1. Prove that exponentiation to a positive odd power defines a strictly increasing function. For \( n \in \mathbb{N} \), find all solutions to \( x^n = y^n \). (Hint: One possibility is to consider the cases \( x < 0 < y \), \( 0 < x < y \) and \( x < y < 0 \).)

**Solution.**

We use induction on \( n \) to prove this for the power \( 2n + 1 \) (an odd number).

**Basis step** \((n = 1)\): Here exponentiation is the identity function, so \( x < y \) does in fact give that \( x^1 < y^1 \).

**Induction step:** Suppose that exponentiation to the power \( 2n - 1 \) is strictly increasing. Thus if \( x < y \), then \( x^{2n-1} < y^{2n-1} \). \((\ast)\)

If \( 0 \leq x < y \), then \( 0 < x^2 < y^2 \) and multiplying equation \((\ast)\) gives us that \( x^{2n+1} < y^{2n+1} \).

If \( x < 0 \leq y \), then \( x^{2n+1} \) is negative and \( y^{2n+1} \) is nonnegative, so \( x^{2n+1} < y^{2n+1} \).

If \( x < y \leq 0 \), then \( 0 \leq -y < -x \), and we proved that \(( -y)^{2n+1} < (-x)^{2n+1} \). Since an odd power of \(-1\) is \(-1\), this gives us that \( -y^{2n+1} < -x^{2n+1} \), and thus we have \( x^{2n+1} < y^{2n+1} \).

**Solutions to** \( x^n = y^n \). All pairs with \( x = y \) are solutions. When \( n \) is odd, the exponentiation is strictly increasing, and hence in this case there are no other solutions. When \( n \) is even, the solutions are \( x = \pm y \). To show that there are no other solutions, it suffices to show that exponentiation to the \( n \)th power is injective from the set of positive real numbers to itself. This follows by induction almost exactly like that above.

2. (a) Let \( f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \) be defined by \( f(m, n) = 2m + n \). Is the function \( f \) an injection? Is the function \( f \) a surjection? Prove it.

**Solution.**

Let \( f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \) be defined by \( f(m, n) = 2m + n \). The function is not an injection since \((0, 2)\) and \((1, 0)\) both map to 2. The function is surjective. Let \( a \in \mathbb{Z} \). Then the element \((0, a)\) maps to \( a \) satisfying the definition of surjectivity.

(b) Let \( g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \) be defined by \( g(m, n) = 6m + 3n \). Is the function \( g \) an injection? Is the function \( g \) a surjection? Prove it.

**Solution.**

Let \( g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \) be defined by \( g(m, n) = 6m + 3n \). The function is not an injection since \((0, 2)\) and \((1, 0)\) both map to 6. The function is also not surjective. Note that \( 6m + 3n = 3(2m + n) \) and so all of the outputs will be multiples of 3. Thus 2 is not a possible output of \( g \).
3. Let $A$ be the set of subsets of $[n]$ that have even size, and let $B$ be the subsets of $[n]$ that have odd size. Establish a bijection from $A$ to $B$. Such a bijection is suggested below for $n = 3$.

$A = \emptyset \; \{2, 3\} \; \{1, 3\} \; \{1, 2\}$

$B = \{3\} \; \{2\} \; \{1\} \; \{1, 2, 3\}$

**Solution.**

Let $A$ be the collection of even subsets of $[n]$, and let $B$ be the collection of odd subsets. For each $X \in A$, define $f(X)$ as follows:

$$f(X) = \begin{cases} X \setminus \{n\} & \text{if } n \in X \\ X \cup n & \text{if } n \notin X. \end{cases}$$

By this definition, $|X|$ and $|f(X)|$ differ by one, so $f(X)$ is a set of odd size and $f$ maps $A$ to $B$.

We claim that this is a bijection. Consider distinct $X, Y \in A$. If both contain or both omit $n$, then $f(X)$ and $f(Y)$ agree on whether they contain $n$, but (since they were distinct) differ outside of $\{n\}$. If exactly one of either $X$ or $Y$ contains $n$, then exactly on of $f(X)$ or $f(Y)$ contains $n$. Thus $X \neq Y$ implies $f(X) \neq f(Y)$ and so $f$ is injective.

If $Z \in B$, then reversing whether $n$ is present in $Z$ yields a subset $X$ such that $f(X) = Z$. This means $f$ is also surjective. Therefore, $f$ is a bijection and so $|A| = |B|$.

