Entrance Exam, Real Analysis
September 1, 2011

Solve exactly 6 out of the 8 problems

1. Prove that the series
$$\sum_{n=1}^{\infty} \frac{nx^2}{n^3+x^3}$$
converges uniformly on $[0,a]$ for any fixed $a>0$. Does it converge uniformly on $[0+\infty)$? (Give reasons.)

2. Let $f:\mathbb{R}\mapsto\mathbb{R}$ be a continuous function and suppose
$$\lim_{x\to-\infty} f(x) = 0 = \lim_{x\to+\infty} f(x).$$
Prove that f is uniformly continuous on the whole real line.

3. Let $f:[0,1]\mapsto\mathbb{R}$ be absolutely continuous and $f'\in L^2[0,1]$. Show that
$$\lim_{x\to 0+} \frac{f(x)-f(0)}{\sqrt{x}} = 0.$$

4. Show that a function satisfying a Lipschitz condition on $[a,b]$ is absolutely continuous (A function f is said to satisfy a Lipschitz condition on $[a,b]$ if there is a constant M such that $|f(x)-f(y)|\le M|x-y|$ for all x,y in $[a,b]$.)

5. Let $E_1\supset E_2\supset\ldots$ be an infinite sequence of measurable subsets of $\mathbb{R}$ and assume that $\cap_{n=1}^{\infty}E_n=\emptyset$.
   (a) Show that if $m(E_1)<\infty$, then $\lim_{n\to\infty} m(E_n)=0$.
   (b) Give an example showing that the conclusion of (a) may be false if $m(E_1)=\infty$.

6. Show that
$$\lim_{n\to\infty}\int \frac{x^n+1}{x^n+2}\,dx$$
exists and find its value.

7. Let $f\in L^\infty(0,1)$. Prove that $\lim_{p\to\infty}||f||_p=||f||_\infty$.

8. Compute the Lebesgue integral
$$\int_0^1 \limsup_{n\to\infty} x\cos(2\pi nx)\,dx.$$

Entrance Exam, Real Analysis

September 2, 2010

Solve exactly 6 out of the 8 problems.

1. Let $f_n(x) = x^n - x^{2n}$, $0 \le x \le 1$.
    - Show that $\lim_{n\to\infty} f_n(x) = 0$;
    - Does $f_n(x)$ converge to 0 uniformly? Prove your conclusion.
2. If $f(x)$ is continuous on [0,1] and $f(x) = 0$ almost everywhere. Show that $f(x) \equiv 0$.
3. Let f be a real function defined on $[0,1]$. If f is monotone increasing, and E is the set of points on $[0,1]$ where f is discontinuous, show that $mE = 0$.
4. Let $f_n(x)$ be a sequence of continuous functions on [0,1] and $f_n(x) \le f_{n+1}(x)$ $(n = 1, 2, \cdots)$. If for every $x \in [0,1]$, $f_n(x)$ converges to $f(x) > 0$. Show that there is a $\delta > 0$ such that $f(x) \ge \delta \ \ \forall x \in [0,1]$.
5. Let $f(x)$ be a function defined on [0,1] and
$$f(x) = \int_{(1-x)}^{x^2} \sin(xy^2)\, dy.$$
    - Show $f(x)$ is absolutely continuous;
    - Compute $f'(x)$.
6. Let f be absolutely continuous on [a,b]. Prove
$$\int_a^b |f'(x)|dx = T_a^b(f),$$
where $T_a^b(f)$ is the total variation of f on [a,b].
7. Let f be a bounded measurable function on $[0,1]$. Prove that
$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty.$$
8. Let $f_n(x)$ be a sequence of continuous functions on [0,1] such that $\displaystyle\int_0^1 |f_n(x)|dx \to 0$. Determine if the following statements are true or false. Prove your conclusion.
    - f_n must converge to 0 almost everywhere on [0,1].
    - f_n must converge to 0 in measure on [0,1].

Entrance Exam, Real Analysis

September 1, 2009

Solve exactly 6 out of the 8 problems.

