If $M, \tilde{M}$, we use (28) to conclude

$$\begin{aligned} \mathcal{Q}_T(\tilde{M}) + V_0(\tilde{M}_0) \geq \mathcal{Q}_T(M) + \quad & V_0(M_0) + 2\mathcal{B}_T(M, \tilde{M} - M) + \langle \delta_{L^2} V_0(M_0), \tilde{M}_0 - M_0 \rangle_{\nu_0} \\ & + \mathcal{Q}_T(\tilde{M} - M) + \tfrac{\lambda}{2} \|M_0 - \tilde{M}_0\|^2_{L^2(\nu_0)} \end{aligned} \tag{58}$$

We also use (26) to define

$$\mathbf{A}^s_T(\sigma) = \mathcal{A}^s_T(M) + U_0(\sigma_0).$$

We denote the $L^2(0, t)$-norm of the metric derivative of $M \in AC^2(0, T; L^2(\nu_0))$ by $\|\tilde{M}'\|_{metric,t}$.

Lemma 5.1 (Strict convexity and coercivity of the augmented Lagrangian). *Suppose $\bar{M}_T \in L^2(\nu_0)$, $V_0 : L^2(\nu_0)$ is Frechet differentiable, λ–convex and $\lambda^- \leq a(T) := (1/T - 4T/\pi^2)$. Suppose $s \in [1, 2]$ and $t \in (0, T]$. Then*
(i) $\mathbf{B}^s_t$ is strictly convex on $\mathcal{C}_t(\cdot, \bar{M}_T)$.
(ii) There exists a constant $c > 0$ depending only T, V_0 and $\bar{M}_T$ such that

$$\mathbf{B}^s_t(\tilde{M}) \geq c\Big(\|\tilde{M}'\|^2_{metric,t} - 1\Big) \tag{59}$$

for all $\tilde{M} \in \mathcal{C}_t(\cdot, \bar{M}_T)$.

Proof: Since the function a is decreasing, we have $\lambda^- < a(T) < a(t)$ for all $t \in (0, T)$. Hence, it suffices to prove the lemma in the case $t = T$. The constant c can be shown to be depending only on T.

Suppose $M, \tilde{M} \in \mathcal{C}(\cdot, \bar{M}_T)$ and set $N = \tilde{M} - M$ so that $N_T \equiv 0$. We apply the Poincaré-Wirtinger inequality (17) with N_T in place of s_0 and N in place if σ to obtain $\|N\|^2_{L^2(\nu)} \leq 2T/\pi \|N'\|^2_{L^2(\nu)}$ and so,

$$\|N'\|^2_{L^2(\nu)} - \|N\|^2_{L^2(\nu)} \geq (1 - 4T^2/\pi^2)\|N'\|^2_{L^2(\nu)}. \tag{60}$$

By (15)

$$\|N_0\|^2_{\nu_0} \leq T\|N'\|^2_{metric} \tag{61}$$

Combining (60) and (61) we conclude that

$$2\mathcal{Q}_T(N) - \lambda^- \int_X |N_0|^2 d\nu_0 \geq \big(1 - 4T^2/\pi^2 - T\lambda^-\big)\|N'\|^2_{L^2(\nu)} \tag{62}$$

By (58) and (62) there exists a continuous linear map L_M defined on $AC^2(0,T;L^2(\nu_0))$ such that

$$\mathcal{Q}_T(\tilde{M}) + V_0(\tilde{M}_0) \geq \mathcal{Q}_T(M) + V_0(M_0) + L_M(\tilde{M} - M) + \big(1 - \frac{4T^2}{\pi^2} - \lambda^- T\big)\|N'\|^2_{L^2(\nu)}. \quad (63)$$

Since, $a(T) > \lambda^-$ and $\|N'\|^2_{L^2(\nu)} > 0$ unless $N \equiv 0$, (63) yields that $M \rightarrow \mathcal{Q}_T(M) + V_0(M_0)$ is strictly convex on $\mathcal{C}(\cdot, \bar{M}_T)$. Since the map $M \rightarrow \int_{X_T^2} |M_t x - M_t \bar{x}|^s dt dx d\bar{x}$ is obviously convex on $\mathcal{C}(\cdot, \bar{M}_T)$, we conclude that $\mathbf{B}_T^s$ is strictly convex on $\mathcal{C}(\cdot, \bar{M}_T)$ as the sum of a strictly convex function and a convex function. We set $M_t = \bar{M}_T$ for all $t \in [0,T]$ in (63) and use that $\mathbf{B}_T^s(\tilde{M}) \geq \mathcal{Q}_T(\tilde{M}) + V_0(\tilde{M}_0)$ to conclude the proof of (ii). QED.

