CHAPTER 4

Introduction to Systems of ODE’s

and Solution Using Elimination

1. Introduction and the Predator Prey Model

2. Scalar Systems and “Vector” Equations

3. Solution by Elimination

4. Development of Model for Drug Effectiveness for Social Anxiety
Systems of O.D.E's arise naturally in applications including mass-spring systems, electric circuits, and ecology models such as predator-prey models. Since we have already seen examples of mass-spring systems and electric circuits, we take a brief look at a predator-prey model. Original work in this area was done by Alfred J. Lotka (1880-1949), a biophysicist, and Vito Voltera (1860-1940), an applied analyst.

**Predator-Prey Model**

\[
\begin{align*}
\frac{dH}{dt} &= aH - \alpha HP \\
\frac{dP}{dt} &= -cP + \gamma HP
\end{align*}
\]

**IC's**

\[H(0) = H_0, \quad P(0) = P_0\]

The positive constants \(a\), \(\alpha\), \(c\), and \(\gamma\) can be based on empirical observations of the particular species that are being modeled. These equations are nonlinear and nonlinear equations are generally more complicated than linear ones. To keep the model simple and solvable by techniques we already know, we use a "linearized" model. Obviously it does not model "reality" as well as the nonlinear model, but it has some interesting properties. However, use of the results of this model is not recommended.

**Linearized Model**

\[
\begin{align*}
\frac{dH}{dt} &= aH - \alpha P \\
\frac{dP}{dt} &= -cP + \gamma H
\end{align*}
\]

**IC's**

\[H(0) = H_0, \quad P(0) = P_0\]

Although the behavior exhibited by the "linearized" model may carry over to the nonlinear model, no such claim is made. Investigation of the nonlinear model and how well the...
linearized and the nonlinear model actually models nature is left to you, if you are interested.

The skills concerning ODE systems that you are to master are.
1) Learn to solve an ODE system by elimination, that is, by eliminating one variable to obtain a single second order scalar equation which can be solved by techniques that you already know. This requires that you compute an “equivalent” second order scalar equation.
2) Learn to use matrix notation to rewrite a system of scalar ODEs as a "vector" equation.
3) Learn to convert a second order scalar equation into an “equivalent” first order system which you can write as either a scalar system or a “vector” equation.
4) Learn to solve a "vector" ODE system by using a vector (or matrix) technique.

We do 2) immediately. 1) is in keeping with our policy of applying known techniques to new problems. 4) is covered by homework problems. 3) makes further use of our review and preview of linear theory.
WRITING TWO SCALAR EQUATIONS AS ONE ‘VECTOR’ EQUATION

The linearized scalar predator-prey system

\[
\begin{align*}
\frac{dH}{dt} &= a H - \alpha P \\
\frac{dP}{dt} &= -c P + \gamma H
\end{align*}
\]

can be written as a “vector” equation as

\[
\begin{bmatrix}
\frac{dH}{dt} \\
\frac{dP}{dt}
\end{bmatrix} = \begin{bmatrix}
a & -b \\
b & a
\end{bmatrix} \begin{bmatrix}
P \\
H
\end{bmatrix}
\]

More generally, we consider

\[
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = \begin{bmatrix}
p_{11}(t) & p_{12}(t) \\
p_{21}(t) & p_{22}(t)
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

Letting \( \hat{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \), \( \hat{\hat{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \), \( \overrightarrow{P} = \begin{bmatrix} p_{11}(t) & p_{12}(t) \\
p_{21}(t) & p_{22}(t) \end{bmatrix} \)

\( \hat{x} - \overrightarrow{P} \hat{\hat{x}} = \overrightarrow{0} \)

which is a first order linear homogeneous system.

WRITING A SECOND ORDER EQUATION AS A FIRST ORDER SYSTEM.

Although this may move us further away from an application, a second order scalar ODE may be written as a first order system.