1. Let n be positive integers.

   - Sketch the graph of the function $f(x)$:

   $$f(x) = \lim_{n\to\infty} e^{-x^{2n}}.$$

   - Compute the following and justify your computation:

   $$\lim_{n\to\infty} \int_{-\infty}^{\infty} e^{-x^{2n}} \mathrm{d}x.$$

2. Let f be a real function defined on $[0,1]$ with $f(0) = 1$. If the set $A = \{x : f(x) > 0\}$ is both open and close in [0,1], then is there a $\delta > 0$ such that $f(x) \geq \delta$ on $[0,1]$? Prove your conclusion.

3. Let f be a real function defined on $[0,1]$. If f is monotone increasing, and E be the set of points on $[0,1]$ where f is discontinuous. Show that $mE = 0$.

4. Let $f \in L(0,\infty)$, show that

   $$\lim_{\lambda\to\infty} \int_0^{\infty} f(x)\cos(\lambda x)dx = 0.$$

5. Let $\{f_n\}$ be a sequence of measurable functions on (a,b), and

   $$E = \{x \in (a,b) : \ f_n(x) \text{ is convergent}\}.$$

   Show that E is measurable.

6. Let f, g be two absolutely continuous functions on $[0,1]$. Prove that fg is absolutely continuous on $[0,1]$. Is it also true if the interval $[0,1]$ is replaced by $(-\infty,\infty)$? Prove your conclusion.

7. Let $\{f_n\}_n$ be a sequence of integrable functions on $[0,1]$ such that $f_n(x) \to x$ a.e. on $[0,1]$ and

   $$\int_0^1 f_n(x)dx \to \frac{1}{2}$$

   Does $f_n(x)$ converge to x in $L^1[0,1]$? Prove your conclusion.

8. Let $f \in L^{\infty}(0,1)$. Prove $\lim_{p\to\infty} \|f\|_p = \|f\|_{\infty}$.

Ph.D. Entrance Exam — Real Analysis

April 18, 2007

Attempt to solve all 8 problems

1. Let $\{f_n\}$ be a sequence of continuous functions which converges uniformly to f on a set $E \subset \mathbb{R}$. Prove: for every sequence $\{x_n\}$ in E that is convergent to $x \in E$, there holds $\lim_{n\to\infty} f_n(x_n) = f(x)$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, and $\{K_n\}_{n=1}^{\infty}$ be a decreasing sequence of compact subsets of $\mathbb{R}$. Show that $f(\bigcap_{n=1}^{\infty} K_n) = \bigcap_{n=1}^{\infty} f(K_n)$.

3. Let m denote the Lebesque measure on $\mathbb{R}$ and let $E \subset \mathbb{R}$ be a Lebesque measurable subset. Suppose $0 < \alpha < m(E)$. Show that there is a compact subset K of $\mathbb{R}$ such that $K \subset E$ and $m(K) = \alpha$.

4. (a) Prove: If f is measurable in E, then for every $\alpha \in \mathbb{R}$, the set $\{x \in E : \ f(x) = \alpha\}$ is measurable.

   (b) Give an example to show that the converse of (a) is not true.

5. Let f be a nonnegative integrable function on E. Prove: for any $\epsilon > 0$, there exists $\delta > 0$ such that for every subset $A \subset E$ with $m(A) < \delta$, there holds $\int_A f < \epsilon$.

6. Let $\{f_n\}$ be a sequence of integrable functions on $[0, 1]$ such that $f_n \to f$ a.e. on $[0, 1]$ with f integrable. Prove: $\lim_{n\to\infty} \int_0^1 |f_n - f| = 0$ if and only if $\lim_{n\to\infty} \int_0^1 |f_n| = \int_0^1 |f|$.

7. Let f be a nonnegative integrable function on a measurable set E with $m(E) > 0$.

   (a) Prove: If $\int_E f = 0$, then $f = 0$ a.e. on E.

   (b) Prove: If $\int_E f^n = \int_E f > 0$ for all $n = 1, 2, \cdots$, then $f = \chi_F$ a.e. on E for some measurable set $F \subset E$ with $m(F) > 0$ (Here χ_F denotes the characteristic function of the subset F).