Theorem 5.2. *Suppose $\bar{M}_T \in L^2(\nu_0)$ and $\lambda^- \leq a(T) := (1/T - 4T/\pi^2)$. Fix $\mu \in \mathcal{P}_2(I\!R)$. Then*
(i) *(55) has a unique minimizer σ^1 and σ^1 satisfies the Euler-Poisson equation of theorem 25.*
(ii) *If M_t^1 is the optimal map pushing ν_0 forward to σ_t^1, then M_t^1 satisfies the Euler-Lagrange equation (44). Furthermore,*

$$\big(M^1\big)'_0 = \delta_{L^2} V_0(M_0^1). \quad (64)$$

Proof: As in the previous lemma, it suffices to prove this theorem for $t = T$. We also assume $1 \leq s \leq 2$. Let $\mu \in \mathcal{P}_2(I\!R)$ and let $\bar{M}_T \in \mathcal{M}on$ be such that $\bar{M}_{T\#}\nu_0 = \mu$. Choose $\{M^n\}_{n=1}^\infty$ a minimizing sequence for $\mathbf{B}_T^s$ over $\mathcal{C}_T(\cdot, \bar{M}_T)$. The coercivity of $\mathbf{B}_T^s$ obtained in (59) and the Poincaré-Wirtinger inequality (17) ensures that $\{M^n\}_{n=1}^\infty$ is bounded in $AC^2(0,T;L^2(\nu_0))$ independently of n and $s \in [1,2]$. Just as in the proof of theorem 4.3, $\{M^n\}_{n=1}^\infty$ has a point of accumulation $M^s \in \mathcal{C}_T(\cdot, \mu)$ as n tends to $+\infty$. We exploit (58) to readily obtain that $\mathbf{B}_T^s$ is sequentially lower semicontinuous on $\mathcal{C}_T(\cdot, \bar{M}_T)$. Indeed the missing term

$$M \rightarrow \int_{X_T^2} |M_t \bar{x} - M_t x|^s dt dx d\bar{x}$$

in that inequality, is easily shown to be s equentially lower semicontinuous on bounded subsets of $AC^2(0,T;L^2(\nu_0))$. As in the proof of theorem 4.3, we conclude that M^s minimizes $\mathbf{B}_T^s$ over $\mathcal{C}_T(\cdot, \bar{M}_T)$. The strict convexity of $\mathbf{B}_T^s$ proven in lemma 5.1 makes M^s the unique minimizer. As pointed out in the proof of theorem 4.3, among all the paths $t \rightarrow M_t$ such that $M_{t\#}\nu_0$ is prescribed, the kinetic energy $\int_{X_T} |M_t'|^2 dx dt$ is minimized when M_t is monotone nondecreasing. Thus, M_t^s is monotone nondecreasing. Set $\sigma_t^s = M^s_{t\#}\nu_0$. As in corollary 4.8, we conclude that σ^s is the unique minimize of $\mathcal{A}_T^s$ over $\mathcal{C}_T(\cdot, \mu)$. Note that σ^s minimizes $\mathcal{A}_T^s$ over $\mathcal{C}_T(\sigma_0^1, \mu)$. This completes the proof of (i).

It is also apparent that M^s minimizes $\mathcal{Q} + \mathcal{C}^s$ over $\mathcal{C}_T(M_0^1, M_T)$. Thus (40) holds. Now let us consider variations of the form

$$N_t^\epsilon(x) := M_t^s(x) + \epsilon\xi(t)\zeta(x) \quad \text{with} \quad \xi \in C_c^2([0,T)), \zeta \in C_c(I\!R).$$

