Entrance Exam, Real Analysis

September 1, 2017

Solve exactly 6 out of the 8 problems

1. Prove by definition (in $\epsilon - \delta$ language) that $f(x) = \sqrt{1+x^2}$ is uniformly continuous in (0, 1). Is $f(x)$ uniformly continuous in $(1, \infty)$? Prove your conclusion.

2. Let $f_n(x) = \dfrac{nx}{n+x}$.

   (a) Find $f(x) = \lim_{n\to\infty} f_n(x)$;

   (b) Does f_n converge to f uniformly $(0, 1)$? Prove your conclusion.

   (c) Does f_n converge to f uniformly $(1, \infty)$? Prove your conclusion.

3. If $\overline{\lim_{n\to\infty}} s_n \geq 0$, which of the following statements are true? Explain your conclusion.

   (a) $\exists N > 0$ such that $\forall n > N$, $s_n \geq 0$.

   (b) $\forall N > 0$, $\exists n > N$ such that $s_n \geq 0$.

   (c) $\exists \epsilon > 0$ such that $\forall N > 0$, $\exists n > N$ with $s_n \geq 0$.

   (d) $\overline{\lim_{n\to\infty}} s_n^2 \leq (\overline{\lim_{n\to\infty}} s_n)^2$.

4. Let $g(x)$ be defined as follows

$$g(x) = \int_{-\infty}^{\infty} \frac{\cos(xy^3)}{1+y^2} dy.$$

   (a) Prove $g(x)$ is continuous in $(-\infty, \infty)$.

   (b) Is $g(x)$ uniformly continuous in $(-\infty, \infty)$? absolutely continuous in $(-\infty, \infty)$? Prove your conclusion.

5. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers satisfying $\sum_{n=1}^{\infty} |a_n| < \infty$. Define f by $f(x) = \sum_{n=1}^{\infty} a_n x^n$.

   (a) Show that f is a function of bounded variation on $x \in [-1, 1]$.

   (b) If $a_1 \neq 0$ and $a_n = 0$ for $n \geq 2$, is f absolutely continuous on $x \in \mathbf{R}$. Explain your answer.

   (c) If $a_1 \neq 0$, $a_2 \neq 0$, and $a_n = 0$ for $n \geq 3$, is f absolutely continuous on $x \in \mathbf{R}$. Explain your answer.

6. For $1 \le p < \infty$ define

$$\|f\|_p = (\int_E |f|^p)^{1/p}$$

and denote $L^p(E)$ to be the set of functions f for which $\int_E |f|^p < \infty$.

Either prove the statement or show a counter example.

(a) If $E = [0, 1]$, then there is a constant $c > 0$ for which

$$\|f\|_1 \le c\,\|f\|_2 \text{ for all } f \in L^2(E).$$

(b) Suppose $E = [1, \infty)$. If f is bounded and $\int_E |f|^2 < \infty$, then it belongs to $L^1(E)$.

7. Suppose that $f(x)$ is a uniformly continuous and Lebesgue integrable function on $\mathbf{R}$. Show that

$$\lim_{|x| \to \infty} f(x) = 0.$$

8. Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of Lebesgue measurable functions on $[0, 1]$ and assume $\lim_{n\to\infty} u_n(x) = 0$ a.e. on $[0, 1]$, and also $\|u_n\|_{L^2[0,1]} \le 1$ for all n. Prove that

$$\lim_{n\to\infty} \|u_n\|_{L^1[0,1]} = 0.$$

Entrance Exam, Real Analysis

April 21, 2017

Solve exactly 6 out of the 8 problems

1. Is $f(x) = x^3$ uniformly continuous in (-5, 1)? in $(1, \infty)$? Prove your conclusion.

2. Let $f_n(x) = \dfrac{nx}{n+x}$.

   (a) Find $f(x) = \lim_{n\to\infty} f_n(x)$;

   (b) Does f_n converge to f uniformly $(0, 1)$? Prove your conclusion.

   (c) Does f_n converge to f uniformly $(1, \infty)$? Prove your conclusion.

3. If $\overline{\lim_{n\to\infty}} s_n \leq 0$, which of the following statements are true? Explain your conclusion.

   (a) $\exists N > 0$ such that $\forall n > N$, $s_n \leq 0$.

