

PHD ENTRANCE EXAM, SPRING 2016: ORDINARY DIFFERENTIAL EQUATIONS

Name (Print). _____ Show all Work. Draw and Explain. All problems carry the same weight.

Solve any 6 (six) out of the 8 (eight) problems and clearly indicate which problems are to be graded.

1 Let n be a positive integer and A be a real, skew-symmetric $n \times n$ matrix (i.e. $A^T = -A$). Let $\alpha \in [0, \infty)$ and consider the differential system

$$y' = Ay - |y|^\alpha y,$$

where $|\cdot|$ denotes the Euclidean norm in \mathbb{R}^n (here $t \in \mathbb{R}$ and $y(t) \in \mathbb{R}^n$).

(a) Show that for any $y_0 \in \mathbb{R}^n$ the initial value problem with $y(0) = y_0$ has a solution for all $t > 0$.

(b) Show that the origin is the only equilibrium point.

(c) Show that the equilibrium point is *globally asymptotically stable*.

Hint: Recall that for any square matrix M and any vectors u, v we have $Mu \cdot v = u \cdot M^T v$ (here \cdot denotes the Euclidean dot product).

2 Consider the differential system (where x, y are scalar functions of the real variable t)

$$x' = -xy + y, \quad y' = x - x^2y.$$

(a) Find all the equilibria and discuss their stability.

(b) Along what curves is dy/dx zero or infinity (plus or minus)? Draw these curves in the xy -plane.

(c) Sketch the phase diagram and illustrate the direction of increasing t .

3 Let $a, b > 0$ and define for $t, t_0 \in \mathbb{R}$ and $y, \eta \in \mathbb{R}^n$ the rectangular box

$$D := \{(t, y) \mid 0 \leq t - t_0 \leq a, |y - \eta| \leq b\}.$$

Assume that $f : D \rightarrow \mathbb{R}^n$ is continuous and there exist real constants K, M such that

$$|f(t, y) - f(t, z)| \leq K |y - z|, \quad |f(t, y)| \leq M \text{ for all } (t, y), (t, z) \in D.$$

(a) Prove under the above assumptions the existence and uniqueness of the solution to the initial value problem

$$y' = f(t, y), \quad y(t_0) = \eta$$

on some interval $[t_0, q]$ using the method of successive approximations. Show in detail the successive approximations convergence.

(b) Determine for the scalar initial value problem

$$y' = \frac{y^2}{y-3}, \quad y(1) = 4$$

numerical values for a, b, q and justify them.

4 (a) Find an obvious, polynomial solution for

$$-t^2 x'' + (t^2 + mt)x' - (t + m)x = 0 \quad (t > 0),$$

where m is any real parameter.

(b) Use the substitution $x(t) = tu(t)$ to find a particular solution for

$$(0.1) \quad -t^2 x'' + (t^2 + t)x' - (t + 1)x = t^2 \quad (t > 0).$$

(c) Find the *general solution* for (0.1). (In your computations you may use the function $F(t) = \int_1^t s^{-1} e^s ds$ for any $t > 0$; do not attempt to compute it explicitly!)

5 (a) Let $f(t)$, $g(t)$, $K(t)$ be nonnegative continuous scalar functions on the interval $[a, b]$ and that the following inequality holds:

$$f(t) \leq g(t) + \int_a^t K(s)f(s)ds, \quad t \in [a, b].$$

Prove that

$$(0.2) \quad f(t) \leq g(t) + \int_a^t g(s)K(s) \exp \left[\int_s^t K(u)du \right] ds.$$

(b) Utilize (0.2) to prove that if

$$f(t) \leq \int_a^t |\sin(f(s))| ds, \quad t \in [a, b]$$

then $f(t) \equiv 0$ for $t \in [a, b]$.

Hint: Prove that $u(\theta) = \sin(\theta)/\theta$ can be defined as a continuous bounded function on \mathbb{R} .

6 Assume $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is differentiable with continuous partial derivatives of first order, $L \in \mathbb{R}^n$ and $f(L) = \mathbf{0} \in \mathbb{R}^n$. Assume also that $y(t)$ is a non-constant vector solution to the differential system $y' = f(y)$ on some interval $[t_0, \infty)$ such that $\lim_{t \rightarrow b^-} y(t) = L$.

(a) Prove that $b^- = \infty$.

(b) Given the scalar differential equation

$$x' = x^2(x - 3),$$

determine all critical points and show that none of them is globally asymptotically stable as $t \rightarrow \infty$.

(c) Assume the scalar differential equation

$$x' = x^k + \sum_{l=0}^{k-1} c_l x^l, \quad c_l \in \mathbb{R}$$

to possess more than one critical point. Prove that no critical point is globally asymptotically stable as $t \rightarrow \infty$.

7 Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$ and $f(x) = \begin{bmatrix} x_1^3 \\ 4x_1^2 x_2 \end{bmatrix}$, where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$.

Prove that the differential system $x' = Ax - f(x)$ has a non-constant, periodic solution.

8 It is given that all solutions to the scalar initial value problem below are periodic:

$$u'' + 4u^3 = 0, \quad u(0) = \alpha, \quad u'(0) = \beta.$$

(a) In the phase plane, sketch the orbit of each solution and the direction of flow along the orbit as t increases.

(b) Derive an expression for the smallest positive period $T(m)$ that shows its dependence on $m := \frac{\beta^2}{2} + \alpha^4$.

PhD Entrance Exam, Differential Equations; Spring 2015

Name (Print).-----

Do 6 out of the 8 problems and mark the 6 problems you want to be graded. Failure to do so will result in automatically grading the first 6 problems. Show all work. Draw and explain. All problems carry the same weight.

1. Let $A(t)$ be a variable $n \times n$ matrix with entries $a_{j,k}(t)$ continuous on (a, b) and let $M(t)$ be the $n \times n$ matrix solution of $M'(t) = A(t)M(t)$. Denote by $D(t)$ the determinant of $M(t)$ on (a, b) .

(a) Prove Abel's theorem, i.e. show that for any $t_0, t \in (a, b)$ we have

$$D(t) = D(t_0)e^{\int_{t_0}^t [\sum_{j=0}^n a_{j,j}(s)] ds}.$$

What is the significance and usefulness of this theorem to the theory of linear homogeneous systems of ODE's?

(b) Given for $n = 3$ that

$$A(t) = \begin{bmatrix} t & t^2 & t^3 \\ t^4 & -2t & t^5 \\ t^6 & t^7 & t \end{bmatrix}, \quad M(t_0) = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix},$$

show that $D(t)$ is bounded on $(-\infty, \infty)$.