We have $N^\epsilon \in \mathcal{C}_T(M_0^1, M_T)$. For $s > 1$ we have that $\mathbf{B}_T^s(N^\epsilon)$ is a differentiable function of ϵ and we can easily obtain $\big(M^s\big)'_0 = \delta_{L^2} V_0(M_0^s)$ by writing that the derivative of $\mathbf{B}_T^s(N^\epsilon)$ with respect to ϵ vanishes when $\epsilon = 0$. From the previous paragraph of the proof, we obtain that $\{M^s |\ s \in [1,2]\}$ is bounded in $AC^2(0,T;L^2(\nu_0))$. By proposition 2.14, there exists a subsequence $\{M^{s_j}\}_{j=1}^\infty$ of $\{M^s|\ s \in [1,2]\}$ such that for each $t \in [0,T]$, $\{M^{s_j}|\}_{j=1}^\infty$ converges weakly in $L^2(\nu_0)$ to $\tilde{M}_t^1$ for some $\tilde{M}^1 \in AC^2(0,T;L^2(\nu_0))$. By lemma 2.4, $\{M^{s_j}\}_{j=1}^\infty$ converges weakly to $\tilde{M}^1$ in $L^2(\nu)$ and $\{(\tilde{M}^{s_j})'\}_{j=1}^\infty$ converges weakly to $(\tilde{M}^1)'$ in $L^2(\nu)$. We use that $\int_X |M_0^s - M_T|^2 dx \leq T\int_0^T dt \int_X \|(M^s)'_t\|^2 dx$ to conclude that $\{(M_0^{s_j})\}_{j=1}^\infty$ is bounded in $L^2(\nu_0)$. This, together with (38) gives that M_0^s is locally bounded in $BV(X)$. Extracting if necessary a subsequence we do not relabel, we may assume that $\{(M_0^{s_j})\}_{j=1}^\infty$ converges locally to $\tilde{M}_0^1$ in $L^2(\nu_0)$ as s tends to 1. We argue as in the proof of theorem 4.6 to obtain that $\tilde{M}^1$ minimizes $\mathbf{B}_T^1$ over $\mathcal{C}_T(\cdot,\bar{M}_T)$. The strict convexity of $\mathbf{B}_T^1$ gives $\tilde{M}^1 = M^1$. Letting s tend to 1 in $\big(M^s\big)'_0 = \delta_{L^2}V_0(M_0^s)$ we obtain (64). QED.

5.2 An infinite dimensional Hamilton-Jacobi equation; viscosity solutions

The notion of viscosity sub and super solutions can be defined in terms of subdifferential and superdifferential of function. The definition of sub (super) differential we give here coincides with that of [2] for λ–convex function. Otherwise, in general, they differ.

Definition 5.3. *Let V be a real valued proper functional defined on $\mathcal{P}_2(I\!R)$ with values in $I\!R\cup\{\pm\infty\}$. Let $\mu_0 \in \mathcal{P}_2(I\!R)$ and $\xi \in \mathcal{T}_{\mu_0}\mathcal{P}_2(I\!R)$. We say that ξ belongs to the subdifferential of V at μ_0 and we write $\xi \in \partial_\cdot V(\mu_0)$ if*

$$V(\mu) - V(\mu_0) \geq \sup_{\Gamma_o(\mu_0,\mu)} \iint_{I\!R\times I\!R} \xi(\bar{y})(y-\bar{y})d\gamma(\bar{y},y) + o\big(W_2(\mu_0,\mu)\big)$$

for all $\mu \in \mathcal{P}_2(I\!R)$. Here $\Gamma_o(\mu_0,\mu)$ is the set of optimal plans between μ_0 and μ.
(ii) We say that ξ belongs to the superdifferential of V at μ_0 and we write $\xi \in \partial^\cdot V(\mu_0)$ if $-\xi \in \partial_\cdot(-V)(\mu_0)$.