Example. Write \( \ddot{z} + 3\dot{z} - z = g(t) \)

\( z + 3z - 2z = g(t) \) with \( z(0) = 4 \) and \( \dot{z}(0) = 5 \) as a first order system.

Solution: Let \( \ddot{x} = \begin{bmatrix} x \\ y \end{bmatrix} \)

and \( x = z \) \( \rightarrow \ddot{\dot{z}} = \ddot{x} = 0 \) so \( \dot{z} \) at \( t = 0 \) = \( y \).

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From the equation, $\ddot{z} = -z + 2z + g(t)$. Since $\dot{z} = \dot{y}$, we have $y = -3y + 2x + g(t)$.

Hence our system is

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= 2x - 3y + g(t)
\end{align*}
\]

or in vector form

\[
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
2 & -3
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} +
\begin{bmatrix}
0 \\
g(t)
\end{bmatrix}
\]
EXAMPLE #1. Solve \( \dot{x} + A \dot{x} = \ddot{g}(t) \)

where \( A = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} \) \( \ddot{g}(t) = \begin{bmatrix} 0 \\ -\alpha \end{bmatrix} \).

That is, solve

\[
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\alpha \end{bmatrix}
\]

Writing this vector equation as two scalar equations we obtain

\begin{align*}
\text{a)} & \quad \dot{x}_1 = x_1 - 2x_2 \quad & x_1(0) = 2 \\
\text{b)} & \quad \dot{x}_2 = x_1 - x_2 - \alpha(\text{harvesting}) \quad & x_2(0) = 1
\end{align*}

We eliminate \( x_2 \).

\begin{center}
\begin{tabular}{ll}
\text{STATEMENT} & \text{REASON} \\
\hline
\text{ODE} & \text{Derivative of Eq. a)} \\
\dot{x}_1 = x_1 - 2x_2 & \text{Substitute into Eq. b)} \\
= \dot{x}_1 - 2(x_1 - x_2 - \alpha) & \text{Algebra} \\
= \dot{x}_1 - 2x_1 + 2x_2 + 2\alpha & \text{Solve for } x_2 \text{ in Eq. a) and substitute.} \\
\dot{x}_1 + x_1 = 2\alpha & \text{Algebra} \\
\hline
\text{IC.} & \text{Given I.C.} \\
x_1(0) = 2 & \text{From Eq. a)} \\
\dot{x}_1(0) = x_1(0) - 2x_2(0) & \text{From given IC's} \\
= 2 - 2(1) = 0 & \\
\end{tabular}
\end{center}

Equivalent second order IVP

\[
\ddot{x}_1 + x_1 = 2\alpha
\]

Homogeneous Equation. Auxiliary Equation: \( r^2 + 1 = 0 \Rightarrow r = \pm i \)

\[
\rightarrow x_i(t) = c_1 \sin t + c_2 \cos t
\]

Particular Solutions of Nonhomogeneous

1) \( x_{i_p} = A \)

2) \( \dot{x}_{i_p} = 0 \)

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3) \( \dot{x}_1 = 0 \)

\[ \ddot{x}_1 + \dot{x}_1 + A = 2a \Rightarrow x_1 = 2a \]

\[ \Rightarrow x_1 = c_1 \sin t + c_2 \cos t + 2a \]

\[ \dot{x}_1 = c_1 \cos t - c_2 \sin t \]

Apply IC.

\[ 2 = c_2 + 2a \Rightarrow c_1 = 0, c_2 = 2 - 2a. \]

\[ 0 = c_1 \]

\[ \Rightarrow x_1 = (2 - 2a) \cos t + 2a \]

\[ \dot{x}_1 = -(2 - 2a) \sin t \]

Case 1  \( \alpha = 0 \Rightarrow x_1 = 2 \cos t \)

Case 2  \( \alpha = 1 \Rightarrow x = 2 \)

Case 3  \( \alpha = 3 \Rightarrow x_1 = -4 \cos t + 6 \)

From Eq. a) (Solved for \( x_2 \))

\[ x_2 = \frac{1}{2} (\dot{x}_1 + x_1) \]

\[ = \frac{1}{2} (-2a \sin t + (2 - 2a) \cos t - 2a) \]

\[ = (1 - a) (\cos t \sin t) + a \]

\[ = \sqrt{2} (1 - a) \cos (t - \pi/4) \]

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Step 1. UNDERSTAND THE APPLICATION AREA.