Alternatively, we could define $g : B \to A$ by the same rule used to define $f$ (switching the domain and target), and observe that $g \circ f$ is the identity function on $A$ and $f \circ g$ is the identity function on $B$. This implies that $g$ is the inverse of $f$ and thus that $f$ is a bijection and therefore $|A| = |B|$. Without knowing $|A| = |B|$ beforehand, it does not suffice to show that one of the compositions is the identity (we must show both).

**Bonus 1.** Determine which cubic polynomials from $\mathbb{R}$ to $\mathbb{R}$ are injective.

(Hint: This is easy if calculus is allowed (but it’s not!). To avoid calculus, first use geometric arguments to reduce the problem to the case $x^3 + rx$.)

(Hint 2: I wrote an article in *Mathematics Teacher*, v101 n 6 p408-11 (Feb 2008) that may help, but it is more general than what is needed here. It is available in the Evansdale Library as well as online.)

**Solution.**

The paper referenced provides the key ideas in the following geometric arguments to reduce the problem to the case $x^3 + rx$.

The formula for the value of a general cubic polynomial at $x$ is $f(x) = ax^3 + bx^2 + cx + d$ where the coefficients $a, b, c$ and $d$ are given with $a \neq 0$. Since multiplying the function by $-1$ doesn’t affect injectivity (one-to-one), we may assume that $a > 0$.

We assert that every cubic polynomial has rotational symmetry about a point. This is similar to the idea of an odd polynomial, except the rotational symmetry is about a point $(m, n)$ instead of $(0, 0)$. The function $f(x) = ax^3 + bx^2 + cx + d$ has rotational symmetry
about a point \((m, n)\) if and only if a function \(g\) given by \(g(x) = f(x + m) - n\) is an odd function. Notice that this transformation moves the point \((m, n)\) on the graph of \(f\) to the origin.

Note that a cubic function is odd if and only if
\[
ax^3 + bx^2 + cx + d = -a(-x)^3 + b(-x)^2 + c(-x) + d
= ax^3 - bx^2 + cx - d.
\]
which is true if and only if \(b = d = 0\). Further, note that \(n = f(m) = am^3 + bm^2 + cm + d\).

Thus, we can rewrite \(g\) as follows:
\[
g(x) = f(x + m) - n
= a(x + m)^3 + b(x + m)^2 + c(x + m) + d - (am^3 + bm^2 + cm + d)
= ax^3 + 3ax^2m + 3axm^2 + bx^2 + 2bxm + cx
= ax^3 + (3am + b)x^2 + (3am^2 + 2bm + c)x.
\]
Recall, that the coefficient of \(x^2\) must be 0, which means
\[
m = -\frac{b}{3a}.
\]
Thus we know that \(f\) has rotational symmetry about the point
\[
\left(-\frac{b}{3a}, f\left(-\frac{b}{3a}\right)\right).
\]
Thus, we can rewrite \(g\) as follows:
\[
g(x) = ax^3 + (3am^2 + 2bm + c)x
= ax^3 + \left(3a \left(-\frac{b}{3a}\right) + 2b \left(-\frac{b}{3a}\right) + c\right)x
= ax^3 + \left(3a \left(\frac{b^2}{(3a)(3a)}\right) - \frac{2b^2}{3a} + \frac{3ac}{3a}\right)
= ax^3 + \left(\frac{3ac - b^2}{3a}\right).
\]
Now, dividing by \(a\) does not affect the injectivity of a function and so we can consider the function \(h\) defined by \(g(x)/a\). In other words, \(h(x) = x^3 + rx\) where \(r = \frac{3am^2 + 2bm + c}{a}\).

We can now proceed three different ways.

**Method #1:** We could realize that Problem 1 gives us that this function is the sum of two injective functions which we can prove is injective.

**Method #2:** First, we realize that for the function to be injective, the maximum and minimum value must be the same. The paper then provides a method of determining the
coordinates of the maximum and minimum value which we then would set equal to each other for the condition.

**Method #3:** Notice that if \( x^3 + rx = (x')^3 + r x' \) for some distinct \( x \) and \( x' \), then dividing by \( x - x' \) yields \( x^2 + xx' + (x')^2 = -r \).

If \( r \) is negative, then \((x, x') = (0, \sqrt{-r})\) is a solution, and so the function is not injective. If \( r \) is 0, then there is no solution with \( x \neq x' \) (since cubing is injective). If \( r \) is positive, then again, there is no solution, because \( x^2 + yx' + (x')^2 \) is never negative (since \( a^2 + b^2 \geq 2 |a||b| \)).

Thus, \( h \) is injective if and only if \( r \geq 0 \) and this determines whether \( f \) is injective. This means that \( 3ac - b^2 \geq 0 \).

Therefore, the requirement for injectivity of a general cubic function \( b^2 - 3ac \leq 0 \). ■