Remark 5.4. *As expected, when the sets $\partial_\cdot V(\mu_0)$ and $\partial^\cdot V(\mu_0)$ are both nonempty then coincide and consist of a single element. Indeed, suppose $\xi_1 \in \partial_\cdot V(\mu_0)$ and $\xi_2 \in \partial^\cdot V(\mu_0)$. Let $\varphi \in C_c^1(I\!R)$ which we assume to be distinct from the null function to avoid trivialities. The map $M^\varepsilon := \mathbf{id}+\varepsilon\varphi$ is smooth and increasing for $|\epsilon| << 1$ and so, it is the gradient of a convex function. Thus, setting $\mu_\varepsilon := M^\varepsilon_\# \mu_0$, we have that $\gamma^\varepsilon := (\mathbf{id}\times M^\varepsilon)_\#\mu_0 \in \Gamma_o(\mu_0,\mu_\varepsilon)$. One checks that,*

$$0 \geq \lim_{\varepsilon\to 0} \frac{\int_{I\!R^2}(\xi_1(y)-\xi_2(y))(\bar{y}-y)d\gamma^\varepsilon(y,\bar{y})}{W_2(\mu_0,\mu^\varepsilon)} = \int_{I\!R}(\xi_1-\xi_2)\frac{\varphi}{||\varphi||_{\mu_0}}d\mu_0.$$

Since φ is arbitrary, we obtain that $\xi_1 \equiv \xi_2$ μ_0–almost everywhere.

In case the convex set $\partial_\cdot V(\mu_0)$ is nonempty, its unique element of minimal $\|\cdot\|_{\mu_0}$–norm is denoted by $\nabla_\mu V(\mu_0)$. It is called the *gradient* of V with respect to the Wasserstein distance at μ_0. In particular, if $\partial_\cdot V(\mu_0)\cap\partial^\cdot V(\mu_0) \neq \emptyset$ then $\nabla_\mu V(\mu_0)$ is the unique element of the intersection.

Our goal is to show that U from (55) is a *viscosity solution* on

$$Q_T := \mathcal{P}_2(I\!R)\times(0,T)$$

for the following infinite-dimensional Hamilton-Jacobi equation (54). By analogy with the standard finite-dimensional theory, we have the following definitions:

Definition 5.5. *Let $U : Q_T \to I\!R$ be jointly continuous.*
(i) We say that U is a viscosity subsolution *for (54) if*

$$U(\cdot,0) \le U_0, \text{ and } \theta + H(\mu,\zeta) \le 0 \text{ for all } (\mu,t) \in Q_T \text{ and all } (\zeta,\theta) \in \partial^{\cdot} U(\mu,t). \tag{65}$$

(ii) We say that U is a viscosity supersolution *for (54) if*

$$U(\cdot,0) \ge U_0, \text{ and } \theta + H(\mu,\zeta) \ge 0 \text{ for all } (\mu,t) \in Q_T \text{ and all } (\zeta,\theta) \in \partial_{\cdot} U(\mu,t). \tag{66}$$

Remark 5.6. *If U is a viscosity solution, then, in view of remark 5.4, we deduce that (54) is satisfied at all points $(\mu,t) \in Q_T$ where $\partial_{\cdot} U(\mu,t) \cap \partial^{\cdot} U(\mu,t) \ne \emptyset$, which are precisely the points where U is differentiable.*

Theorem 5.7. *Assume that $0 < T < \pi/2$, that $U_0 : \mathcal{P}_2(I\!R) \to I\!R$ and V_0 is given by (56). Suppose that* (H1) *and* (H2) *hold. Then the functional U given by (55) is a viscosity solution for (54).*

Proof: Clearly, (55) implies $U(\mu,0) = U_0(\mu)$ for all $\mu \in \mathcal{P}_2(I\!R)$, so the first inequalities in (65) and (66) hold trivially.
Step 1: Let $(\mu,t) \in Q_T$ and assume that $(\zeta,\theta) \in \partial^{\cdot} U(\mu,t)$. Next take an arbitrary $\varphi \in C_c^\infty(I\!R)$ and $\varepsilon > 0$ small enough such that $M^\varepsilon := \mathbf{id} + \varepsilon\varphi$ is nondecreasing and define $\mu_\varepsilon := M^\varepsilon_{\#}\mu$. The geodesic connecting μ_ε at time $t-\varepsilon$ and μ at time t is given by $\sigma(\tau) := [\mathbf{id} + (t-\tau)\varphi]_{\#}\mu$, $\tau \in [t-\varepsilon, t]$. If we extend the optimal path in the definition (55) of $U(\mu_\varepsilon, t-\varepsilon)$ by this geodesic we clearly obtain a path in $\mathcal{C}_t(\cdot,\mu)$ which is at most optimal, therefore,

$$U(\mu,t) \le U(\mu_\varepsilon, t-\varepsilon) + \int_{t-\varepsilon}^t L(\sigma(\tau),\dot\sigma(\tau))d\tau. \tag{67}$$