1. We accept the Liebowitz scale as a measure of social anxiety. 
   Total is \(0 - 2(3)(24) = 144\) (or Average 0-3)
   Interpretation:
   - 50-65: \(0.735294117 - 0.955882352\) Moderate social phobia
   - 65-80: \(0.955882352 - 1.323529412\) Marked social phobia
   - 80-95: \(1.323529412 - 1.397058824\) Severe social phobia
   - >95: \(>1.397058824\) Very severe social phobia

2. We believe that a normal person has some measure, say \(u, >0\), of social anxiety. That is a normal person will have some social anxiety. (For computational convenience, we assume \(u_n = 1\) instead of say 0.5)

3. We are looking for a treatment strategy for a temporary stress that causes one anxiety level to increase. We wish to know the “best” treatment strategy (e.g., drug dosage).

Step 2. DEVELOPMENT OF MODEL.

We wish to develop a model that can be used to evaluate treatment strategies (e.g., drug dosage).

Nomenclature:
- \(u_1\) = Current dosage of drug (e.g., Paxel)
- \(u_2\) = Social Anxiety of Patient (or average for class)
- \(u_1(0) = u_1^0\) = Initial dosage of drug.
- \(u_2(0) = u_2^0\) = Initial social anxiety.
- \(u_n\) = Normal anxiety level
- \(u_b\) = Initial anxiety level that we wish to bring back to normal
- \(a_{11}, a_{12}, a_{21}, a_{12}\) = model parameters, see below.

MODEL:

Scalar Form:

\[
\begin{align*}
\frac{du_1}{dt} &= -a_{11}u_1 + a_{12}(u_2 - u_b) \\
\frac{du_2}{dt} &= -a_{21}u_1 - a_{22}(u_2 - u_n)
\end{align*}
\]

\(u_1(0) = 0\) \hspace{1cm} \(u_2(0) = u_0\)

Vector Form:

\[
\frac{d\tilde{u}}{dt} = A\tilde{u} + \tilde{g}_b
\]

\(\tilde{u}(0) = \tilde{u}_0\)

where

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Analysis of model:

A. $a_{11}$ and $a_{12}$ are treatment strategy parameters.
1. $a_{11} > 0$ so that $-a_{11} < 0$ since if the dosage is high (and social anxiety is low) we would like to decrease the dosage.
2. $a_{12} > 0$ since if $u_2 > u_n$, we increase the dosage.

B. $a_{21}$ and $a_{22}$ are physiological constants for the patient (or the general population).
1. $a_{21} > 0$ so that $-a_{21} < 0$. $a_{21}$ is the effectiveness of the drug; that is, the rate at which the drug can control anxiety. If $u_1$ increases, since $-a_{21} < 0$, $u_2$ will go down.
2. $a_{22} > 0$ so that $-a_{22} < 0$. We expect that even if no drug is given, the body's defenses will cause the anxiety to go down.

Steps 3 and 4. SOLUTION OF MODEL and INTERPRETATION OF RESULTS.