8. Let $\{f_n\}$ be a sequence in $L^p[a, b]$ for some $p > 1$, and let q satisfy $1/p + 1/q = 1$.

   (a) If f_n is convergent in $L^p[a, b]$, is it true that $\int_a^b f_n g$ is convergent for any $g \in L^q[a, b]$? Prove your conclusion.

   (b) If $\int_a^b f_n g$ is convergent for every $g \in L^q[a, b]$, is it true that f_n is convergent in $L^p[a, b]$? Prove your conclusion.

Ph.D. Entrance Exam — Real Analysis

April 2005

Choose six of the following:

1. For a bounded set E, define

$$m_*(E) = b - a - m^*([a, b] \setminus E),$$

where $[a, b]$ is an interval containing E, and m^* denotes the usual outer measure. Prove the following statements.

   (a) If E be the set of all irrational numbers in $[0, 1]$, then $m_*(E) = 1$.

   (b) $m_*(E)$ is independent of the choice of $[a, b]$, as long as it contains E.

   (c) $m_*(E) \leq m^*(E)$.

2. Let E be a measurable set in $[0, 1]$ with $mE = c$ $(\frac{1}{2} < c < 1)$. Let $E_1 = E + E = \{x + y; x, y \in E\}$. Show that there exists a measurable set $E_2 \subset E_1$ such that $mE_2 = 1$.

3. Let $f(x)$ be monotone increasing on $[0, 1]$ with $f(0) = 0$ and $f(1) = 1$. If the set $\{f(x); x \in [0, 1]\}$ is dense in $[0, 1]$, show that f is a continuous function on $[0, 1]$. Is it absolutely continuous on $[0, 1]$? Prove your conclusion.

4. Let $f_n(x)$ be a sequence of continuous functions on [0,1] and $f_n(x) \geq f_{n+1}(x)$ $(n = 1, 2, \cdots)$. For every $x \in [0, 1]$, $\lim_{n\to\infty} f_n(x) < 0$. Determine and prove if there is a $\delta > 0$ such that

$$\lim_{n\to\infty} f_n(x) \leq -\delta \ \ \forall x \in [0, 1].$$

5. Let $f \in L^1(R^1)$ and define

$$F(t) \equiv \int_{-\infty}^{\infty} f(x) \sin(xt) dx.$$

Prove $F(t)$ is continuous in R. Is $F(t)$ uniformly continuous in R? Prove your conclusion.

6. Let $\{f_n\}$ be a sequence of measurable functions on (a, b), and

$$E = \{x \in (a, b) : \ f_n(x) \text{ is convergent}\}.$$

Show that E is measurable.

7. (a) State Fatou's Lemma.

   (b) Show by an example that the strict inequality in Fatou's Lemma is possible.

   (c) Show that Fatou's Lemma can be derived from the Monotone Convergence Theorem.

8. Suppose f is a non-negative integrable function on $[0, 1]$. If

$$\int_0^1 f^n = \int_0^1 f \quad \text{for all } n = 1, 2, \cdots,$$

then $f(x)$ must be the characteristic function of some measurable set $E \subset [0, 1]$.

Ph.D. Entrance Exam — Real Analysis

August 30, 2004

(Choose exactly six of the following eight problems.)

1. Use the definition of Lebesgue measure show that the set of all rational numbers in [0,1] is a Lebesgue measurable set.

2. Let f, g be two absolutely continuous functions on $[0,1]$. Prove that fg is absolutely continuous on $[0,1]$. Is it also true if the interval $[0,1]$ is replaced by $(-\infty, \infty)$? Justify your conclusion.

3. If $f(x)$ is integrable over $[0,1]$, then

$$\lim_{\lambda \to \infty} \int_0^1 f(x) cos(\lambda x) dx = 0.$$

4. Let f be a function in (0,1) defined by setting

$$f(x) = \begin{cases} 0 & \text{if } x \text{ irrational} \\ \sin(\frac{1}{q}) & \text{if } x = \frac{p}{q} \text{ in lowest terms.} \end{cases}$$

   - Find the set C of points where f is continuous and the set D of points where f is discontinuous. Justify your conclusion.
   - Is function f Riemann integrable on (0,1)? Lebesgue integrable on (0,1)? Justify your conclusion.