2. Given $y \in \mathbb{R}^2, \nu > 0, \sigma < 0$ and that

$$\frac{dy(t)}{dt} = \begin{bmatrix} \sigma & \nu \\ -\nu & \sigma \end{bmatrix} y(t), \quad (1)$$

determine a fundamental matrix solution to (1) that has real valued entries. Sketch the phase portrait of (1) as $t \rightarrow \infty$. Carefully explain the direction of

arrows along orbits.

3. Let $A(t)$ be a variable $n \times n$ matrix with entries $a_{j,k}(t)$ continuous on $(-\infty, \infty)$ and assume there exists $\omega > 0$ such that $A(t + \omega) = A(t)$ for all $t \in (-\infty, \infty)$. Let $M(t)$ be an $n \times n$ fundamental matrix solution of $M'(t) = A(t)M(t)$. Show that:

(a) $M(t + \omega)$ is also a fundamental matrix solution of the above differential system.

(b) The matrix $C := M^{-1}(t)M(t + \omega)$ exists, is invertible, and actually is a constant independent of t .

(c) It is given that there exists a constant matrix R such that $C = \exp(\omega R)$. Prove that $P(t) := M(t)\exp(-tR)$ is periodic with period ω and that the transformation $M(t) = P(t)Z(t)$ takes the differential system into the system $Z'(t) = RZ(t)$.

(d) Determine the matrices C and $P(t)$ if

$$\omega = 2\pi, \quad A(t) = \begin{bmatrix} \cos t & 1 \\ 0 & \cos t \end{bmatrix}, \quad M(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Can you find the matrix R as well?

4. Consider the differential system

$$y' = f(t, y), \quad y(t_0) = \eta, \quad y, \eta, f \in \mathbb{R}^n, \quad (2)$$

and assume:

(i) there exist $b, \delta > 0$ such that $f(t, y)$ is a continuous vector function in the set of points (t, y) given by

$$REC.BOX := \{|t - t_0| \leq \delta, \quad |y - \eta| \leq b\} \quad (3)$$

and $|f(t, y)| \leq M < \infty$ for all $(t, y) \in REC.BOX$.

(ii) the entries of the Jacobian matrix

$$JM := \left(\frac{\partial f_i}{\partial y_j} \right) = \begin{bmatrix} \nabla f_1 \\ \nabla f_2 \\ \cdot \\ \cdot \\ \cdot \\ \nabla f_n \end{bmatrix}$$

are continuous in the *REC.BOX*.

(iii) there exists a solution $y = \phi(t, t_0, \eta)$ to the initial value problem (2) on the interval $[t_0 - q, t_0 + q]$, where $q = \min\{\delta, b/M\}$.

Given all the above, prove that the solution $y = \phi(t, t_0, \eta)$ is unique.

5. Denote by \mathcal{M}_n the space of all $n \times n$ real matrices. Let $A \in \mathcal{M}_n$ such that all its eigenvalues have negative real parts, and let $B : [0, \infty) \rightarrow \mathcal{M}_n$ be continuous and such that $\int_0^\infty |B(t)| dt < \infty$ (where $|\cdot|$ represents any matrix norm). Prove that the null solution of the system $\mathbf{x}' = A\mathbf{x} + B(t)\mathbf{x}$ is asymptotically stable.

6. Discuss the stability of the null solution for the system

$$x' = -x + e^{-t}y + \frac{z^2}{1 + |t|}, \quad y' = -y + e^{-t^2}x^2, \quad z' = e^{-2t}x - 2z.$$

7. Consider the differential equation $x'' + x^3 = 0$.

(a) For what initial data $(x(0), x'(0))$ is it true that $\lim_{t \rightarrow \infty} x(t) = 0$?

(b) For what initial data $(x(0), x'(0))$ is it true that $\lim_{t \rightarrow -\infty} x'(t) = 0$?

8. Consider the system

$$x' = 3x - y - 4x^3 - 2y^3, \quad y' = 3x + y - 2x^3 - 4y^3.$$

(a) Show that $V(x, y) = x^2 - xy + y^2$ is positive definite around the origin.

(b) Prove that the system has a nontrivial periodic solution whose orbit in the phase plane lies between the ellipses $V(x, y) = 1/4$ and $V(x, y) = 3/2$.

Name (Print):_____

Show all Work. Draw and Explain. All problems carry the same weight.

Do 6 out of the 8 problems and mark the 6 problems you want to be checked.

1. (a) Given A a constant n by n matrix with n distinct eigenvalues $\lambda_j, j = 1, \dots, n$ such that $\text{Re } \lambda_j \leq 0, j = 1, 2, \dots, n$, prove that the zero vector solution of $y' = Ay$ is stable.

(b) Discuss the stability and asymptotic stability of the zero vector solution of

$$y' = \begin{bmatrix} -1 + e^{-t} & 2 + e^{-t} & 3 + (t^2 + 1)^{-1} \\ (t^2 + 1)^{-1} & -2 & 5 \\ e^{-2t} & \frac{\ln(t^2+1)}{t^2+1} & -3 \end{bmatrix} y.$$

2. Consider the differential system $y' = f(t, y)$.

(i) Formulate an existence and uniqueness theorem for the above nonlinear system. Explain the method of successive approximations.

(ii) Estimate an interval of existence for solutions of the initial value problem

$$y_1' = y_1 y_2, y_2' = y_1^2 + y_1 y_2 + 1, y_1(1) = 3, y_2(1) = 0.$$

(iii) Provide a best estimate for the maximal interval of validity (existence) of the solution for the initial value problem below, without explicitly finding the solution:

$$y' = \begin{bmatrix} t^{-2}(t-2)^{-3} & e^t \\ t & t^2 \end{bmatrix} y, y(1) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

3. Consider the initial value problem

$$y' = A(t)y + g(t), \quad y(t_0) = \eta. \tag{0.1}$$

where the $n \times n$ matrix $A(t)$ and the n column vector $g(t)$ are continuous on $(-\infty, \infty)$ and are such that

$$\int_{-\infty}^{\infty} [|A(s)| + |g(s)|] ds < \infty.$$

Prove then that the initial value problem for any given $t_0 \in \mathbb{R}, \eta \in \mathbb{R}^n$ possesses a unique solution on $(-\infty, \infty)$.

4. Consider the scalar initial value problem

$$2x'' + 10x^9 = 0, x(0) = \alpha, x'(0) = \beta, \alpha, \beta \in \mathbb{R}. \tag{0.2}$$

(i) Show that if $x(t)$ is a solution then

$$[x'(t)]^2 + x^{10}(t) = \beta^2 + \alpha^{10}. \tag{0.3}$$

Note that it (0.3) implies that (0.2) possesses a bounded solution with a bounded derivative.

