Entrance Exam, Real Analysis
August 28, 2012

Solve exactly 6 out of the 8 problems

1. Let function $F(x)$ $(x > 0)$ be defined by

$$F(x) = \int_{x}^{x^2} e^{-t^2 x^3} dt.$$

Compute $F'(x)$ without evaluating the integral.

2. Let function $f(x)$ be continuous in $[0, 1)$ and $\lim_{x\to 1} f(x)$ exists. Is $f(x)$ uniformly continuous in $[0, 1)$? Prove your conclusion.

3. Let f be Lebesgue integrable over $(0, 1)$ and for all $\alpha \in (0, 1)$,

$$\int_0^{\alpha} f(t)dt = 0.$$

Prove $f(t) = 0$ a.e. in (0,1).

4. Let f be monotone increasing on [0,1] with $f(0) = 0$ and $f(1) = 1$. Let E be the image set: $E = f([0, 1])$, and the Lebesgue measure $mE = 1$. Prove f is continuous on [0,1]. If $mE < 1$, does it imply that f must have discontinuous point in [0,1]? Prove your conclusion.

5. Compute the limit. Justify your computation

$$\lim_{n\to\infty} \int_0^4 \frac{x^2}{1 + x^n} dx.$$

6. Let f be of bounded variation on $[a, b]$, and define $v(x) = TV(f_{[a,x]})$ for all $x \in [a, b]$. Here, a and b are finite.

   (a) Show that $|f'| \le v'$ a.e. on $[a, b]$, and infer from this that

   $$\int_a^b |f'| \le TV(f).$$

   (b) Show that the above is an equality if f is absolutely continuous on $[a, b]$.

Entrance Exam, Real Analysis

August 27, 2013

Solve exactly 6 out of the 8 problems

1. Given function $\phi(x)$ with $\phi(1) = 1$ and $\phi'(x) = e^{-x^2}$. The plane curve Γ is defined by the equation
$$y = \phi(1 + xy + y^2).$$
Find the equation for the tangent line to the curve Γ at the point $(x, y) = (-1, 1)$.

2. Does
$$f_n(x) = x^n \sin\left(\frac{1-x}{x}\right)$$
converge uniformly on (0,1)? Prove your conclusion.

3. Let $\{x_n = p_n/q_n \in \mathbb{Q}\}$ be a sequence of rational numbers and $x_n \to \alpha \in \mathbb{R}\backslash\mathbb{Q}$ (α is irrational). Prove $q_n \to \infty$.

4. Let E be a Lebesgue measurable set in $\mathbb{R}$, and f_n be a sequence of nonnegative Lebesgue integrable functions on E. If $f_n \to f$ a.e. on E, is it always true that
$$\int_E f dx \leq \underline{\lim}_{n\to\infty} \int_E f_n dx?$$
Prove your conclusion.

5. Let a and b are real numbers satisfying $a < b$. Prove or disprove the following.
   (a) If $f(x)$ is a differentiable function on $a < x < b$, it is absolutely continuous on $a < x < b$.
   (b) If $f(x)$ is absolutely continuous on $a \leq x \leq b$, it is Lipschitz on $a \leq x \leq b$.

6. A family $\mathcal{F}$ of measurable functions on E of finite measure is said to be uniformly integrable over E provided that for each $\varepsilon > 0$, there is a $\delta > 0$ such that for each $f \in \mathcal{F}$ and for each measurable set $A \subseteq E$ with $mA < \delta$, $\int_A |f| < \varepsilon$. Either prove or disprove the following statements.
   (a) If $\mathcal{G}$ is the family of measurable functions f on $[0, 1]$, each of which is integrable over $[0, 1]$ and has $\int_0^1 |f| < 1$, then $\mathcal{G}$ is uniformly integrable over $[0, 1]$.
   (b) If $\{f_n\}$ is a sequence of nonnegative, measurable, and integrable functions that converges pointwise a.e. on E to $h = 0$, then $\{f_n\}$ is uniformly integrable.

7. For $1 \leq p < \infty$ define

$$\|f\|_p = (\int_E |f|^p)^{1/p}$$

and denote $L^p(E)$ to be the set of functions f for which $\int_E |f|^p < \infty$. Either prove the statement or show a counter example.

(a) If $E = [0, 1]$, then there is a constant $c \geq 0$ for which

$$\|f\|_2 \leq c \|f\|_1 \text{ for all } f \in L^1(E).$$

(b) Suppose $E = [1, \infty)$. If f is bounded and $\int_E |f|^2 < \infty$, then it belongs to $L^1(E)$.

8. Prove or disprove that for every $\varepsilon > 0$ and for every $f \in L^\infty[a, b]$, where a and b are finite, there is a $g \in C[a, b]$ such that

$$\text{ess sup}_{a \leq x \leq b} |f - g| < \varepsilon.$$

7. For f in $C[a,b]$, define

$$\|f\|_p = (\int_a^b |f|^p)^{1/p}, \quad \|f\|_{\max} = \max_{a\le x\le b} |f(x)|,$$

where a and b are finite, and $1 \le p < \infty$. For (a), (b), and (c) prove the statements or show a counter example.

(a) There is a constant $c \ge 0$ for which

$$\|f\|_p \le c\,\|f\|_{\max} \text{ for all } f \in C[a,b].$$

(b) There is a constant $c \ge 0$ for which

$$\|f\|_{\max} \le c\,\|f\|_1 \text{ for all } f \in C[a,b].$$

(c) There is a constant $c \ge 0$ for which

$$\|f\|_1 \le c\,\|f\|_2 \text{ for all } f \in C[a,b].$$

8. Show the following.

(a) Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions with the same domain. Prove that $\underline{\lim} f_n$ is measurable.

(b) Let g be an integrable function on a set E and suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions such that $|f_n(x)| \le g(x)$ a.e. on E. Show that

$$\int_E \underline{\lim} f_n \le \underline{\lim} \int_E f_n.$$