5. Let $E \subset [0,1]$ be closed and with no interior point. Is it true that the measure $m(E) = 0$? Justify your conclusion.

6. Let $f \in L^1(R^1)$ and $g \in L^\infty(R^1)$, and define

$$F(t) \equiv \int_{-\infty}^{\infty} g(x) f(x+t) dx.$$

   Prove $F(t)$ is continuous. Is $F(t)$ uniformly continuous on R^1? Justify your conclusion.

7. Let $f(x) = x \cos \dfrac{\pi}{x}$ for $0 < x \leq 1$ and $f(0) = 0$.

   (a) Is f continuous on $[0,1]$?

   (b) Is f uniformly continuous on $[0,1]$?

   (c) Is f absolutely continuous on $[0,1]$?

   Justify your conclusion.

8. Let $\{f_n\}$ be a sequence of real Lebesgue measurable functions on $[0,1]$. If for any real $g(x) \in L^2[0,1]$, the sequence of real numbers

$$\int_0^1 g(x) f_n(x) dx$$

   converges. Does $f_n(x)$ converge to a function $f(x)$ in $L^2[0,1]$? Justify your conclusion.

NAME (print): ______________________

Analysis Ph.D. Entrance Exam, August 29, 2003

Solve **five** exercises from the following list. Write a solution of each exercise on a separate page. This is a two hours exam.

In what follows $\mathbb{R}$ stands for the real line and m for the Lebesgue measure.

Ex. 1. Let $E_1 \supset E_2 \supset \cdots$ be an infinite sequence of measurable subset of $\mathbb{R}$ and assume that $\bigcap_{n=1}^{\infty} E_n = \emptyset$.

(a) Show that if $m(E_1) < \infty$ then $\lim_{n\to\infty} m(E_n) = 0$.

(b) Give an example showing that the conclusion of (a) may be false when $m(E_1) = \infty$.

Ex. 2. Let $f_n\colon (0,1] \to [0,\infty)$ be a decreasing sequence of continuous functions converging pointwise to a zero function θ. Must f_n converge uniformly?

Ex. 3. Is the product of two integrable functions from $\mathbb{R}$ to $\mathbb{R}$ integrable? Prove it or give a counterexample.

Ex. 4. Show that

$$\lim_{n\to\infty} \int_0^1 \frac{x^n+1}{x^n+2}$$

exists and find its value.

Ex. 5. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions on $[0,1]$. Describe the three concepts of convergence stated in (i)–(iii) and give any implications between them. The implications must be proved. (Sketch is enough.) The lack of each implication must be supported by a counter example.

(i) $f_n \to 0$ in measure as $n \to \infty$

(ii) $f_n \to 0$ a.e. as $n \to \infty$

(iii) $||f_n||_1 \to 0$ as $n \to \infty$.

Ex. 6. Suppose that f is a continuous function on $[0,1]$ for which

$$\int_{[0,1]} f(t)t^n\,dt = 0, \qquad \forall n = 0,1,2,3,\ldots.$$

Show that f is the zero function.

Ph.D. Entrance Exam — Real Analysis

August 28, 2001

Complete all problems:

1. Let E be a set in R with $m^*E = \beta > 0$. Show that for any $\alpha \in (0, \beta)$, there exists a set $E_\alpha \subset E$ with $m^*E_\alpha = \alpha$.

2. Suppose E_1, E_2, $\cdots$, E_n are n measurable sets in $[0,1]$, and every $x \in [0,1]$ belongs to at least q of these sets. Show that, there is at least an E_k such that $mE_k \geq q/n$.

3. Let f be nonnegative and measurable on E, and $E_n = \{x \in E : \; f(x) \geq n\}$. Show that if $\sum_{n=1}^{\infty} n \cdot mE_n < \infty$, then f is integrable on E, but the converse is not true.