   (b) $\forall N > 0$, $\exists n > N$ such that $s_n \leq 0$.

   (c) $\exists \epsilon > 0$ such that $\forall N > 0$, $\exists n > N$ with $s_n > -\epsilon$.

   (d) $\overline{\lim_{n\to\infty}} s_n^2 > (\overline{\lim_{n\to\infty}} s_n)^2$.

4. Let $g(x)$ be defined as follows

$$g(x) = \int_{-\infty}^{\infty} \frac{\cos^2(xy)}{1+y^2} dy.$$

   (a) Prove $g(x)$ is continuous in $(-\infty, \infty)$.

   (b) Is $g(x)$ uniformly continuous in $(-\infty, \infty)$? absolutely continuous in $(-\infty, \infty)$? Prove your conclusion.

5. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers satisfying $\sum_{n=1}^{\infty} |a_n| < \infty$. Define $f : [-1, 1] \to \mathbf{R}$ by $f(x) = \sum_{n=1}^{\infty} a_n x^n$. Show that f is a function of bounded variation.

6. Suppose that $f : [0, \infty) \to [0, \infty)$ is measurable and that $\int_0^{\infty} f(x) < \infty$. Prove that

$$\lim_{n\to\infty} \int_0^{\infty} \frac{x^n f(x)}{1+x^n} dx = \int_1^{\infty} f(x) dx$$

7. Suppose that $f(x)$ is a uniformly continuous and Lebesgue integrable function on $\mathbf{R}$. Show that

$$\lim_{|x|\to\infty} f(x) = 0.$$

8. Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of Lebesgue measurable functions on $[0, 1]$ and assume $\lim_{n\to\infty} u_n(x) = 0$ a.e. on $[0, 1]$, and also $\|u_n\|_{L^2[0,1]} \leq 1$ for all n. Prove that

$$\lim_{n\to\infty} \|u_n\|_{L^1[0,1]} = 0.$$

Entrance Exam, Real Analysis

August 31, 2016

Solve exactly 6 out of the 8 problems

1. (a) Prove by definition that x^2 is uniformly continuous in $(-7, 2)$.

   (b) Prove by definition that x^2 is not uniformly continuous in $[0, \infty)$.

2. If $\overline{\lim_{n\to\infty}} s_n = 0$, which of the following statements are true? Explain your conclusion.

   (a) $\exists N > 0$ such that $\forall n > N$, $s_n \le 0$.

   (b) $\forall N > 0$, $\exists n > N$ such that $s_n \le 0$.

   (c) $\forall \epsilon > 0$, $\exists N > 0$, such that $\forall n > N$, $s_n > -\epsilon$.

   (d) $\exists \epsilon > 0$ such that $\forall N > 0$, $\exists n > N$ with $s_n > -\epsilon$.

3. Let $f(x)$ be a function of bounded variation defined on [0,1]. Prove that the set of all discontinuity points of $f(x)$ is countable.

4. Let $f(x)$ be a measurable function defined in $\mathbb{R}$ and satisfying:

   (a) $f(0) = 1$;

   (b) If $f(x_0) > 0$, then $\exists \delta > 0$ such that$f(x) > 0$ in $(x_0 - \delta, x_0 + \delta)$;

   (c) If $f(x_n) > 0$ and $x_n \to a$, then $f(a) > 0)$.

   Is it true $f(x) > 0$ $\forall x \in \mathbb{R}$? Prove your conclusion.

5. Suppose that $\{f_n\}$ is a sequence of real valued measurable functions defined on the interval $[0, 1]$ and suppose that

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ a.e. on } [0, 1].$$

   Let $p > 1$, $M > 0$, and that$\|f_n\|_{L^p(R)} \le M$ for all n.
   (a) Prove that $f \in L^p(R)$ and that $\|f\|_{L^p(R)} \le M$.

   (b) Prove that $\lim_{n\to\infty} \|f - f_n\|_{L^p(R)} = 0$.