The optimal map pushing ν_0 forward to $\sigma(\tau)$ is the composition $[\mathbf{id} + (t-\tau)\varphi] \circ \tilde M_t$, where $\tilde M_t$ is optimal such that $\tilde M_{t\#}\nu_0 = \mu$. Thus, we compute

$$L(\sigma(\tau),\dot\sigma(\tau)) = \frac{1}{2}\int_{I\!R} |\varphi|^2 d\mu - \frac{1}{2}W_2^2(\mu,\nu_0) - \frac{(t-\tau)^2}{2}\int_{I\!R}|\varphi|^2 d\mu + (t-\tau)\int_X (\tilde M_t - \mathbf{id})\ \varphi \circ \tilde M_t d\nu_0.$$

In view of (67), this implies

$$U(\mu,t) - U(\mu_\varepsilon,t-\varepsilon) \le \frac{\varepsilon}{2}\Big(\int_{I\!R}|\varphi|^2 d\mu - W_2^2(\mu,\nu_0)\Big) + O\big(\varepsilon^2\big)$$

which yields

$$\limsup_{\varepsilon\downarrow 0} \frac{U(\mu,t) - U(\mu_\varepsilon,t-\varepsilon)}{\varepsilon} \le \frac{1}{2}\int_{I\!R}|\varphi|^2 d\mu - \frac{1}{2}W_2^2(\mu,\nu_0). \tag{68}$$

Since $(\zeta,\theta) \in \partial^{\cdot} U(\mu,t)$ we also have

$$\begin{aligned} U(\mu,t) - U(\mu_\varepsilon,t-\varepsilon) \ \ge&\ \iint_{I\!R\times I\!R} \zeta(\bar y)(\bar y - y)d\gamma_\varepsilon(\bar y,y) + \varepsilon\theta + o(W_2(\mu,\mu_\varepsilon)) + o(\varepsilon) \\ =&\ \int_{I\!R} \zeta\ (\mathbf{id} - M^\varepsilon)d\mu + \varepsilon\theta + o(\varepsilon) \\ =&\ -\varepsilon\int_{I\!R}\zeta\varphi d\mu + \varepsilon\theta + o(\varepsilon), \end{aligned}$$

where we took into account that $W_2^2(\mu,\mu_\varepsilon)=\varepsilon^2\int_{I\!R}|\varphi|^2d\mu$ and the optimal plan γ_ε between μ and μ_ε is $\big(\mathbf{id}\times M^\varepsilon\big)_{\#}\mu$. Thus,

$$\liminf_{\varepsilon\downarrow 0}\frac{U(\mu,t)-U(\mu_\varepsilon,t-\varepsilon)}{\varepsilon}\ge\theta-\int_{I\!R}\zeta\varphi d\mu. \tag{69}$$

Combine (68) and (69) to deduce $\theta+H(\mu,\zeta)\le 1/2\int_{I\!R}|\zeta+\varphi|^2d\mu$. Since φ is an arbitrary smooth, compactly supported function and $\zeta\in\mathcal{T}_\mu\mathcal{P}_2(I\!R)=L^2(\mu)$, we infer the second inequality in (65).
Step 2: Consider now $\mu\in\mathcal{P}_2(I\!R)$, $t\in(0,T)$ and the corresponding minimizing path $\tilde\sigma\in\mathcal{C}_t(\cdot,\mu)$. Also, assume $(\zeta,\theta)\in\partial.U(\mu,t)$. We set $\sigma:=\tilde\sigma$ to unburden notation. It is standard to observe that, given any $0<\varepsilon<t$, we have

$$U(\mu,t)=U(\sigma(t-\varepsilon),t-\varepsilon)+\int_{t-\varepsilon}^t L(\sigma(\tau),\dot\sigma(\tau),)d\tau. \tag{70}$$

