1 The Problems Motivating The Research

We consider finite and simple digraphs. Usually, we use $G$ to denote a graph and $D$ to a digraph. Undefined terms and notations will follow [6] for graphs and [3] for digraphs. In particular, $\kappa(G), \kappa'(G), \alpha(G)$ ans $\alpha'(G)$ denote the connectivity, the edge connectivity, the independence number, and the matching number of a graph $G$; and $\lambda(D)$ denotes the arc-strong connectivity of a digraph $D$ as well as $\text{diam}(D) = \text{diam}(G(D))$; $\text{circ}(D) = \text{length of a longest dicircuit of } D$. The independence number and the matching number of a digraph $D$ are respectively defined as

$$\alpha(D) = \alpha(G(D)) \text{ and } \alpha'(G(D)).$$

Motivated by the Chinese Postman Problem, Boesch, Suffel, and Tindell [5] in 1977 proposed the supereulerian problem, which seeks to characterize graphs that have spanning Eulerian subgraphs, and they indicated such that problem would be very difficult. Pulleyblank [13] later in 1979 proved that determining whether a graph is supereulerian, even within planar graphs, is NP-complete. Since then, there have been lots of researches on this topic. Catlin [7] in 1992 presented the first survey on supereulerian graphs. Later Chen et al [8] gave an update in 1995, specifically on the reduction method associated with the supereulerian problem. A recent survey on supereulerian graphs is given in [12].

It is a natural to consider the supereulerian problem in digraphs. A strong digraph $D$ is **eulerian** if for any $v \in V(D), d^+_D(v) = d^-_D(v)$. A strong-connected digraph $D$ is **supereulerian** if $D$ contains a spanning eulerian subdigraph. Several efforts have been made to determine supereulerian digraphs. The earlier studies were done by Gutin ([9],[10]). Recently, the following have been obtained.

Bang-Jensen and Thomassé ([2], also see [4]) made the following conjecture.

**Conjecture 1** Let $D$ be a digraph . If $\lambda(D) \geq \alpha(D)$, then $D$ is supereulerian.

As an effort towards this conjecture Bang-Jensen and Maddaloni [4] characterized supereulerian digraphs belonging to particular classes such as semicomplete multipartite digraphs and quasi-transitive digraphs by the following definitions and theorems:

**Definition 2** Given a digraph $D$, an eulerian factor of $D$ is a collection of arc-disjoint cycles spanning $V(D)$. 
Theorem 3 Let \( D \) be a digraph. If \( \lambda(D) \geq \alpha(D) \) then \( D \) has an eulerian factor.

Definition 4 A digraph \( D = (V, A) \) is semicomplete multipartite if there is a partition \( V_1, V_2, \ldots, V_c \) of \( V \) into independent sets so that every vertex in \( V_i \) shares an arc with every vertex in \( V_j \) for \( 1 \leq i < j \leq c \).

Theorem 5 Let \( D \) be a semicomplete multipartite digraph. \( D \) is supereulerian if and only if it is strong and has an eulerian factor.

Combining Theorem 3 and Theorem 5 they gave:

Theorem 6 Let \( D \) be a semicomplete multipartite digraph. If \( \lambda(D) \geq \alpha(D) \), then \( D \) is supereulerian.

Definition 7 A digraph \( D \) is quasi-transitive if, for every triple of distinct vertices \( x, y, z \in V(D) \), with \( xy, yz \in A(D) \), there is at least one arc between \( x \) and \( z \).

Theorem 8 Let \( D \) be a quasi transitive digraph. If \( \lambda(D) \geq \alpha(D) \), then \( D \) is supereulerian.

Furthermore, Bang-Jensen and Maddaloni [4] proved the following two theorems:

Theorem 9 A strong digraph such that \( d^+(x) + d^-(y) \geq 2n - 3 \) for all pairs of non-adjacent vertices \( x, y \) is supereulerian.

Theorem 10 A strong digraph such that \( d^+(x) + d^-(y) \geq n - 1 \) for all ordered pairs \( (x, y) \) of dominated or dominating non-adjacent vertices is supereulerian.

Hong, Lai and Liu [11] gave a necessary condition for the existence of a spanning eulerian subdigraph after presented the following:

Definition 11 Let \( D \) be a strong digraph and \( U \subset V(D) \). Then in \( D[U] \), we can find some ditrails \( P_1, \ldots, P_t \) such that \( \bigcup_{i=1}^t V(P_i) = U \) and \( A(P_i) \cap A(P_j) = \emptyset \) for any \( i \neq j \). Let \( \tau(U) \) be minimum value of such \( t \). Then \( c(G[D[U]]) \leq \tau(U) \leq |U| \), where \( c(G[D[U]]) \) is the number of component of the underlying graph of \( D[U] \). For any \( A \subseteq V(D) - U \), denote \( B := V(D) - U - A \) and let

\[
\begin{align*}
h(U, A) & : = \min\{|\partial^+_D(A)|, |\partial^-_D(A)|\} + \min\{|(U, B)_D|, |(B, U)_D|\} - \tau(U), \text{ and} \\
h(U) & : = \min\{h(U, A) : A \cap U = \emptyset\}.
\end{align*}
\]

Then they gave the following theorem:

Theorem 12 If \( D \) has a spanning Eulerian subdigraph, then for any \( U \subset V(D) \), \( h(U) \geq 0 \).