6. Suppose that $f \in L^1(R)$ is a uniformly continuous function. Show that

$$\lim_{|x|\to\infty} f(x) = 0.$$

7. For $1 \le p < \infty$ define

$$\|f\|_p = (\int_E |f|^p)^{1/p}$$

and denote $L^p(E)$ to be the set of functions f for which $\int_E |f|^p < \infty$. For each statement below, either prove it or show a counter example.

(a) If $E = [0,1]$, then there is a constant $c \ge 0$ for which

$$\|f\|_2 \le c\,\|f\|_1 \text{ for all } f \in L^1(E).$$

(b) Suppose $E = [1,\infty)$. If f is bounded and $\int_E |f|^2 < \infty$, then it belongs to $L^1(E)$.

8. (a) For a Lebesgue integrable function f on [0,1] define $F(x) = \int_0^x f(t)dt$ for $x \in [0,1]$. Prove that F is absolutely continuous on [0,1].

(b) Give an explicit example of a continuous function F of bounded variation on [0,1] that is not absolutely continuous on [0,1]. It is not necessary to verify your assertion.

Entrance Exam, Real Analysis

April 21, 2016

Solve exactly 6 out of the 8 problems

1. Let f be a monotone function defined on [0,1] with $f(0) = 0$ and $f(1) = 1$. Show that f has at most countable discontinuous points.

2. Answer the following questions and prove your conclusion.

   (a) Is x^2 uniformly continuous in the open interval $(-5, 2)$?

   (b) Is x^2 uniformly continuous in $[0, \infty)$?

3. If $\overline{\lim_{n\to\infty}} s_n > 0$, which of the following statements are true? Prove your conclusion.

   (a) $\exists N > 0$ such that $\forall n > N$, $s_n > 0$.

   (b) $\exists \epsilon > 0$ such that $\forall N > 0$, $\exists n > N$ with $s_n > \epsilon$.

   (c) $\forall \epsilon > 0$, $\exists N > 0$, such that $\forall n > N$, $s_n > -\epsilon$.

   (d) $\overline{\lim_{n\to\infty}} s_n^2 = (\overline{\lim_{n\to\infty}} s_n)^2$.

4. Let $f \in L^1(-\infty, \infty)$, and $g(x)$ be defined as follows

$$g(x) = \int_{-\infty}^{\infty} \cos^2(xy) f(y) dy.$$

   (a) Prove $g(x)$ is continuous in $(-\infty, \infty)$.

   (b) Is $g(x)$ uniformly continuous in $(-\infty, \infty)$? absolutely continuous in $(-\infty, \infty)$? Prove your conclusion.

5. Let f and g be real valued measurable functions on $[0, 1]$ with the property that g is differentiable at every x on $[0, 1]$ and

$$g'(x) = (f(x))^2.$$

   Show that $f \in L^1[0, 1]$.

6. (a) Let m be the Lebesgue measure on $\mathbb{R}$ and let $E \in \mathbb{R}$ be measurable. Also, suppose $f \in L^1(E)$ and that $f > 0$ a.e. on E. Show that

$$\lim_{n\to\infty} \int_E |f(x)|^{1/n} dx = m(E).$$

   (b) Let $E = [a, b]$, where a and b are real numbers satisfying $a < b$. With the same assumptions on f as in (a), show that

$$\lim_{n\to\infty} \left(\int_a^b |f(x)|^{1/n} dx\right)^n = \infty \text{ if } b - a > 1$$

   and

$$\lim_{n\to\infty} \left(\int_a^b |f(x)|^{1/n} dx\right)^n = 0 \text{ if } b - a < 1.$$

7. Suppose that $f : [0,\infty) \to [0,\infty)$ is measurable and that $\int_0^\infty f(x) < \infty$. Prove that

$$\lim_{n\to\infty} \int_0^\infty \frac{x^n f(x)}{1+x^n} dx = \int_1^\infty f(x)dx$$

8. Let $\{u_n\}_{n=1}^\infty$ be a sequence of Lebesgue measurable functions on $[0,1]$ and assume $\lim_{n\to\infty} u_n(x) = 0$ a.e. on $[0,1]$, and also $\|u_n\|_{L^2[0,1]} \leq 1$ for all n. Prove that

$$\lim_{n\to\infty} \|u_n\|_{L^1[0,1]} = 0.$$

Entrance Exam, Real Analysis

April 17, 2014

Solve exactly 6 out of the 8 problems.