Due to the definition of $(\zeta,\theta)\in\partial.U(\mu,t)$, we write

$$U(\mu,t)-U(\sigma(t-\varepsilon),t-\varepsilon)\le\iint_{I\!R\times I\!R}\zeta(\bar y)(\bar y-y)d\gamma_\varepsilon(\bar y,y)+\varepsilon\theta+o(W_2(\mu,\sigma(t-\varepsilon)))+o(\varepsilon), \tag{71}$$

where γ_ε is any optimal plan between $\mu=\sigma_t$ and $\sigma(t-\varepsilon)$. Taking $\gamma_\varepsilon=(M_{t-\varepsilon}\times M_t)_{\#}\nu_0$, one has

$$\lim_{\varepsilon\downarrow 0}\iint_{I\!R\times I\!R}\zeta(\bar y)\frac{y-\bar y}{\varepsilon}d\gamma_\varepsilon(\bar y,y)=-\int_X\zeta(M_tx)(M_t'x)dx,$$

Here $M\in H^2(0,t;L^2(\nu_0))$ is such that M_τ is monotone nondecreasing and $M_{\tau\#}\nu_0=\sigma(\tau)$, $0<\tau<t$. Also, note that $W_2(\sigma_t,\sigma(t-\varepsilon))/\varepsilon\to\|M_t'\|_{L^2(\nu_0)}$ as $\varepsilon\downarrow 0$. Consequently, the inequality (71) yields

$$\limsup_{\varepsilon\downarrow 0}\frac{U(\mu,t)-U(\sigma(t-\varepsilon),t-\varepsilon)}{\varepsilon}\le\theta+\int_X(M_t'x)\zeta(M_tx)dx, \tag{72}$$

which, in view of (70), leads to

$$\theta+\langle M_t',\zeta\circ M_t\rangle_{L^2(\nu_0)}-\frac12\|M_t'\|^2_{L^2(\nu_0)}+\frac12W_2^2(\mu,\nu_0)\ge 0.$$

Therefore,

$$\theta+H(\mu,\zeta)\ge\frac12\|M_t'-\zeta\circ M_t\|^2_{L^2(\nu_0)},$$

which is stronger than the second inequality in (66). QED.

5.3 Connection with the finite dimensional case

The following result connects our infinite-dimensional objects with the corresponding finite-dimensional, classical restrictions. It is the analogue of the closedness principle for $\mathcal{P}_{\cdot n}$. In this subsection, we will not make a distinction between ν_0 and its restriction to subsets of X. We use the same convention for ν.

Proposition 5.8. *Suppose that $0 < T < \pi/2$, that $U_0 : \mathcal{P}_2(I\!R) \to I\!R$ and V_0 is given by (56). Suppose that* (H1) *and* (H2) *hold. Then the viscosity solution U given by (55) for our infinite-dimensional Hamilton-Jacobi with initial data U_0 has the following property: if $\mu^n_{\mathbf{x}} = (1/n)\sum_{i=1}^n \delta_{x_i}$ for some real $x_1,\ x_2, \cdots,\ x_n \in I\!R$, then the optimal trajectory for $U(\mu, t)$ consists of averages of n point masses as well.*

Before we prove this, let us introduce the finite-dimensional version of (54), namely

$$\begin{cases} \partial_t U^n + H^n\big(\mathbf{x}, \nabla U^n\big) = 0 \quad \text{in } Q^n_T := I\!R^n \times (0,T), \\ \qquad\qquad U^n(\cdot, 0) = U^n_0 \text{ on } I\!R^n, \end{cases} \tag{73}$$

where

$$\begin{aligned} H^n(\mathbf{x}, \mathbf{p}) := H(\mu^n_{\mathbf{x}}, \mathbf{p}) &= \frac{|\mathbf{p}|^2}{2n} + \frac{1}{2}W_2^2(\mu^n_{\mathbf{x}}, \nu_0) \\ &= \frac{|\mathbf{p}|^2}{2n} + \frac{|\mathbf{x}|^2}{2n} - \frac{1}{4n^2}\sum_{i,j=1}^n |x_i - x_j| + \frac{1}{24} \text{ for } \mathbf{x} \in I\!R^n, \mathbf{p} \in I\!R^n \simeq \mathcal{T}_{\mu^n_{\mathbf{x}}}\mathcal{P}_2(I\!R). \end{aligned}$$

