REU Site in Graph Theory and Combinatorics

at

West Virginia University

June 8 – August 1, 2015

Students Presentations

July 29 and 30, 2015

Sponsored by NSA, NSF, and Department of Mathematics at WVU

- ## Student Participants

1. Jordan Almeter, College of William and Mary
2. Jon Ashbrock, University of Dayton
3. Samet Demircan, West Virginia University
4. Ethan Gegner, Taylor University
5. Rachel Gouveia, University of Rhode Island
6. Andrew Kallmeyer, University of Miami-Ohio
7. Sarah Locke, University of Tennessee at Martin
8. Kate Lorenzen, Juniata College
9. William Noland, North Central College
10. Joshua Thompson, Iowa State University
11. Drea Trice, West Virginia University
12. Robert Winslow, The University of Kansa

- ## Mentors

1. Jian Cheng
2. John Goldwasser
3. Hong-Jian Lai
4. Rong Luo
5. Kevin Milans
6. Cun-Quang Zhang
7. Meng Zhang

Schedule

- **Wednesday, July 29, 2015**

10:30-10:55

Packing and Covering in the Integer Lattice

Ethan Gegner, William Noland, Robert Winslow

Mentor: John Goldwasser

11:00-11:25

R-hued Coloring Sparse and Planar Graphs

Joshua C. Thompson and Kate J. Lorenzen

Mentors: Jian Cheng, Rong Luo, Hong-Jian Lai, Cun-Quan Zhang

11:30-11:55

Adjacency Lemma for Delta-Critical Graphs

Kate Lorenzen and Joshua Thompson

Mentors: Jian Cheng, Rong Luo, Cun-Quan Zhang

- **Thursday, July 30, 2015**

10:30-10:55

An exploration of a graph ranking variant

Jordan Almeter, Samet Demircan, Andrew Kallmeyer

Mentor: Kevin Milans

11:00-11:25

Determining all graphs G such that $h(G) = h$, for small integers h

Sarah Locke, Andrea Trice,

Mentors: Meng Zhang, Hong-Jian Lai

11:30-12:55

Cyclic Base Orderings and Uniformly Dense Graphs

Jon Ashbrock, Rachel Gouveia

Mentor: Hong-Jian Lai

Abstracts

Packing and Covering in the Integer Lattice

Ethan Gegner, William Noland, Robert Winslow

Mentor: John Goldwasser

A somewhat well-known problem in discrete geometry is the rectangle-free problem in a lattice, in which one tries to select the maximum number of points from an integer lattice without selecting the four corners of any axis-aligned rectangle. This is equivalent to selecting the smallest set such that each axis-aligned rectangle has at least one of its corners selected. This is called a covering set of the rectangles. We explore the less restrictive problem of find the minimal density of points needed to cover the axis-aligned rectangles of some pair of dimensions ($A \times B$,$C \times D$). We find optimal covering densities for a number of different categories of rectangle pairs, including (1×1,2×2), ($A \times B$,$B \times A$), ($A \times B$, $A \times D$)

In addition to exploring coverings, we also explore packings, which are a selection of points such that no two selected points is part of the same rectangle. We find an equivalence between 1/4 density packings and coverings. Further, we explore the idea of a rainbow packings, in which the lattice is labeled with a minimum number of colors such that the specified rectangles have different colors on each corner. We also find the optimal density for the covering of 1×1 axis aligned squares and $\sqrt{2} \times \sqrt{2}$ non-axis-aligned squares. This leads to a result about the optimal packing density of Manhattan distance balls. Finally, we consider the problem of finding the maximum number of 2×2 submatrices in an $n \times n$ $(0, 1)$ array that contain exactly one 0.

R-hued Coloring Sparse and Planar Graphs

Joshua C. Thompson and Kate J. Lorenzen

Mentors: Jian Cheng, Rong Luo, Hong-Jian Lai, Cun-Quan Zhang

A vertex coloring for a graph G is an assignment of colors to each vertex such that each vertex is colored differently from its neighbors. An r-hued coloring is a vertex coloring where for each vertex v with $d(v)$ neighbors, these neighbors are colored with at least $min\{r, d(v)\}$ different colors. The smallest number of colors necessary for a graph G to be r-hued colored is $\chi_r(G)$, the r-hued chromatic number of G. The maximum average degree of a graph G, $mad(G)$, is the maximum of the average degrees of all subgraphs H of G. We proved that graphs with $mad(G) < \frac{8}{3}$ satisfy $\chi_2(G) \leq 4$ unless $G = C_5$, graphs with $mad(G) < \frac{12}{5}$ satisfy $\chi_3(G) \leq 6$, and graphs with $mad(G) < \frac{7}{3}$ satisfy $\chi_3(G) \leq 5$ by a discharging method. For planar graphs, it is known that $\chi_2(G) \leq 5$. We also proved that for all planar graphs, $\chi_3(G) \leq 8$.