(ii) It is given further that $\alpha > 0, \beta > 0$. Show that the solution $x(t)$ attains a (relative) local maximum at a time $t_{max} > 0$. Determine precisely the values of $x(t_{max}), x'(t_{max}), x''(t_{max})$ in terms of α, β . Show that there exists a time $t_{min} > t_{max} > 0$ where $x(t)$ attains a local minimum. Determine precisely the values of $x(t_{min}), x'(t_{min}), x''(t_{min})$.

(iii) Sketch the orbit of this solution for $t_{max} \leq t \leq t_{min}$ in the phase space. Also sketch in the (t, x) plane the graph of $x(t)$ on the interval $t_{max} \leq t \leq t_{min}$. What is the sign of $x'(t)$ on this interval? Conclude that there is a one to one correspondence between the values of t and $x(t)$ on $t_{max} \leq t \leq t_{min}$.

(iv) Determine p, q that makes the relation below valid

$$\int_{t_{max}}^{t_{min}} dt = t_{min} - t_{max} = \int_q^p -[\beta^2 + \alpha^{10} - x^{10}]^{-\frac{1}{2}} dx.$$

(v) Prove that the solution to (0.2) is periodic and find an expression for its period.

5. (a) Find and classify all the equilibria of the system

$$\dot{x} = y^3, \quad \dot{y} = -x^3 + 3x^2 - 3x + 1.$$

(b) Sketch the phase diagram.

6. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous and consider the differential equation

$$\dot{y} = f(y), \quad \text{where } \dot{y} := \frac{dy}{dt}. \tag{DE}$$

(a) Show that if a solution φ of (DE) satisfies $\lim_{t \rightarrow \infty} \varphi(t) = c \in \mathbb{R}$, then c is a critical point for (DE).

(b) Assume further that $f(0) = 0$ and $xf(x) < 0$ for all $x \in \mathbb{R} \setminus \{0\}$. Show that for any $y_0 \in \mathbb{R}$ there exists a solution y of (DE) on $(0, \infty)$ satisfying $y(0) = y_0$.

(c) Under the assumptions from (b) above, show that the equilibrium solution (why is that unique?) is globally asymptotically stable.

7. Consider the system

$$\dot{x} = x - y - x|x|, \quad \dot{y} = x + y - y|y|.$$

(a) Discuss the stability of the origin.

(b) Determine if there is a periodic solution. In either case, show your proof.

8. Solve the initial value problem

$$\begin{aligned} xy'' + (x+1)y' + y &= 2x, \\ y(0) &= 1, \\ y'(0) &= -1. \end{aligned}$$

GRADUATE TEST-SPRING 2012, ODE'S

Solve any 6 (six) problems

1. Let I_n be the $n \times n$ identity matrix and J be the $n \times n$ Jordan block

$$J = \lambda I_n + H$$

with λ a given real constant and H the $n \times n$ matrix $H := (h_{j,k}) =$

$$\begin{bmatrix} 0 & 1 & 0 & & 0 & 0 \\ & 0 & 1 & & & 0 \\ & & \cdot & \cdot & & \\ \cdot & & & \cdot & \cdot & \\ & 0 & 0 & 0 & & 0 & 1 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \end{bmatrix}.$$

Namely,

$$H := (h_{j,k}), j, k = 1, \dots, n, h_{j,k} = 1 \text{ if } k = j+1, j = 1, 2, \dots, n-1, h_{j,k} = 0 \text{ if } k \neq j+1.$$

- a) Prove that $M = \exp(tJ)$ is a fundamental matrix solution of the system

$$\frac{d}{dt}M(t) = JM(t).$$

- b) Prove that

$$\exp(tJ) = \exp(t\lambda) \left(I_n + \sum_{k=1}^{n-1} \frac{t^k}{k!} H^k \right).$$

- c) Determine the elements of the $n \times n$ matrix

$$\lim_{t \rightarrow \infty} t^{-(n-1)} \left(I_n + \sum_{k=1}^{n-1} \frac{t^k}{k!} H^k \right).$$

2. A differential system is called complete if for any $y_0 \in \mathbb{R}^n$ and any $t_0 \in \mathbb{R}$ there exists a solution $y(t)$ defined for all $t \in \mathbb{R}$ and such that $y(t_0) = y_0$. Given the differential system with unknown $y = (y_1, y_2) \in \mathbb{R}^2$

$$\dot{y}_1 = [\cos(t + y_1 y_2)]^{23} + 3 \sin(y_1^5 + y_2^{2012}),$$

$$\dot{y}_2 = \sin[\log(1 + y_1^2 y_2^4)] + 3 \cos(y_1^5 + y_2^{2012}),$$

prove that it is complete.

3. Given the autonomous differential system

$$\dot{y} = f(y), \text{ where } f \in C^1(\mathbb{R}^n),$$

let $y(t)$ be a $n \times 1$ column real-valued vector solution defined on \mathbb{R} . Prove that if $y(t_0 + P) = y(t_0)$ for some $t_0 \in \mathbb{R}$ and some $P > 0$, then

$$y(t + P) = y(t), \text{ for all } t \in \mathbb{R},$$

i.e. $y(t)$ is a periodic function of $t \in \mathbb{R}$.

4. Given the autonomous differential system

$$(0.1) \quad \dot{y} = f(y), \quad \text{where } f \in C^1(\mathbb{R}^n),$$

let $y(t)$ be an $n \times 1$ column real-valued vector solution defined on (a, b) . Prove the following:

(i) If $\lim_{t \rightarrow \infty} y(t) = L$ (L being a constant vector in \mathbb{R}^n), then L is a critical point for the system.

(ii) Assume \dot{y} is not identically zero and that $\lim_{t \rightarrow b^+} y(t) = L$, where L is a critical point for the system. Then $b = \infty$.

5. Let $f(x, y) = x^2y + y^3$ and $g(x, y) = y(1 - x^2 - 2y^2) - 2x^3 - 2xy^2$.

(a) Prove that the system below has periodic solutions

$$\dot{x} = f(x, y), \quad \dot{y} = g(x, y).$$

(b) Prove that the system below has non-periodic, non-constant solutions

$$\dot{x} = -x + f(x, y), \quad \dot{y} = g(x, y).$$

6. Consider the system

$$\dot{x} = x + xy, \quad \dot{y} = y + xy.$$

(a) Find the equilibria and study their stability.

(b) Prove that $x = 0$, $y = 0$ and $x = y$ are the only invariant lines.

(c) Find the null and vertical (possibly curvilinear) clines.

(d) Draw the phase diagram and use arrows to indicate the direction of increasing time.

7. (a) Solve the initial value problem

$$\dot{x} = x + y + z, \quad \dot{y} = y + z, \quad \dot{z} = -z \quad \text{with } x(0) = y(0) = z(0) = 1.$$

(b) Find all initial vectors $(x(0), y(0), z(0))$ for which the solution converges to zero as t converges to infinity.