1. Answer the following questions and prove your conclusion.
   - If A is a closed subset of [0,1] and $A \neq [0,1]$. Is it possible that $mA = 1$?
   - If B is an open subset of [0,1] and dense in $[0,1]$. Is it possible that $mB < 1$?

2. Let $\{p_k\}$ be all the rational numbers in [0,1], and $H(x)$ be the Heaviside function defined as
$$H(x) = \begin{cases} 1, & x > 0, \\ 0, & x \leq 0. \end{cases}$$
   Define $f(x)$ in [0,1] as
$$f(x) = \sum_{k=1}^{\infty} \frac{(-1)^k}{2^k} H(x - p_k).$$
   - Prove by definition (in $\epsilon - \delta$ language) that $f(x)$ is continuous at all irrational numbers, and discontinuous at all rational numbers in [0,1].
   - Determine and explain: (i). Is $f(x)$ Riemann integrable in [0,1]? (ii). Is $f(x)$ Lebegue integrable? (iii) Is $f(x)$ of bounded total variation?

3.
   - Let f be Lebegue integrable in [0,b] with $b < \infty$. Prove
$$\lim_{n\to\infty} \int_0^b f(x) \cos nx \, dx = 0.$$
   - Is it also true if $b = \infty$? Prove your conclusion.

4.
   - If f is absolutely continuous on [0,1] and $f(x) > 0$ on [0,1]. Prove by definition that $1/f$ is absolutely continuous on [0,1].
   - Is $f(x) = \sqrt{x}$ absolutely continuous on [0,1]? In $[1, \infty)$? Prove your conclusion.

5. Prove or disprove that for every $\varepsilon > 0$ and for every $f \in L^\infty[a,b]$, where a and b are finite, there is a $g \in C[a,b]$ such that
$$\text{ess sup}_{a \leq x \leq b} |f - g| < \varepsilon.$$

6. Compute the Lebesgue integral
$$\int_0^1 \liminf_{n\to\infty} xe^x \sin^2(\pi n x) \, dx.$$

7. Prove or disprove the following statement.

   (a) Let g be an integrable function on $[0; 1]$. Then there is a bounded measurable function f such that

   $$\int_0^1 fg = \|g\|_1 \, \|f\|_\infty \, .$$

   (b) Let $\{f_n\}$ be a sequence of functions in $L^1[0,1]$, which converge almost everywhere to a function f in L^1, and suppose that there is a constant M such that $\|f_n\|_1 \leq M$ all n. Then for each function g in L^∞ we have

   $$\int_0^1 fg = \lim_{n\to\infty} \int_0^1 f_n g.$$

8. Let $\{a_n\}_{n=1}^\infty$ be a sequence of real numbers satisfying $\sum_{n=1}^\infty |a_n| < \infty$. Define $f : [-1,1] \to R$ by $f(x) = \sum_{n=1}^\infty a_n^2 x^n$. Show that f is a function of bounded variation in $[-1,1]$.

Entrance Exam, Real Analysis

April 22, 2015

Solve exactly 6 out of the 8 problems.

1. Let $A \subset \mathbb{R}$ be an open set with the following property: $\forall \{x_n\} \subset A$, $\exists$ a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x_0 \in A$.
   Which of the following statements is true?
   $mA = 0$? $mA = \ln 2$? $mA = \sqrt{\pi}$? $mA = \infty$? mA is not uniquely determined and could be any positive number?
   Prove your conclusion.

2. Answer the following questions and prove your conclusion.

   - Let $A \subset [0,1]$ be an open set. If $\exists \delta < 1$ such that $mA \leq \delta$, is it possible that A is dense in [0,1]?
   - Let $B \subset [0,1]$ and $B \neq [0,1]$. If $\forall \epsilon > 0$, $mB > 1 - \epsilon$, is it possible that B is closed?

3. Let $f \in L^1(-\infty, \infty)$, and $g(x)$ be defined as follows

$$g(x) = \int_{-\infty}^{\infty} \cos(xy) f(y) dy.$$

   - Prove $g(x)$ is continuous in $(-\infty, \infty)$.
   - Is $g(x)$ uniformly continuous in $(-\infty, \infty)$? Prove your conclusion.