4. Let $f(x,y)$ be a bounded function on the unit square $Q = (0,1) \times (0,1)$. Suppose for each y, that f is a measurable function of x. For each $(x,y) \in Q$, let the partial derivative $\dfrac{\partial f}{\partial x}$ exist. Under the assumption that $\dfrac{\partial f}{\partial x}$ is bounded in Q, prove that

$$\frac{d}{dy} \int_0^1 f(x,y)\; dx = \int_0^1 \frac{\partial f}{\partial y}(x,y)\; dx.$$

5. Prove:

   (a) If f is absolutely continuous on $[a,b]$, then for any set $E \subset [a,b]$ with $mE = 0$, there holds $m(f(E)) = 0$.

   (b) For a continuous and increasing function f on $[a,b]$, if $m(f(E)) = 0$ for every E in $[a,b]$ with $mE = 0$, then f is absolutely continuous on $[a,b]$.

6. For $f \in L^p[a,b]$ $(p > 1)$, set $f = 0$ outside of $[a,b]$ and define

$$f_h(x) = \frac{1}{2h} \int_{x-h}^{x+h} f(t)dt \quad \text{for} \quad h > 0.$$

   Show that

$$\|f_h\|_p \leq \|f\|_p \quad \text{and} \quad \lim_{h \to 0^+} \|f_h - f\|_p = 0.$$

   (Note: You can use the fact, without giving its proof, that for integrable ϕ, there holds $\int_a^b |\phi_h(x)|dx \leq \int_a^b |\phi(x)|dx$.)

Ph.D. Entrance Exam — Real Analysis

August 25, 2000

Complete all problems.

1. For each statement below, either prove it (if true) or give a counter example (if false).

   (a) E is measurable if and only if $m^*(P \cup Q) = m^*(P) + m^*(Q)$ for any $P \subset E$ and any $Q \subset \tilde{E}$.

   (b) If E is a countable set, then E is measurable and $mE = 0$.

   (c) If E is measurable with $mE = 0$, then E is a countable set.

   (d) If f is measurable, then so is $|f|$.

   (e) If $|f|$ is measurable, then so is f.

   (f) If $f \geq 0$ is measurable on E and $\int_E f = 0$, then $f = 0$ a.e. on E.

   (g) If f is continuous on $[a, b]$, then f is of bounded variation on $[a, b]$.

2. Suppose $\{f_n\}$ is a sequence of nonnegative measurable functions, and $\sum_{n=1}^{\infty} f_n$ converges a.e. on E. Show that $\int_E \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int_E f_n$.

3. Show that a function satisfying a Lipschitz condition on $[a, b]$ is absolutely continuous. (*Note*: A function f is said to satisfy a Lipschitz condition on $[a, b]$ if there is a constant M such that $|f(x) - f(y)| \leq M|x - y|$ for all x, y in $[a, b]$.)

4. Suppose $\{f_n\}$ and f are functions in $L^p[0, 1]$ ($p \geq 1$), and $f_n \to f$ a.e. Show that $\{f_n\}$ converges to f in L^p if and only if $\|f_n\|_p \to \|f\|_p$.

Ph.D. Entrance Exam — Real Analysis

August 20, 1997

Instruction: Complete 6 of the following 7 problems. *In all these problems "measurable" and "integable" are in the sense of Lebesque, m^* denotes the Lebesque outer measure, and m the Lebesque measure, and $\int$ the Lebesque integral.*

1. For a set $E \subset [a, b]$ (a bounded interval) with $m^*E = \beta > 0$, show that for any $\alpha \in (0, \beta)$, there exists a set $E_\alpha \subset E$ with $m^*E_\alpha = \alpha$.

2. For measurable sets $E_n \subset E$ with $\lim_{n\to\infty} mE_n = mE < \infty$, prove that there holds $\lim_{n\to\infty} \int_{E_n} f = \int_E f$ for every integrable function f on E.

3. (a) State the definition of a measurable function.