(c) Find all initial vectors $(x(0), y(0), z(0))$ for which the solution converges to zero as t converges to negative infinity.

8. Consider the equation

$$\ddot{x} + \cos x = 0.$$

(a) Find all the critical points and discuss their stability.

(b) Find the level set of the Hamiltonian which contains all the unstable critical points, i.e. find the constant $c \in \mathbb{R}$ such that the planar curve of equation $H(x, y) = c$ contains all the unstable critical points.

(c) Draw the phase diagram and use arrows to indicate the direction of increasing time (independent variable).

ENTRANCE EXAM: *Differential Equations* (August 30, 2012)

There are 8 (eight) problems on this test. Solve any 6 (six) of them.
Clearly motivate your answers/solutions.

1. Let $M(t)$ be a square matrix and consider the matrix differential equation

$$\frac{dM(t)}{dt} = AM(t), \quad A = \begin{bmatrix} 0 & 0 & 0 \\ -1 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

(a) Determine a fundamental solution.

(b) Determine all column vector solutions $y(t)$ to

$$\frac{dy(t)}{dt} = Ay(t)$$

that converge to the origin as $t \rightarrow \infty$.

(c) Determine all solutions that are bounded on $(-\infty, \infty)$.

2. A differential system is called complete if all of its real-valued solutions to all of its initial value problems exist for all $t \in \mathbb{R}$. Consider the differential system ($y_1, y_2 \in \mathbb{R}$)

$$\begin{cases} \dot{y}_1 = 3 \cos(\exp(y_1 y_2)) + y_1 y_2 (1 + y_1^2 y_2^2)^{-1}, \\ \dot{y}_2 = \cos(1 + y_1^2 y_2^4) + \sin(y_1^5 + y_2^{20}). \end{cases} \quad (1)$$

Prove that (1) is complete.

3. Let $f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ be a continuously differentiable vector field $f = f(y, p)$ with $y \in \mathbb{R}^n$, $p \in \mathbb{R}$. Let $y(t, p)$ be an n by 1 vector valued solution of the IVP:

$$\frac{dy}{dt}(t, p) = f(y(t, p), p), \quad t, p \in \mathbb{R}, \quad \text{with } y(t_0, p) = y_0 \in \mathbb{R}^n \text{ for all } p \in \mathbb{R}. \quad (2)$$

Prove that

$$\frac{\partial y}{\partial p}(t, p) = \int_{t_0}^t \frac{\partial f}{\partial p}(y(s, p), p) ds + \int_{t_0}^t J \frac{\partial y}{\partial p}(s, p) ds,$$

where J is the Jacobian matrix

$$J = (a_{k,j}), \quad a_{k,j} := \frac{\partial f_k}{\partial y_j}(y, p), \quad k, j = 1, \dots, n.$$

Hint: Convert (2) into an integral equation.

4. Let

$$A(t) := \begin{bmatrix} t & \sin t \\ \cos t & -t \end{bmatrix}.$$

Denote by $M(t)$ a 2 by 2 matrix solution of the linear matrix equation

$$\frac{dM(t)}{dt} = A(t)M(t), \quad -\infty < t < \infty.$$

Prove that $\det M(t)$, the determinant of $M(t)$, is a constant on $(-\infty, \infty)$.

5. (a) Find and classify all the equilibria of the system

$$\dot{x} = -6y + 2xy - 8, \quad \dot{y} = -x^2 + y^2.$$

(b) Sketch the phase diagram of the system from (a) (make sure to draw arrows to show direction of increasing time).

6. Consider the system

$$\dot{x} = x - xy, \quad \dot{y} = y - xy.$$

(a) Find the equilibria and study their stability.

(b) Prove that $x = 0$, $y = 0$ and $x = y$ are the only invariant lines.

(c) Find the null and vertical (possibly curvilinear) clines.

(d) Draw the phase diagram and use arrows to indicate the direction of increasing time.

7. Consider the system

$$\dot{x} = x - y - 2x^2, \quad \dot{y} = x + y - 2y^2.$$

(a) Discuss the stability of the origin.

(b) Determine if there is a periodic solution. In either case, show your proof.

8. Consider the initial value problem (IVP)

$$\ddot{x} - 2x - 2x^3 = 0, \quad x(0) = \alpha, \quad \dot{x}(0) = \beta.$$

(a) Use the Hamiltonian energy of the solution $x(t)$ to show that if $\alpha = 1$, $\beta = 0$, then $x(t) \geq 1$ on its maximal interval of existence.

(b) Find the solution with $\alpha = 0$, $\beta = 1$. What is its maximal interval of existence? (Recall: the *maximal interval of existence* is the interval around $t = 0$ beyond which the solution is undefined.)

Graduate exam 2013, ODE's

Do any 6 problems. All problems carry the same weight. Explain your solutions.

1. Solve the initial value problem

$$\dot{y} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} y - \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad y(7) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

Determine $\lim_{t \rightarrow \infty} y(t)$.

2. Let A be a constant n by n matrix with eigenvalues λ_j , $j = 1, 2, \dots, n$. Let $\rho := \max\{\operatorname{Re}(\lambda_j) : j = 1, 2, \dots, n\}$. Consider a fixed vector solution of

$$\dot{y} = Ay(t)$$

on the interval $[a, \infty)$. Prove that for every $\epsilon > 0$, there exists $K > 0$ such that

$$|y(t)| \leq Ke^{(\rho+\epsilon)t} \quad \text{for all } t \in [a, \infty),$$

where $|\cdot|$ is a suitable norm.

3. Consider the autonomous differential system

$$\dot{y} = f(y), \quad \text{where } f \in C^1(\mathbb{R}^n). \quad (1)$$

Let $y(t)$ be a continuous $n \times 1$ column vector solution of (1) on (a, b) . Prove the following:

(i) Assume \dot{y} is not identically zero and that $\lim_{t \rightarrow b^+} y(t) = L$, where L is a critical point of (1). Then $b = \infty$.

(ii) $y(t + \omega)$ is also a vector solution of (1) on $(a + \omega, b + \omega)$.

4. (i) Calculate the Jacobian matrix of f if

$$f(t, y) = A(t)y + g(t).$$

(ii) Prove in detail (using the theorem of existence and uniqueness for general systems), that if $A(t)$ and $g(t)$ possess continuous entries on the closed interval $[c, d]$, then every initial value problem

$$\dot{y} = A(t)y + g(t), \quad y(t_0) = \eta, \quad t_0 \in [c, d], \quad \eta \in \mathbb{R}^n$$

possesses a unique solution on $[c, d]$.