The associated Lagrangian is thus

$$\begin{aligned} L^n(\mathbf{x}, \mathbf{v}) := L(\mu^n_{\mathbf{x}}, \mathbf{v}) &= \frac{|\mathbf{v}|^2}{2n} - \frac{1}{2}W_2^2(\mu^n_{\mathbf{x}}, \nu_0) \\ &= \frac{|\mathbf{v}|^2}{2n} - \frac{|\mathbf{x}|^2}{2n} + \frac{1}{4n^2}\sum_{i,j=1}^n |x_i - x_j| - \frac{1}{24} \text{ for } \mathbf{x}, \mathbf{v} \in I\!R^n. \end{aligned}$$

Note that, since L^n is invariant to coordinate permutations, we have that

$$U^n_0(\sigma^\kappa(0)) + \int_0^t L^n(\sigma^\kappa(\tau), \dot{\sigma^\kappa}(\tau))d\tau$$

is independent of the permutation $\kappa \in \Sigma_n$, where $\mathbf{x}^\kappa = (x_{\kappa(1)}, x_{\kappa(2)}, ..., x_{\kappa(n)})$. Due to U^n_0 sharing this property, it follows that

$$U^n(\mathbf{x}, t) = U^n(\mathbf{x}^\kappa, t) \text{ for all } \mathbf{x} \in I\!R^n,\ \kappa \in \Sigma_n,\ t \in [0, T). \tag{74}$$

Here U^n is the viscosity solution for (73) given by

$$U^n(\mathbf{x}, t) := \min_{\sigma \in AC^2(0,t;I\!R^n)} \left\{ U^n_0(\sigma_0) + \int_0^t L^n(\sigma(\tau), \dot\sigma(\tau))d\tau \;\middle|\; \sigma_t = \mathbf{x} \right\}. \tag{75}$$

The optimal trajectory for $U^n(\mathbf{x}^\kappa, t)$ is obviously σ^κ_0, where σ_0 is optimal for $U^n(\mathbf{x}, t)$. Note that, according to the proof of Proposition 5.8 (see below), the solution for (54) at $(\mu^n_{\mathbf{x}}, t)$ is

$$U(\mu^n_{\mathbf{x}}, t) = \min_{\kappa \in \sum_n} U^n(\mathbf{x}^\kappa, t). \tag{76}$$

Along with (74), (76) implies the result stated next.

Theorem 5.9. *If $U_0 \in \mathcal{U}_T$ for some $T \in (0, \pi/2)$, then for any integer $n \geq 1$ one has*

$$U\big(\mu^n_{\mathbf{x}}, t\big) = U^n(\mathbf{x}, t) \text{ for all } \mathbf{x} \in I\!R^n \text{ and all } t \in [0, T],$$

where U and U^n are the variational solutions for (54) and (73) respectively.

We shall next give a proof of Proposition 5.8.
Proof: (of Proposition 5.8) Let us denote

$$\tilde{\mathcal{A}}(\mu) := \mathcal{A}(\mu) + U_0(\mu_0), \;\; \mu \in AC^2(0, t; L^2(\nu_0)).$$

W.l.o.g. we assume $x_1 \leq x_2 \leq ... \leq x_n$. One may argue as in the proof of Theorem 2.14 that $\mathbf{B}^1_t$ has a unique minimizer $\tilde{M}$ over $AC^2(0, t; L_{\cdot n})$ among the paths of prescribed terminal point $\bar{M}_t$. Here, $\bar{M}_t$ is the monotone nondecreasing map such that $M_{t\#}\nu_0 = \mu^n_{\mathbf{x}}$. Since, in particular, $\tilde{M}$ minimizes $\mathbf{B}^1_t$ over the set of paths in $AC^2(0, t; L_{\cdot n})$ which have the same initial and terminal point as $\tilde{M}$, if we let $r_i(\tau) := \tilde{M}_\tau(z)$ for $z \in (c_{i-1}, c_i)$, $i = 1, ..., n$, we deduce $\mathbf{r}(t) = \mathbf{x}$ and $r_1(\tau) \leq r_2(\tau) \leq, \cdots, \leq r_n(\tau)$ for all $\tau \in [0, t]$. Also, $\mathbf{r} \in H^2(0, t; I\!R^n)$. For $\mathcal{L}^1$–almost every τ point of twice differentiability of $\mathbf{r}$,

$$\tilde{M}''_\tau x = \ddot{r}_i(\tau) \text{ for all } x \in (c_{i-1}, c_i). \tag{77}$$