Moreover, they proved the following theorem:

Theorem 13 Let \( D \) be a strong digraph with \( |V(D)| = n \) and \( \delta^+(D) \geq 4 \) and \( \delta^-(D) \geq 4 \); If \( \delta^+(D) + \delta^-(D) \geq n - 3 \) then \( D \) is supereulerian.
A well known theorem of Chvátal and Erdős states that if \( \kappa(G) \geq \alpha(D) \), then \( G \) is hamiltonian. Thomassen [14] gave an infinite family of non hamiltonian (but supereulerian) digraphs such that \( \kappa(D) = \alpha(D) = 2 \), showing that the Chvátal-Erdős Theorem does not extend to digraphs. This motivated Bang-Jensen and Thommassé as we mentioned before to make Conjecture 1.

Carsten Thomassen [14] presented two more kinds of the independence number of a digraph \( D \), \( i_2(D) \), \( i_3(D) \) to be the maximum cardinality of a vertex set \( A \) of \( D \) such that \( D(A) \) has no cycle, no 2-cycle respectively. However, Thomassen gave infinitely many examples such that \( \lambda(D) \geq i_2(D) \), and \( D \) is non-Hamiltonian and the same for \( i_3(D) \). This motivated Professor Lai to consider the following conjectures:

**Conjecture 14** (Lai) Let \( D \) be a digraph. If \( \lambda(D) \geq i_2(D) \), then \( D \) is supereulerian.

**Conjecture 15** (Lai) Let \( D \) be a digraph. If \( \lambda(D) \geq i_3(D) \), then \( D \) is supereulerian.

These three conjectures encourage us to search the sufficient conditions for a digraph to be a supereulerian digraph.

## 2 My Progresses and Preliminary Results

We proved in [1] the following conjectures (raised by Professor Lai):

**Conjecture 16** Let \( D \) be a digraph. If \( \lambda(D) \geq \alpha'(D) \), then \( D \) is supereulerian.

**Conjecture 17** Let \( D \) be a digraph. If \( \kappa(D) \geq \alpha'(D) \), then \( D \) is supereulerian.

To prove these, we first show the following:

**Lemma 18** Let \( k > 0 \) be an integer and \( D \) be a digraph with a matching \( M \) such that \( |M| = k \). Suppose that \( V(D) - V(M) \) has a subset \( X \) with \( |X| \geq 2 \) such that there exist two vertices \( v, u \in X \) with \( \max\{d^+(v),d^-(v)\} \geq k \) and \( \max\{d^+(u),d^-(u)\} \geq k+1 \), then there exists a matching \( M' \) in \( D \) such that \( |M| < |M'| \).

**Corollary 19** Let \( D \) be a digraph. If \( \lambda(D) \geq \alpha'(D) \), then

(i) \( \lambda(D) = \alpha'(D) \) when \( |V(D)| \geq 2k + 2 \).

(ii) \( \lambda(D) \leq 2\alpha'(D) \) when \( 2k \leq |V(D)| \leq 2k + 1 \).

**Lemma 20** Let \( k > 0 \) be an integer, \( D \) be a digraph on \( n \geq 2k+2 \) vertices and \( M \) be a maximum matching of \( D \) with \( |M| = k \). Suppose that

\[
\min\{d^+(v),d^-(v)\} \geq k, \text{ for every vertex } v \in V(D).
\]

Let \( X = V(D) - V(M) \). Then

for every \( x \in X \), we have \( d^+(x) = d^-(x) = k \).
Moreover, if \( n \geq 2k + 3 \) or if \( n = 2k + 2 \) and \( D \) is strong, then for any \( e = [u, v] \in M \), and for any \( x \in X \), we have the following conclusions.

(i) There exists exactly one \( v(e) \in \{u, v\} \) such that both \((v(e), x)\) and \((x, v(e))\) are in \( A(D) \), and the vertex \( u(e) \in \{u, v\} - \{v(e)\} \) is not adjacent to any vertex in \( X \).

(ii) The set \( \{u(e) : e \in M\} \) is an independent set in \( D \) such that \( d^+(u(e)) = d^-(u(e)) = k \) for any \( e \in M \) and such that for any \( e, e' \in M \), \((u(e), v(e')), (v(e'), u(e))\) \( \in A(D) \).

**Theorem 21** Let \( D \) be a digraph on \( n \geq 2k \) vertices with \( \alpha'(D) = k \). Suppose that if \( n \leq 2k + 2 \), then \( D \) is strong. If

\[
\min\{d^+(v), d^-(v)\} \geq k, \text{ for every vertex } v \in V(D).
\]

then each of the following holds.

(i) \( D \) is supereulerian.