5. Discuss the stability of the zero solution for

$$\dot{x} = -x \cos y + xy + ye^{-t}, \quad \dot{y} = \sin x - 2y + 2x \sin y + x \sin \frac{1}{1+t^2}.$$

6. (i) (2 points) Show that the origin is the only equilibrium solution for

$$\dot{x} = -x^3 - xy^4, \quad \dot{y} = -y^3 - x^2y.$$

(ii) Show that the equilibrium is *globally* asymptotically stable (which means that every solution, no matter where it starts, approaches this equilibrium forward in time).

(iii) Does this system have nonconstant *periodic solutions*?

7. Consider the scalar equation $\ddot{x} + 2x^3 = 0$.

(i) Sketch the phase diagram, find the critical points and study their stability.

(ii) Assume $x(0) = 1$, $\dot{x}(0) = \sqrt{3}$. Prove that the time it takes the corresponding orbit to cross the x -axis in phase plane is $t_0 = \int_0^{\sqrt{2}} \frac{ds}{\sqrt{4-s^4}}$.

(iii) Show that t_0 defined at (c) is finite (note that the integral above is improper, so it is necessary to prove that it is convergent).

8. Consider the system

$$\dot{x} = 2x + y - 2x^3 - 3xy^2, \quad \dot{y} = -2x + 4y - 2x^2y - 4y^3.$$

(i) Find all critical points and discuss their stability. *Hint: First use the Lyapunov function $V(x, y) = 2x^2 + y^2$ to prove that if there are other critical points beside the origin, then they must lie on the unit circle.*

(ii) Are there any periodic solutions? If so, approximately where are their orbits located?

ENTRANCE EXAM: Differential Equations, Fall 2013.

Solve any 6 (six) problems. All problems carry the same weight.

1. Given the initial value problem

$$\begin{aligned} y' &= f(t, y, p), & y(t_0) &= \eta, \\ t, t_0, p &\in \mathbb{R}^1, & y, \eta, f &\in \mathbb{R}^n. \end{aligned}$$

Assume that $f(t, y, p)$ is a continuous vector function and that the entries of the Jacobian matrix

$$Jf := \left(\frac{\partial f_i}{\partial y_j} \right)_{ij} = \begin{bmatrix} \nabla f_1 \\ \nabla f_2 \\ \cdot \\ \cdot \\ \cdot \\ \nabla f_n \end{bmatrix}$$

are continuous in the set of points (t, y, p) , where

$$t, t_0, p \in \mathbb{R}^1, \quad y, \eta, f \in \mathbb{R}^n.$$

Consider the solution $y = \phi(t, t_0, \eta, p)$ as a vector function of the variables (t, t_0, η, p) . Determine an equation that is satisfied by a) $\frac{\partial \phi}{\partial t_0}$, b) $\frac{\partial \phi}{\partial p}$. What information could be derived from these equations? Why is that information important?

2. Convert the scalar ode $x'' + x^7 = 0$ into a vector system of differential equations. Show, that a solution to any initial value problem of the ode exists on any interval $[a, b]$.

3. a) Explain the essence of the method of successive approximations and under what assumptions it is valid. b) Calculate the first 3 successive approximations for the initial value problem

$$y' = \begin{bmatrix} -y_1 + y_2^2 \\ y_1 + y_2 \end{bmatrix}, \quad y(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

4. Provide the best estimates for the maximal intervals of existence of solutions of

$$y' = \begin{bmatrix} (t+5)^{-2} & t \\ t^2 & (t-1)^{-3} \end{bmatrix} y, \quad y(t_0) = \eta,$$

with a) $t_0 = -10$, b) $t_0 = 0$, c) $t_0 = 2$.

5. (a) Find and classify all the equilibria of the system

$$\dot{x} = -3y + xy - 4, \quad \dot{y} = x^2 - xy.$$

(b) Sketch the phase diagram.

6. Consider the system

$$\dot{x} = x - xy, \quad \dot{y} = y + xy.$$

(a) Find the equilibria and study their stability.

(b) Find all the invariant lines.

(c) Find the null and vertical (possibly curvilinear) clines.

(d) Draw the phase diagram and use arrows to indicate the direction of increasing time.

7. Consider the system

$$\dot{x} = 2x + y - 2x^3 - 3xy^2, \quad \dot{y} = -2x + 4y - 2x^2y - 4y^3.$$

(a) (6 points) Find all critical points and discuss their stability. *Hint: First use the Lyapunov function $V(x, y) = 2x^2 + y^2$ to prove that if there are other critical points beside the origin, then they must lie on the unit circle.*

(b) (4 points) Are there any periodic solutions? If so, approximately where are their orbits located?

8. Consider the linear system $\dot{\mathbf{x}} = A\mathbf{x}$, where A is an $n \times n$ nonsingular, antisymmetric (i.e. $A^T = -A$), real matrix.

(a) Show that every solution satisfies $|\mathbf{x}(t)| = \text{const}$ for all $t \in \mathbb{R}$.

(b) Find all critical points and study their stability.

(c) If $n = 2$, are there any non-periodic solutions?

PhD Entrance Exam, Differential Equations; Fall 2014
Solve any six problems!

1. Let $y^T = (y_1, y_2, \dots, y_n)$ denote a row vector that is the transpose of $y \in \mathbb{R}^n$. In particular let $\hat{0}^T = (0, \dots, 0)$ be the transpose of the zero vector. Let $f^T(y, t) := (f_1(y, t), f_2(y, t), \dots, f_n(y, t))$ be a vector field in \mathbb{R}^n where $f_k(y, t) \in \mathbb{R}$, $k = 1, 2, \dots, n$ and $t \in [a, b]$. Formulate (without proof) an existence and uniqueness theorem that will ensure that the initial value problem

$$\frac{dy}{dt}(t) = f(y, t), \quad y(t_0) = \eta, \quad t_0 \in [a, b], \quad (1)$$

possesses a unique solution on some subinterval of $[a, b]$. Based on this formulation prove that if

$$|f(y, t)| \leq A|y| + B, \quad y \in \mathbb{R}^n, t \in [a, b],$$

with some nonnegative constants A and B , then the initial value problem (1) possesses a unique solution on $[t_0, b]$.

2. (a) Determine the value of all constants c that will guarantee the existence of a bounded non trivial (not zero) vector solution $y(t)$ on $(-\infty, \infty)$ to the linear system with constant coefficients

$$y' = \begin{bmatrix} 1 & c \\ 3 & 2 \end{bmatrix} y.$$

(b) Let $A = A^T$ be a constant symmetric $n \times n$ matrix with real entries. Prove that the system

$$\frac{dy}{dt}(t) = Ay(t),$$

possesses a bounded non trivial (non zero) vector solution $y(t)$ on $(-\infty, \infty)$ iff the matrix A has a zero eigenvalue. *Hint: Use properties of symmetric matrices.*

3. Prove that the equation below has periodic solutions

$$\ddot{x} + (\dot{x})^5 + x^2\dot{x} - \dot{x} + x^3 = 0.$$

4. Solve the initial value problem

$$\dot{y} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} y - \begin{bmatrix} 4 \\ 4 \\ 10 \end{bmatrix}, \quad y(6) = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}.$$

5. Provide examples for the following:

(a) An initial value problem $\dot{y} = f(y)$, $y(0) = 0$ with $f(y) \in C(\mathbb{R})$ such that the initial value problem does not have a unique solution.