Adjacency Lemma for $\Delta-$Critical Graphs

Kate Lorenzen and Joshua Thompson

Mentors: Jian Cheng, Rong Luo, Cun-Quan Zhang

An edge coloring of $k-$colors of G is a map $c : E(G) \to \{1, ..., k\}$ such that no two edges incident with the same vertex are colored the same. The chromatic index $\chi'(G)$ is the smallest integer k such that G has an edge coloring. Let $\Delta_G = max\{d(v) : v \in V(G)\}$. In 1965, Vizing proved $\Delta \leq \chi'(G) \leq \Delta + 1$. A graph is said to be Class One if $\chi'(G) = \Delta$ and Class Two if $\chi'(G) = \Delta + 1$. A $\Delta-$critical graph is defined as a Class Two graph and for any edge $e \in E(G)$, $G - \{e\}$ is a Class One graph. We prove if $xy \in E(G)$ such that $d(x) + d(y) = \Delta + 2$ and z be a common neighbor of x, y then,

(1) every vertex with distance three from x or y has degree at least $\Delta - 1$, and

(2) every vertex distance two from z has degree Δ.

This lemma completes the proof given in Miao and Pang's paper about 4-critical graphs.

An exploration of a graph ranking variant

Jordan Almeter, Samet Demircan, Andrew Kallmeyer

Mentor: Kevin Milans

In a graph whose vertices are assigned integer ranks, a path is good if the endpoints have distinct ranks or an interior point has a higher rank than the endpoints. A graph ranking is an assignment such that all paths are good. Rankings are well studied and have applications in chip design and certain problems in online algorithms. A k-ranking is a relaxation of rankings in which all paths of length at most k are good. The k-ranking number of a graph, denoted by χ_k, is the minimum number of ranks among its k-rankings. For the n-dimensional cube Q_n, we prove $\chi_2 = n + 1$. We provide bounds on χ_2 for the cartesian product of various graphs. For subcubic graphs, we provide upper bounds on χ_2. We also prove the existence of graphs with maximum degree k and $\chi_2 \geq \frac{k^2}{\log(k)}$.

Determining all graphs G such that $h(G) = h$, for small integers h

Sarah Locke, Andrea Trice,

Mentors: Meng Zhang, Hong-Jian Lai

A model of uniformly dense networks has been proposed for secure networks, which is associated with the cyclically base orderings of a graph. We define $h(G)$ as the largest number of consecutive edges in a cyclic ordering where the edges do not form a closed loop. It has been conjectured in [Discrete Math., 72 (1988),187-194] that a connected network G is uniformly dense if and only if $h(G) = n - 1$, where n is the number of vertices in G. Hence, $h(G)$ is considered a measure of how close a network is to being uniformly dense. In this paper, we determine all connected, loopless graphs G with small values of $h(G)$, specifically $h(G) = 1$ and $h(G) = 2$. Also, a formula is provided to determine $h(G)$ for a particular graph structure.

Cyclic Base Orderings and Uniformly Dense Graphs

Jon Ashbrock, Rachel Gouveia

Mentor: Hong-Jian Lai

A cyclic base ordering of a graph G is a cyclic ordering on E(G) such that every $|V(G)| - 1$ consecutive elements induces a forest. For a cyclic ordering, o, let $h(o) = max\{k$: every k consecutive edges in o induces a forest$\}$ and $h(G) = max\{h(o) : \forall o$ on $E(G)\}$. Let $\omega(G)$ be the number of components of G. Define $\beta = |E(G)| - |V(G)| + \omega(G)$ for a graph, G. The quantities $g(G), \gamma(G)$ are defined as $g(G) = \frac{|E(G)|}{|V(G)| - \omega(G)}$ and $\gamma(G) = max\{g(H) : H \subseteq G\}$. We say a graph is uniformly dense if $g(G) = \gamma(G)$. In [Discrete Math., 72 (1988), 187 194], it is conjectured that a graph is uniformly dense if and only if it has a cyclic base ordering. We show the conjecture holds for graphs with small β. Further, we completely characterize the quantity $h(G)$ for all connected graphs with $\beta \leq 3$.