4. Let $f(x) > 0$ and be absolutely continuous on [0,1]. Let $g(x) = \dfrac{1}{\sqrt{f(x)}}$. Prove $g(x)$ is absolutely continuous on [0,1].

5. (a) For a bounded Lebesgue integrable function f on [0,1] define $F(x) = \int_0^x f(t)dt$ for $x \in [0,1]$. Prove that F is absolutely continuous on [0,1].

   (b) Give an explicit example of a continuous function F of bounded variation on [0,1] that is not absolutely continuous on [0,1]. It is not necessary to verify your assertion.

6. (a) Suppose that $f \in L^1[0,1]$ and let m denote Lebesgue measure. Prove that for $c > 0$

$$m\{|f(x)| \geq c\} \leq \frac{1}{c}\int_0^1 |f(x)|dx.$$

(b) Suppose that $\{f_n\} \subseteq L^p[0,1]$ is a sequence of functions satisfying $\|f_n\|_{L^p[0,1]} \leq M$, where $1 < p < \infty$. Prove that

$$\lim_{c\to\infty} \int_{|f_n(x)|\geq c} |f_n(x)|dx = 0,$$

uniformly in n.

7. Assume that $n \geq 1$ is an integer and let $f \in \cap_{n=1}^{\infty} L^n[0,1]$. Prove that if

$$\sum_{n=1}^{\infty} \|f\|_{L^n[0,1]} < \infty,$$

then $f = 0$ a.e.

8. Suppose that $\{f_n\}$ is a sequence in $L^1(R)$ with $\|f_n\|_{L^1(R)} \leq 1$ for all n and

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ a.e.}$$

(a) Prove that $f \in L^1(R)$ and that $\|f\|_{L^1(R)} \leq 1$.

(b) Show that

$$\lim_{n\to\infty} (\|f - f_n\|_{L^1(R)} - \|f_n\|_{L^1(R)} + \|f\|_{L^1(R)}) = 0.$$

Hint: The following inequality might be useful. For any numbers a and b

$$0 \leq |a - b| - |a| + |b| \leq 2|b|$$

Entrance Exam, Real Analysis

August 27, 2013

Solve exactly 6 out of the 8 problems

1. Given function $\phi(x)$ with $\phi(1) = 1$ and $\phi'(x) = e^{-x^2}$. The plane curve Γ is defined by the equation
$$y = \phi(1 + xy + y^2).$$
Find the equation for the tangent line to the curve Γ at the point $(x, y) = (-1, 1)$.

2. Does
$$f_n(x) = x^n \sin\left(\frac{1-x}{x}\right)$$
converge uniformly on (0,1)? Prove your conclusion.

3. Let $\{x_n = p_n/q_n \in \mathbb{Q}\}$ be a sequence of rational numbers and $x_n \to \alpha \in \mathbb{R}\backslash\mathbb{Q}$ (α is irrational). Prove $q_n \to \infty$.

4. Let E be a Lebesgue measurable set in $\mathbb{R}$, and f_n be a sequence of nonnegative Lebesgue integrable functions on E. If $f_n \to f$ a.e. on E, is it always true that
$$\int_E f dx \leq \underline{\lim}_{n\to\infty} \int_E f_n dx?$$
Prove your conclusion.

5. Let a and b are real numbers satisfying $a < b$. Prove or disprove the following.
   (a) If $f(x)$ is a differentiable function on $a < x < b$, it is absolutely continuous on $a < x < b$.
   (b) If $f(x)$ is absolutely continuous on $a \leq x \leq b$, it is Lipschitz on $a \leq x \leq b$.

6. A family $\mathcal{F}$ of measurable functions on E of finite measure is said to be uniformly integrable over E provided that for each $\varepsilon > 0$, there is a $\delta > 0$ such that for each $f \in \mathcal{F}$ and for each measurable set $A \subseteq E$ with $mA < \delta$, $\int_A |f| < \varepsilon$. Either prove or disprove the following statements.
   (a) If $\mathcal{G}$ is the family of measurable functions f on $[0, 1]$, each of which is integrable over $[0, 1]$ and has $\int_0^1 |f| < 1$, then $\mathcal{G}$ is uniformly integrable over $[0, 1]$.
   (b) If $\{f_n\}$ is a sequence of nonnegative, measurable, and integrable functions that converges pointwise a.e. on E to $h = 0$, then $\{f_n\}$ is uniformly integrable.