(b) An initial value problem $\dot{y} = f(y)$, $y(0) = y_0$ with $f(y) \in C(\mathbb{R})$, with solutions that do not exist on \mathbb{R} .

(c) A scalar singular differential equation such that non trivial solutions exist on \mathbb{R} .

6. Consider the system

$$\dot{x} = y + x^2y, \quad \dot{y} = -4x + x^3 - xy^2.$$

(a) Show that the system is Hamiltonian and find its Hamiltonian function.

(b) Find the critical points and discuss their stability.

(c) Sketch the phase diagram (make sure to put arrows to indicate the direction of motion along orbits).

7. Consider the differential system

$$\dot{x} = x^3 + x^2y, \quad \dot{y} = y^3 + y^2z, \quad \dot{z} = z^3 + z^2x.$$

(a) Show that all solutions exist for all negative times;

(b) Show that all solutions converge to the origin as $t \rightarrow -\infty$;

(c) Show that with the exception of the stationary solutions (find them!), all other solutions blow up in finite time;

(d) Study the stability of the critical points.

8. (a) Discuss the stability of the null solution for the system:

$$\dot{x} = x - e^{-t}y, \quad \dot{y} = y - z(1 + e^{-t}), \quad \dot{z} = 2z + xyz^2.$$

(b) Is the origin asymptotically stable for the system:

$$\dot{x} = x^2 - y^2z, \quad \dot{y} = xy^3 - z, \quad \dot{z} = -2xy \quad ?$$

Hint for (b): Study the evolution of a point in a certain octant.

GRAD TEST, ODE'S, SPRING 09

Show all work. Justify your answers. Do 6 problems of your choice.

1) i) Show that the system of equations

$$(0.1) \quad y_1' = y_1^2 + y_2^3, y_1(0) = 1,$$

$$(0.2) \quad y_2' = y_1 y_2, y_2(0) = 2,$$

possesses a unique solution on some interval $(-b, b)$, with b a positive number. Provide an estimate for this b . Justify!

ii) Give an example of an initial value problem, with a scalar ordinary differential equation $y' = f(y)$, $y(0) = 0$, such that;

a) the initial value problem does not have a unique solution .

b) has a solution that does not exist on $[0, \infty)$.

iii) Provide an exact solution to the following initial value problem

$$y_1' = y_1^2 + y_2^3 + y_3 \exp(y_1 y_2 y_3), y_1(0) = 0,$$

$$y_2' = y_1^2 y_2^3 + y_3, y_2(0) = 0,$$

$$(0.3) \quad y_3' = y_1^2 y_3 + y_2^3, y_3(0) = 0.$$

Is the solution unique?

2) Prove Gronwal's inequality. Let K be a nonnegative constant and let f and g be continuous nonnegative functions on some interval $\alpha \leq t \leq \beta$, satisfying the inequality

$$(0.4) \quad f(t) \leq K + \int_{\alpha}^t f(s)g(s)ds,$$

for $\alpha \leq t \leq \beta$. Then,

$$(0.5) \quad f(t) \leq K \exp\left(\int_{\alpha}^t g(s)ds\right),$$

for $\alpha \leq t \leq \beta$.

Explain the role of Gronwal's inequality in the theory of ordinary differential systems.

3) Let $g(u)$ be a continuously differentiable scalar function for $-\infty < u < \infty$ such that $g(0) = 0$, $g'(0) = -\frac{3}{16}$. Consider the ode system

$$y_1' = -y_1 + y_2$$

$$y_2' = g(y_1),$$

and assume that the origin $y_1 = y_2 = 0$ is an isolated critical point. Show that this critical point is stable.

Show that $u = y_1$ is a solution of the scalar ode $u'' + u' = g(u)$.

4) Determine explicitly, in finite terms, the solution to the initial value problem

$$y' = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} y + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, y(0) = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix},$$

α, β, γ , some scalars.

Differential Equations Exam, 2009S. Continued.

[5] Let x be real variable. Consider the differential equation

$$\ddot{x} = 2x^3 - 2x, \quad x = x(t), \quad \dot{x} = \frac{d^2x}{dt^2}.$$

- (a) Take \dot{x} as the vertical axis and x as the horizontal axis to draw the phase diagram.
- (b) Suppose $x(0) = 0$ and $\dot{x}(0) = 1$. Find the solution. Draw the orbit of the solution in the phase diagram.
- (c) Suppose $x(0) = 0$ and $\dot{x}(0) = 0$. Find the solution. Draw the orbit of the solution in the phase diagram.
- (d) Explain if the periodic solutions exist?

[6] Let x and y be real variables. Consider the system

$$\begin{aligned}\dot{x} &= y + x^2y, \\ \dot{y} &= xy + x.\end{aligned}$$

- (a) Find the critical points and discuss their stability.
- (b) Sketch the phase diagram. Be sure to put "Arrows" in orbits (or paths) to indicate the direction of time. Along what curves (or lines) $\frac{dy}{dx}$ is zero or infinite? Draw these curves in the xy -plane.

[7] Let x and y be real variables. Discuss the stability of critical points.

(a)

$$\begin{aligned}\dot{x} &= xy^4 + x^3 \\ \dot{y} &= -x^2y - y^3.\end{aligned}$$

(b)

$$\begin{aligned}\dot{x} &= 4x^2 - y^2 \\ \dot{y} &= -2x + xy - 4.\end{aligned}$$

[8] Let x and y be real variables.

(a) Show if there is a periodic orbit (limit cycle) for

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -x^3 - (1 - x^2 - y^2)y.\end{aligned}$$

(b) Show if there is a periodic orbit (limit cycle) for

$$\begin{aligned}\dot{x} &= x - y - x^3 \\ \dot{y} &= x + y - y^3.\end{aligned}$$

Entrance Exam, Differential Equations
April, 2007

(Solve exactly 6 out of the 8 problems)

1. Consider the following initial value problem:

$$\begin{cases} y' + 2y + y \cos(x^2 y) = 0, \\ y(0) = y_0. \end{cases}$$

Find all the values y_0 such that the solution $y(x)$ satisfies

$$\lim_{x \rightarrow \infty} y(x) = 0.$$

Prove your conclusion.