Dosage strategy #1 Constant dosage ($a_{11} = a_{12} = 0$)
If we assume $a_{11} = a_{12} = 0$, then the ODE’s are uncoupled and

$$\frac{du_1}{dt} = 0$$
$$\frac{du_2}{dt} = -a_{21} u_1 - a_{22} (u_2 - u_n) \quad a_{21} > 0, \quad a_{22} > 0$$

$u_1(0) = 0$ \quad $u_2(0) = u_0$

Hence $u_1$ = constant. However, we do not assume that since $u_1(0) = 0$, that $u_1$ is always 0. Instead we assume that there is a jump discontinuity at $t = 0$ to some value $u_1$ to be decided later. It remains to solve

$$\frac{du_2}{dt} = -a_{21} u_1 - a_{22} (u_2 - u_n) \quad a_{21} > 0, \quad a_{22} > 0$$

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or
\[
\frac{du_2}{dt} + a_{22} u_2 = a_{22} u_n - a_{21} u_1, \quad a_{21} > 0, \quad a_{22} > 0
\]
\[
u_2(0) = u_0
\]

Since this problem is now first order linear, we could use an integrating factor. However, we note that it has constant coefficients and the right hand side is “nice”. Hence we can (and will) use the techniques we learned for second order ODE’s with constant coefficients and “nice” right hand sides.

Homogeneous: \( \frac{du_2}{dt} + a_{22} u_2 = 0 \).

Auxiliary equation: \( r + a_{22} = 0 \). Hence \( r = -a_{22} \) so that \( u_{2c} = c_2 e^{a_{22}t} \)

Particular Solution of Nonhomogeneous: \( \frac{du_2}{dt} + a_{22} u_2 = a_{22} u_n - a_{21} u_1 \).

Let
\[a_{22}\] 1) \( \frac{du_{2p}}{dt} = 0 \).

\[
\frac{du_{2p}}{dt} + a_{22} u_{2p} = a_{22} A = a_{22} u_n - a_{21} u_1.
\]

\[
A = u_n - (a_{21}/a_{22}) u_1.
\]

\[
u_2 = u_n - (a_{21}/a_{22}) u_1 + c_2 e^{a_{22}t}
\]

We apply the initial condition to obtain \( c_2 \).

\[
u_0 = u_n - (a_{21}/a_{22}) u_1 + c_2 e^{a_{22}t(0)}
\]

\[
c_2 = u_0 - u_n + (a_{21}/a_{22}) u_1.
\]

\[
u_2 = u_n - (a_{21}/a_{22}) u_1 + [u_0 - u_n + (a_{21}/a_{22}) u_1] e^{a_{22}t}
\]

Note:

\[
\lim_{t \to \infty} u_2 = u_n - (a_{21}/a_{22}) u_1 \neq u_n
\]

\( \tau = \text{Decay time constant} = 1/a_{22} \) (\( = \frac{1}{2} \) using the parameters in the specific model that follows).
Suppose \( u_n = u_n \) at \( t = t_1 \). Then \( t_1 \) satisfies

\[
\begin{aligned}
\dot{u}_n &= u_n - \left( \frac{a_{21}}{a_{22}} \right) u_1 + \left[ u_0 - u_n + \left( \frac{a_{21}}{a_{22}} \right) u_1 \right] e^{-a_{22}t_1} \\
\text{or}
[u_0 - u_n + \left( \frac{a_{21}}{a_{22}} \right) u_1] e^{-a_{22}t_1} &= \left( \frac{a_{21}}{a_{22}} \right) u_1 \\
-a_{22}t_1 &= \ln \frac{\left( \frac{a_{21}}{a_{22}} \right) u_1}{[u_0 - u_n + \left( \frac{a_{21}}{a_{22}} \right) u_1]} \\
t_1 &= \frac{1}{a_{22}} \left[ \frac{u_0 - u_n + \left( \frac{a_{21}}{a_{22}} \right) u_1}{\left( \frac{a_{21}}{a_{22}} \right) u_1} \right] \\
t_1 &= \frac{1}{a_{22}} \ln \left[ \frac{\left( \frac{a_{21}}{a_{22}} \right) u_1}{u_0 - u_n} \right] \\
t_1 &= \frac{1}{a_{22}} \ln \left[ \frac{a_{22}[u_0 - u_n]}{a_{21}u_1} \right]
\end{aligned}
\]