7. For $1 \leq p < \infty$ define

$$\|f\|_p = (\int_E |f|^p)^{1/p}$$

and denote $L^p(E)$ to be the set of functions f for which $\int_E |f|^p < \infty$. Either prove the statement or show a counter example.

(a) If $E = [0, 1]$, then there is a constant $c \geq 0$ for which

$$\|f\|_2 \leq c \|f\|_1 \text{ for all } f \in L^1(E).$$

(b) Suppose $E = [1, \infty)$. If f is bounded and $\int_E |f|^2 < \infty$, then it belongs to $L^1(E)$.

8. Prove or disprove that for every $\varepsilon > 0$ and for every $f \in L^\infty[a, b]$, where a and b are finite, there is a $g \in C[a, b]$ such that

$$\text{ess sup}_{a \leq x \leq b} |f - g| < \varepsilon.$$

Entrance Exam, Real Analysis

April 22, 2013

Solve exactly 6 out of the 8 problems

1. Given function $\phi(x)$ with $\phi(1) = 5$ and $\phi'(x) = e^{-x^2}$. The plane curve Γ is defined by the equation
$$y = \phi(1 + \sin 3(xy)).$$
Find the equation for the tangent line to the curve Γ at the point $(x, y) = (0, 5)$.

2. Does
$$f_n(x) = x^n \sin\left(\frac{1-x}{x}\right)$$
converge uniformly on (0,1)? Prove your conclusion.

3. Evaluate the integral
$$I = \int_0^1 dy \int_y^1 e^{-x^2} dx.$$

4. Let f, g be absolutely continuous in (0,1). Is it true that the product fg is also absolutely continuous in (0,1)? Prove your conclusion.

5. Let $\{f_n\}$ be a sequence of nonnegative integrable functions on E. For each of (a), (b), and (c) either prove the statement or show a counter example.

   (a) If $\lim_{n\to\infty} \int_E f_n = 0$, then $\{f_n\} \to 0$ in measure.

   (b) If $\{f_n\} \to 0$ in measure, then $\lim_{n\to\infty} \int_E f_n = 0$.

   (c) If $\{f_n\} \to 0$ almost everywhere on E, then $\lim_{n\to\infty} \int_E f_n = 0$.

6. For f in $C(E)$, where $C(E)$ is the set of continuous functions on E, define
$$\|f\|_p = (\int_E |f|^p)^{1/p}, \quad \|f\|_{\sup} = \sup_{x\in E} |f(x)|.$$
Here $1 \le p < \infty$. For each of (a), (b), and (c) either prove the statement or show a counter example.

   (a) If $E = [0,1]$ and $1 \le p < q < \infty$, then there is a constant $c \ge 0$ for which
   $$\|f\|_p \le c\,\|f\|_q \text{ for all } f \in C(E).$$

   (b) If $E = [0,1]$, then there is a constant $c \ge 0$ for which
   $$\|f\|_{\sup} \le c\,\|f\|_1 \text{ for all } f \in C(E).$$

   (c) Suppose $E = [1, \infty)$. If $f \in C(E)$ is bounded and $\int_E |f|^2 < \infty$, then it belongs to $L^1(E)$.

7. Prove or disprove that for every $\varepsilon > 0$ and for every $f \in L^\infty[a,b]$, there is a $g \in C[a,b]$ such that

$$\text{ess sup}_{a \le x \le b} |f - g| < \varepsilon.$$

8. (a) Suppose $E \subseteq R$ has Lebesgue measure zero. Prove that if $f : R \to R$ is increasing and absolutely continuous, then $f(E)$ has Lebesgue measure zero. Here, $f(E) = \{y \mid y = f(x) \text{ for some member } x \text{ of } E\}$.

(b) Would the same conclusion hold if we remove "increasing" in the assumption on f? In other words, if $f : R \to R$ is absolutely continuous, then would $f(E)$ have Lebesgue measure zero? Explain your reasoning.