2. Solve the initial value problem

$$\begin{cases} y'' - 3y^2 = 0, \\ y(0) = 2, y'(0) = 4. \end{cases}$$

3. Consider the following second order differential equation:

$$y'' + (1 + \sin t)y' - (1 + t^2)y = 0.$$

Let $f(t)$ be a function satisfying

$$f(0) = 0, f(1) = 1, f(2) = -2.$$

Is it possible that $f(t)$ is a solution? Prove your conclusion.

4. Given a solution $y = t$ ($t > 0$) for the second order differential equation

$$t^2 y'' - t(t+2)y' + (t+2)y = 0.$$

Find the general solution for

$$t^2 y'' - t(t+2)y' + (t+2)y = 2t^3, (t > 0)$$

5. • Solve

$$\dot{x} = Ax = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 3 \\ 1 & 0 & 1 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

- Find e^{At} for this problem.
- Write down the solution formula for

$$\dot{x} = Ax + f(t), \quad x(0) = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

6. Consider

$$\begin{aligned}\dot{x} &= 2y + 2x^2y, \\ \dot{y} &= -2x - 2xy^2 + 2x^3,\end{aligned}$$

where $(x, y) \in \mathbb{R}^2$.

- Show that the system is Hamiltonian.
- Find the Hamiltonian function.
- Find the critical points and name them. Also, discuss their stability.
- Sketch the phase diagram. Be sure to put “Arrows” in orbits (or paths) to indicate the direction of time.

7. For the following system in \mathbb{R}^2 :

$$\begin{aligned}\dot{x} &= x - y - x(x^2 + 2y^2), \\ \dot{y} &= x + y - y(x^2 + 2y^2).\end{aligned}$$

- (a) Discuss the stability of critical points.
- (b) Show if there is a periodic solution.

8. Consider

$$\dot{x} = f(x), \tag{1}$$

where $x \in \mathbb{R}^n$, f is continuously differentiable, $f(0) = 0$, and $x = 0$ is an isolated critical point. Suppose that the linearized system is given by

$$\dot{x} = Ax, \tag{2}$$

where

$$A = \frac{\partial f}{\partial x}(0)$$

is the Jacobian of f evaluated at 0.

Suppose the zero solution of (2) is stable. Either prove that the zero solution of (1) is stable or show a counter example.

Differential Equations Exam, 2006S. Solve five problems.

[1] Let $p(t)$ and $q(t)$ be two continuous functions. Prove that the second order linear equation $y'' + p(t)y' + q(t)y = 0$ has two, and only two linearly independent solutions.

[2] Consider the second order differential equation : $y'' - (\sin t)y' + (1 + \cos t)y = 0$.

It is easy to see that $y = \sin t$ is a solution with period 2π . Are all the solutions must be periodic with period 2π ? Prove your conclusion.

[3] Prove Gronwall inequality: Let $k(t), u(t)$ be continuous in $[a, b]$ with $k(t) \geq 0$, and $C \geq 0$ is a constant. If

$$u(t) \leq C + \int_a^t k(s)u(s)ds \quad \forall t \in [a, b],$$

then

$$u(t) \leq Ce^{\int_a^t k(s)ds} \quad \forall t \in [a, b].$$

[4] (a) Solve

$$\dot{x} = \begin{pmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

(b) Find a Jordan (canonical) form J of the above matrix and the transformation P such that $P^{-1}AP = J$.

(c) Discuss the stability of the system for $t \geq 0$.

$$\dot{x} = \begin{pmatrix} -2 + \frac{2}{1+t^2} & -1 - e^{-t} & -2 \\ -4 & -2 & -4 \\ -2 & -1 + \frac{\ln(t+1)}{1+t^2} & -2 \end{pmatrix} x + \begin{pmatrix} e^{3t} \\ 2 \\ te^{-t} \end{pmatrix}.$$

[5] Consider

$$\begin{aligned} \dot{x} &= 2x - xy + x^3, \\ \dot{y} &= y - xy. \end{aligned}$$

(a) Find the critical points and name them. Also, discuss their stability.

(b) Along what curves (or lines), is $\frac{dy}{dx}$ zero or infinite?

(c) Are the following sets invariant? Circle the invariant sets.

(1) The line $y = 0$. (2) The line $x = 0$. (3) The line $x = 1$. (4) The curve $2 - y + x^2 = 0$.

(d) Sketch the phase diagram. Be sure to put "Arrows" in orbits (or paths) to indicate the direction of time.

[6] Discuss the stability of critical points.

(a)

$$\begin{aligned} \dot{x} &= y^3 + 2x^2y, \\ \dot{y} &= -xy^2 + x^3. \end{aligned}$$

(b)

$$\begin{aligned} \dot{x} &= -x - xy^2, \\ \dot{y} &= x^4y - y^3. \end{aligned}$$

Differential Equations Exam, 2004F. Show and explain all work. Do 6 out of the 7 problems.

[1] Consider the planar system

$$\begin{aligned}\frac{dx}{dt} &= x + y - x^3, \\ \frac{dy}{dt} &= -x + y - y^3.\end{aligned}$$

- (i) Determine if the origin is stable, asymptotically stable, or unstable.
- (ii) Determine if there is a periodic solution. Either case, show your proof.

[2] Consider the planar system

$$\begin{aligned}\frac{dx}{dt} &= -y + x[\mu - (1 - x^2 - y^2)^2](1 - x^2 - y^2), \\ \frac{dy}{dt} &= x + y[\mu - (1 - x^2 - y^2)^2](1 - x^2 - y^2),\end{aligned}$$

where μ is a parameter independent of t , x , and y .

- (i) Rewrite the planar system in polar coordinates.
- (ii) Show that there is a pitchfork bifurcation of a periodic orbit as the parameter μ is changed.
- (iii) Sketch the phase diagrams according to the values of μ .

[3] Consider the planar system

$$\begin{aligned}\frac{dx}{dt} &= y - axy^2, \\ \frac{dy}{dt} &= -x - by,\end{aligned}$$

where a and b are positive constants.

- (i) Find all critical points and classify them according to stable, asymptotically stable, and unstable.
- (ii) Determine the ω -limit set and α -limit set for every point in the plane. If there is no such set, state so.

[4] (i) Determine a fundamental solution to the system of ODE's

$$\frac{dy}{dt} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} y, \quad -\infty < t < \infty. \quad (1)$$

- (ii) Are there non trivial (non zero) column vector solutions of (1) that converge to $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ as

$t \rightarrow -\infty$? If yes determine all of them.

(iii) Solve the initial value problem

$$\frac{dy}{dt} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} y + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad y(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

[5] Let $Y = Y(t)$ be an n by n matrix solution $Y(t)$ of

$$\frac{dY}{dt} = A(t)Y,$$

where $A(t) = [a_{jk}(t)]$, $j, k = 1, \dots, n$, and $A(t)$ is an n by n continuous matrix function on (a, b) . Denote by $q(t)$ the determinant of $Y(t)$.