General Dosage Strategy for a Specific Model \( (a_{11} = 4, \ a_{12} = 1, \ a_{21} = 1, \ a_{22} = 2) \)

For convenience of computation, assume the physiological constants are \( a_{21} = 1, \ a_{12} = 2 \). Also for computational convenience assume \( u_n = 1 \) and \( u_0 = 2 \). Now choose \( a_{11} = 4, \ a_{12} = 1 \). Hence our system becomes:

\[
\begin{aligned}
\frac{du_{11}}{dt} &= -4u_1 + (u_2 - 1) \\
\frac{du_{22}}{dt} &= -u_1 - 2(u_2 - 1) \\
u_{11}(0) &= 0 \quad u_{21}(0) = 2
\end{aligned}
\]

or, using dot notation

\[
\begin{aligned}
\dot{u}_1 &= -4u_1 + u_2 - 1 \\
\dot{u}_2 &= -u_1 - 2u_2 + 2
\end{aligned}
\]
\[ u_1(0) = 0 \quad u_2(0) = 2 \]

Solving the first equation for \( u_2 \) we obtain

\[ u_2 = \ddot{u}_1 + 4 u_1 + 1 \]

Once we have solved for \( u_1 \), substitution into this equation gives \( u_2 \). To obtain \( u_2(0) \) we use the first equation directly:

\[ \ddot{u}_1 \quad (0) = - 4 u_1(0) + u_2(0) - 1 = - 4 (0) + 2 - 1 = 1 \]

Taking the derivative of the first equation and substituting the second equation in for \( u_2 \), we obtain:

\[
\ddot{u}_1 = - \ddot{u}_1 + u_2 - 4 \dot{u}_1 + (- u_1 - 2 u_2 + 2) \\
= - 4 \ddot{u}_1 - u_1 - 2 u_2 + 2 \\
= - 4 \ddot{u}_1 - u_1 - \ddot{u}_1 + 4 u_1 + 1 + 2 \\
= - 4 \ddot{u}_1 - u_1 - \ddot{u}_1 - 8 u_1 - 2 + 2 \\
= - 6 \ddot{u}_1 - 9 u_1
\]

or

\[ \ddot{u}_1 + \ddot{u}_1 + 9 u_1 = 0 \]

with the initial conditions \( u_1(0) = 0 \) and \( \ddot{u}_1 \quad (0) = 1 \)

The auxiliary equation is \( r^2 + 6 r + 9 = 0 \) or \( r + 3 = 0 \)

Hence the general solution is

\[ u_1 = c_1 e^{-3 t} + c_2 t e^{-3 t} \]

\[ \ddot{u}_1 = - 3 c_1 e^{-3 t} + c_2 e^{-3 t} - 3 c_2 t e^{-3 t} \]

Applying the initial conditions we obtain

\[ 0 = c_1 e^{-3 (0)} + c_2 (0) e^{-3 (0)} \]
\[ 1 = - 3 c_1 e^{-3 (0)} + c_2 e^{-3 (0)} - 3 c_2 (0) e^{-3 (0)} \]

or

\[ c_1 = 0 \]
\[ - 3 c_1 + c_2 = 1 \]
Hence \( c_1 = 0 \) and \( c_2 = 1 \) so that

\[ u_1 = t e^{-3t} \]

\[ \dot{u}_1 = e^{-3t} - 3t e^{-3t} \]

\[ u_2 = \dot{u}_1 + 4u_1 + 1 = e^{-3t} - 3t e^{-3t} + 4t e^{-3t} + 1 \]

\[ = (1 + t) e^{-3t} + 1 \]

\( \tau = \) Decay time constant = \( 1/3 \) (Compare to \( 1/2 \) for the constant dosage model).

\[
\lim_{t \to \infty} u_2 = 1 = u_n.
\]