Consider $M \in \mathcal{C}_t(\cdot, \bar{M}_t)$ a path such that M_τ are monotone nondecreasing maps and let $M_{\tau\#}\nu_0 =: \mu_\tau$ for $0 \leq \tau < t$. The convexity of $e \to G(e) = |e|/2$ implies

$$\begin{aligned}
2\tilde{\mathcal{A}}(\mu) + \frac{1}{12} \;&=\; \int_{X_t} (|M'|^2 - |M|^2) d\nu + \frac{1}{2}\int_{X_t^2} |M_\tau x - M_\tau \bar{x}| dx d\bar{x} d\tau + 2V_0(M_0) \\
&\geq\; 2\tilde{\mathcal{A}}(\tilde{\mu}) + \frac{1}{12} + \int_{X_t} (|M' - \tilde{M}'|^2 - |M - \tilde{M}|^2) d\nu + 2[V_0(M_0) - V_0(\tilde{M}_0)] \\
&+\; 2\sum_{i=1}^n \int_{c_{i-1}}^{c_i} d\nu_0 \int_0^t \Big(\big(M'_\tau - \dot{r}_i(\tau)\big)\dot{r}_i(\tau) - \big(M_\tau - r_i(\tau)\big) r_i(\tau)\Big) d\tau \qquad (78) \\
&+\; \frac{1}{2}\sum_{j \neq i}^n \int_{X_{ij}} dx d\bar{x} \int_0^t w_{ij}(\tau)\Big((M_\tau x - r_i(\tau)) - (M_\tau \bar{x} - r_j(\tau))\Big) d\tau. \qquad (79)
\end{aligned}$$

where we have set $X_{ij} = (c_{i-1}, c_i) \times (c_{j-1}, c_j)$ and w_{ij} are from (51). Due to the minimizing property of $\tilde{M}$ over $AC^2(0, t; L_{\cdot n})$ with terminal endpoint fixed $\bar{M}_t$, one can easily infer (as in the proof of theorem 5.2) that (64) holds. Thus, we can replace $\delta_{L^2} V_0(\tilde{M}_0)$ by $\tilde{M}'_0$ and write

$$\langle \tilde{M}'_0, M_0 - \tilde{M}_0 \rangle_{L^2(\nu_0)} = \sum_{i=1}^n \int_{c_{i-1}}^{c_i} \dot{r}_i(0)(M_0 - r_i(0)) d\nu_0. \tag{80}$$

Since $\tilde{M}_t = M_t$, we use

$$\int_{X_t} (|M' - \tilde{M}'|^2 - |M - \tilde{M}|^2) d\nu \geq \frac{1}{t}\Big(1 - \frac{4t^2}{\pi^2}\Big) \|\tilde{M}_0 - M_0\|^2_{L^2(\nu_0)}$$

again, along with the fact that $U_0 \in \mathcal{U}_T$ for $T \in (0, \pi/2)$ and $t \in (0, T)$, to deduce in light of (80)

$$\int_{X_t} (|M' - \tilde{M}'|^2 - |M - \tilde{M}|^2) d\nu + 2(V_0(M_0) - V_0(\tilde{M}_0)) \geq 2 \sum_{i=1}^{n} \int_{c_{i-1}}^{c_i} \dot{r}_i(0)(M_0 - r_i(0)) d\nu_0. \quad (81)$$

We can integrate by parts the expression in (78) and obtain explicit boundary terms that vanish when we combine the result with (79), (81). Finally, we exploit the fact that $w_{ji} = -w_{ij}$ to obtain

$$\begin{aligned} 2\tilde{\mathcal{A}}(\mu) + \frac{1}{12} &\geq 2\tilde{\mathcal{A}}(\tilde{\mu}) + \frac{1}{12} \\ &- 2\sum_{i=1}^{n} \int_{c_{i-1}}^{c_i} d\nu_0 \int_0^t \Big(\ddot{r}_i(\tau) + r_i(\tau) - \frac{1}{2n} \sum_{j \neq i}^{n} w_{ij}(\tau)\Big)\big(M_\tau - r_i(\tau)\big) d\tau \\ &= 2\tilde{\mathcal{A}}(\tilde{\mu}) + \frac{1}{12}. \end{aligned} \quad (82)$$

We have used (51) to obtain (82). QED.

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