   (b) Use the definition to deduce that, if f is measurable on a measurable set E, then for every $\alpha \in R$, the set $E_\alpha = \{x \in E : f(x) = \alpha\}$ is measurable.

   (c) Construct a function f on $E = (0, 1)$ to show the converse of (b) is not true. (*Note: You may assume the existence of a non-measurable set $S \subset (0, 1)$.*)

4. Use the Hölder inequality to establish the generalized Hölder inequality:

   Let $p_i > 1$ with $\sum_{i=1}^{m} 1/p_i = 1$. Then $\| \prod_{i=1}^{m} f_i \|_1 \le \prod_{i=1}^{m} \|f_i\|_{p_i}$ for any $f_i \in L^{p_i}(0, 1)$.

   (*Note: It would be sufficient if you just show the case $m = 3$.*)

5. (a) State the definition of an absolutely continuous function on a bounded interval $[a, b]$.

   (b) Prove that, if f is absolutely continuous on $[a, b]$, then for any set $E \subset [a, b]$ with $mE = 0$, there holds $m(f(E)) = 0$.

6. Suppose $\{f_n\}$ and f are measurable functions and $f_n \to f$ a.e. in E with $mE < \infty$. Show that there exists a sequence of measurable sets $\{E_k\}_{k=0}^{\infty}$ such that

$$\bigcup_{k=0}^{\infty} E_k = E, \quad mE_0 = 0, \quad \text{and} \quad f_n \to f \text{ uniformly on each } E_k \text{ for } k = 1, 2, \cdots.$$

7. Construct a closed nowhere dense (i.e., Cantor-like) set $K \subset [0, 1]$ with $0 < mK < 1$. (*Note: A set is said to be nowhere dense if its closure contains no nonempty open interval.*)

Ph.D. Entrance Exam — Real Analysis

August 28, 2001

Complete all problems:

1. Let E be a set in R with $m^*E = \beta > 0$. Show that for any $\alpha \in (0, \beta)$, there exists a set $E_\alpha \subset E$ with $m^*E_\alpha = \alpha$.

2. Suppose E_1, E_2, $\cdots$, E_n are n measurable sets in $[0,1]$, and every $x \in [0,1]$ belongs to at least q of these sets. Show that, there is at least an E_k such that $mE_k \geq q/n$.

3. Let f be nonnegative and measurable on E, and $E_n = \{x \in E : \; f(x) \geq n\}$. Show that if $\sum_{n=1}^{\infty} n \cdot mE_n < \infty$, then f is integrable on E, but the converse is not true.

4. Let $f(x,y)$ be a bounded function on the unit square $Q = (0,1) \times (0,1)$. Suppose for each y, that f is a measurable function of x. For each $(x,y) \in Q$, let the partial derivative $\dfrac{\partial f}{\partial x}$ exist. Under the assumption that $\dfrac{\partial f}{\partial x}$ is bounded in Q, prove that
$$\frac{d}{dy} \int_0^1 f(x,y) \; dx = \int_0^1 \frac{\partial f}{\partial y}(x,y) \; dx.$$

5. Prove:

   (a) If f is absolutely continuous on $[a,b]$, then for any set $E \subset [a,b]$ with $mE = 0$, there holds $m(f(E)) = 0$.

   (b) For a continuous and increasing function f on $[a,b]$, if $m(f(E)) = 0$ for every E in $[a,b]$ with $mE = 0$, then f is absolutely continuous on $[a,b]$.

6. For $f \in L^p[a,b]$ $(p > 1)$, set $f = 0$ outside of $[a,b]$ and define
$$f_h(x) = \frac{1}{2h} \int_{x-h}^{x+h} f(t)dt \quad \text{for} \quad h > 0.$$
Show that
$$\|f_h\|_p \leq \|f\|_p \quad \text{and} \quad \lim_{h \to 0^+} \|f_h - f\|_p = 0.$$
(Note: You can use the fact, without giving its proof, that for integrable ϕ, there holds $\int_a^b |\phi_h(x)|dx \leq \int_a^b |\phi(x)|dx$.)