(ii) \( \lambda(D) \geq k \), (in particular \( \lambda(D) = k \) when \( n \geq 2k + 2 \)).

**Corollary 22** Let \( D \) be a digraph . If \( \lambda(D) \geq \alpha'(D) \), then \( D \) is supereulerian.

**Corollary 23** Let \( D \) be a digraph . If \( \kappa(D) \geq \alpha'(D) \), then \( D \) is supereulerian.

It is well known that every connected, locally-connected graph is supereulerian. This motivates the problem whether strong and locally strong digraphs will behave similarly, and how local structure in digraph will warrant the existence of a spanning eulerian subdigraph. Towards this direction we obtain new sufficient conditions for a digraph \( D \) to be supereulerian. The following definitions are our attempt to extend local connectivity to digraphs.

**Definition 24** A vertex \( v \in V(D) \) is \( k^+\)-locally-arc-connected, (or \( k^-\)-locally-arc-connected, or \( k\)-locally-arc-connected, respectively) if \( \lambda(D[N^+(v)]) \geq k \) (\( \lambda(D[N^-(v)]) \geq k \), or \( \lambda(D[N(v)]) \geq k \), respectively).

**Definition 25** A digraph \( D \) is \( k^+\)-locally-arc-connected, (or \( k^-\)-locally-arc-connected, or \( k\)-locally-arc-connected, respectively) if every vertex of \( D \) is \( k^+\)-locally-arc-connected, (or \( k^-\)-locally-arc-connected, or \( k\)-locally-arc-connected, respectively).

Using Theorem 12, we showed that for any integer \( k > 0 \), there exist an infinite family of strong, \( k^+\)-locally-arc-connected non-supereulerian digraphs, and an infinite family of strong, \( k\)-locally-arc-connected non-supereulerian digraphs. It proves that local connectivity is not sufficient for supereulerian digraphs. This further pushes us to consider stronger conditions, which leads to the following definitions. With them we obtained the next few theorems and corollaries:

**Definition 26** A strong digraph \( D \) is locally \((\alpha, \beta)^+\)-dense if \( \forall v \in V(D), d_{D[N^+(v)]}(u) \geq \alpha|N^+(v)| + \beta \), and \( D \) is locally \((\alpha, \beta)^\cdot\)-arc-connected if \( \forall v \in V(D), \lambda(D[N^+(v)]) \geq \alpha|N^+(v)| + \beta \).

**Theorem 27** Every locally \((\frac{4}{3}, \frac{\sqrt{5}}{3})^+\)-dense strong simple digraph is supereulerian.
Corollary 28 Every locally $\left(\frac{2}{3}, 0\right)^+$-arc-connected strong simple digraph is supereulerian.

Theorem 29 Every locally $(\frac{4}{3}, \frac{-2}{3})$-dense strong simple digraph is supereulerian.

Corollary 30 Every locally $\left(\frac{2}{3}, 0\right)$-arc-connected strong simple digraph is supereulerian.

3 Next Goal and Future Work

Throughout this section, $D$ always denotes a simple digraph.

Our ultimate goal is to prove and to investigate the following:

Conjecture 31 (A) (Bang-Jensen and Thomassé Conjecture [2] also see ([4]). If $\lambda(D) \geq \alpha(D)$, then $D$ is supereulerian.

Let $i_2(D), i_3(D)$ be the two different kinds of independence number of a digraph $D$, as defined by Carsten Thomassen in [14].

Conjecture 32 (B) (Lai) Let $D$ be a digraph. If $\lambda(D) \geq i_j(D)$, then $D$ is supereulerian ($j \in \{2, 3\}$).

Problem 33 (Lai) Suppose $D$ is strong. Let $\text{circ}(D)$ be the length of a longest dicycle of $D$. Is there an increasing function $h(k)$ such that for any $k > 0$, every $h(k)$-arc-connected digraph $D$ with $\text{circ}(D) \leq k$ is supereulerian. If such a function $h(k)$ exists, for each $k$, we also want to determine the minimum value of $h(k)$.

Problem 34 (Lai) Let $D$ be a strong digraph and let $\text{diam}(D)$ the diameter of the underlying graph of $D$. Is there an increasing function $f(k)$ such that for any $k > 0$, every $f(k)$-arc-connected digraph $D$ with $\text{diam}(D) \leq k$ is supereulerian. If such a function $f(k)$ exists, for each $k$, we also want to determine the minimum value of $f(k)$.

Problem 35 (Lai) Let $F$ be a family of digraphs. A strong digraph $D$ is locally-$F$ if every arc of $D$ lies in a subdigraph $H$, which is isomorphic to a member in $F$. We want to characterize families $F$ such that every strong, locally-$F$ digraph is supereulerian.

Problem 36 (Lai) Let $F$ be a family of digraphs. A digraph $D$ is $F$-free if $D$ does not have an induced subdigraph $H$ isomorphic to a member in $F$. We want to determine families $F = F(k)$ such that every $k$-arc-strong $F(k)$-free digraph is supereulerian.

References