(i) Show that $q(t)$ is a solution of the first order homogeneous ODE

$$\frac{dq(t)}{dt} = \left[\sum_{j=1}^n a_{jj}(t) \right] q(t). \quad (2)$$

(ii) Determine a particular solution $q(t)$ of (2) that satisfies the initial condition $q(t_0) = q_0$, with $t_0 \in (a, b)$.

(iii) Show that if

$$\lim_{t \rightarrow b^-} \left[\int_{t_0}^t \left[\sum_{j=1}^n a_{jj}(s) \right] ds \right] = -\infty, \text{ then } \lim_{t \rightarrow b^-} [q(t)] = 0.$$

(iv) What is the significance of the solution $q(t)$?

[6] Let A be an n by n constant matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ such that $\operatorname{Re}\{\lambda_j(t)\} < 0$, $j = 1, \dots, n$. Let $y(t)$ and $h(y)$ be column vectors given by $y^T(t) = (y_1(t), y_2(t), \dots, y_n(t))$ and $h^T(y) = (y_1^2, y_2^2, \dots, y_n^2)$. Show that the zero vector solution of

$$\frac{dy}{dt} = Ay + h(y)$$

is asymptotically stable as $t \rightarrow \infty$.

[7] (i) Determine an interval of existence for the solution of the initial value problem

$$\begin{aligned} \frac{dy_1}{dt} &= y_1^2 + 2y_2 + 1, \quad y_1(0) = 0, \\ \frac{dy_2}{dt} &= y_2^2 + 2y_1 + 1, \quad y_2(0) = 0. \end{aligned}$$

(ii) Explain what theorem guaranteeing existence and uniqueness you apply.

(iii) Show that the initial value problem

$$t \frac{dy}{dt} = y, \quad y(0) = 0$$

possesses infinitely many solutions. Then explain why a theorem guaranteeing the existence of unique solutions does not apply to this initial value problem.

Differential Equations Exam, 2003F. Solve all five problems.

[1] (a) Solve

$$\dot{x} = \begin{pmatrix} 0 & -1 & 2 \\ 0 & -1 & 2 \\ -1 & 0 & 2 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

The characteristic polynomial of the above coefficient matrix is $\lambda^2(\lambda - 1)$.

- (b) Find a Jordan form J of the above matrix and the transformation P such that $P^{-1}AP = J$.
(c) Find the stable, center, and unstable subspaces if there are.

[2] Consider

$$\begin{aligned} \dot{x}_1 &= x_2 + x_2^2, \\ \dot{x}_2 &= -x_2 + x_1^2. \end{aligned}$$

- (a) Find an approximation for the stable, center, unstable manifolds at the critical point $x = \vec{0}$. If a particular manifold does not exist, state so.
(b) Draw the phase diagram near the origin including the above manifolds.
(c) Is the critical point $x = \vec{0}$ stable or unstable?

[3] Discuss the stability of critical points.

(a)

$$\begin{aligned} \dot{x} &= -\frac{1}{2}x^3 + 2xy^2, \\ \dot{y} &= -y^3. \end{aligned}$$

(b)

$$\begin{aligned} \dot{x} &= 4x^2 - y^2, \\ \dot{y} &= -2x + xy - 4. \end{aligned}$$

[4] Consider

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + y(1 - x^2 - 2y^2). \end{aligned}$$

- (a) Show the existence of a periodic orbit.
(b) Assuming that the periodic solution (a) is unique, show the stability of the periodic orbit.

[5] (a) Derive the variation of constant formula for a system $\dot{x} = A(t)x + f(t)$, where $A(t)$ is an $n \times n$ matrix and $x, f \in R^n$. Assume that a matrix solution $X(t)$ to $\dot{x} = A(t)x$ is given. **Hint:** $X^{-1}(t)$ (the inverse of $X(t)$) satisfies $(\dot{X}^{-1}) = -X^{-1}A(t)$, where $(\dot{X}^{-1}) = \frac{d}{dt}(X^{-1})$.

(b) Find the solution of

$$\dot{x} = \begin{pmatrix} 0 & 2 & 1 \\ -4 & 6 & 2 \\ 4 & -4 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ e^{2t} \end{pmatrix}, \quad x(0) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

The characteristic polynomial of the above coefficient matrix is $(\lambda - 2)^3$.

Ph.D. Entrance Exam — Differential Equations

May 7, 1996

Part I: Ordinary Differential Equations

Instruction: Complete 4 of the 5 problems.

1. Find and classify all equilibrium points of the system

$$\dot{x} = -6y + 2xy - 8, \quad \dot{y} = y^2 - x^2.$$

Sketch the phase diagram of the system.

2. For the autonomous system $\dot{x} = X(x, y)$, $\dot{y} = Y(x, y)$, suppose X and Y satisfy the following conditions:

$$\begin{cases} X(0, 0) = Y(0, 0) = 0, \\ -X_x > |X_y| \quad \text{and} \quad -Y_y > |Y_x| \quad \text{in a neighbourhood of } (0, 0) \end{cases}$$

Use $V(x, y) = \max\{|X(x, y)|, |Y(x, y)|\}$ as a Liapunov function to determine the stability of the zero solution.

3. For the second order equation $\ddot{x} + \epsilon(|x| + |\dot{x}| - 1)\dot{x} + x = 0$ where ϵ is a positive parameter,

(a) Show that there exists at least one periodic solution.

(b) Suppose $x(t)$ is a periodic solution with period T . Show that $x(t)$ has exactly two zeros in $[0, T)$.

(c) When the parameter $\epsilon > 0$ is small, find the approximate amplitude a of the periodic solution by using the energy balance method, and determine whether it is stable.

4. For the equation $\ddot{x} + f(x) = 0$, where $f(x) = \operatorname{sgn}(x)$ when $|x| > 1$ and $f(x) = x$ when $|x| \leq 1$,

(a) show that all solutions are periodic solutions.

(b) Find the amplitude and period of the solution that satisfies the initial conditions $x(0) = 0$ and $\dot{x}(0) = b > 0$.

(c) Obtain the explicit expression for the solution $x(t)$ in (b).

5. Show that the boundary value problem

$$\ddot{x} + x^2 = 0 \quad (0 < t < 1), \quad x(0) = x(1) = 0$$

has a unique nontrivial solution.

Some useful integrals:

$$\int_0^{\pi/2} \sin^k t \, dt = \int_0^{\pi/2} \cos^k t \, dt = \begin{cases} 1 & k = 1 \\ \pi/4 & k = 2 \\ 2/3 & k = 3 \\ 3\pi/16 & k = 4 \end{cases}$$