CHAPTER 7

Introduction to

Determinants

1. Introduction to Computation of Determinants
2. Computation Using Laplace Expansion
3. Computation Using Gauss Elimination
4. Introduction to Cramer’s Rule
Rather than give a fairly complicated definition of the determinant in terms of minors and cofactors, we focus only on two methods for computing the determinant function \( \det: \mathbb{R}^{n \times n} \rightarrow \mathbb{R} \) (or \( \det: \mathbb{C}^{n \times n} \rightarrow \mathbb{C} \)). Let \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \). The we define \( \det(A) = ad - bc \). Later we will show that

\[
A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} a & -b \\ -c & d \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}.
\]

For \( A \in \mathbb{R}^{n \times n} \) (or \( \mathbb{C}^{n \times n} \)), we develop two methods for computing \( \det(A) \): Laplace Expansion and Gauss Elimination.

But first, we give (without proof) several properties of determinants that aid in their evaluation.

**Theorem.** Let \( A \in \mathbb{R}^{n \times n} \) (or \( \mathbb{C}^{n \times n} \)). Then

1. (ERO's of type 1) If \( B \) is obtained from \( A \) by exchanging two rows, then \( \det(B) = -\det(A) \).

2. (ERO's of type 2) If \( B \) is obtained from \( A \) by multiplying a row of \( A \) by a \( \neq 0 \), then \( \det(B) = a \det(A) \).

3. (ERO's of type 3) If \( B \) is obtained from \( A \) by replacing a row of \( A \) by itself plus a scalar multiple of another row, then \( \det(B) = \det(A) \).

4. If \( U \) is the upper triangular matrix obtained from \( A \) by Gauss elimination (forward sweep) using only ERO's of type 3, then \( \det(A) = \det(U) \).

5. If \( U \in \mathbb{R}^{n \times n} \) (or \( \mathbb{C}^{n \times n} \)) is upper triangular, then \( \det(U) \) is equal to the product of the diagonal elements.

6. If \( A \) has a row (or column) of zeros, then \( \det(A) = 0 \).

7. If \( A \) has two rows (or columns) that are equal, then \( \det(A) = 0 \).

8. If \( A \) has one row (column) that is a scalar multiple of another row (column), then \( \det(A) = 0 \).

9. \( \det(AB) = \det(A) \det(B) \).

10. If \( \det(A) \neq 0 \), then \( \det(A^{-1}) = 1/\det(A) \).

11. \( \det(A^T) = \det(A) \).

**Exercises on Introduction of Computation of Determinants**

**Exercise #1.** True or False.

_____ 1. If \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \), then \( \det(A) = ad - bc \).

_____ 2. If \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \), then \( A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} a & -b \\ -c & d \end{bmatrix} \).

_____ 3. If \( A \in \mathbb{R}^{n \times n} \) (or \( \mathbb{C}^{n \times n} \)), there are (at least) two methods for computing \( \det(A) \).

_____ 4. If \( A \in \mathbb{R}^{n \times n} \) (or \( \mathbb{C}^{n \times n} \)), Laplace Expansion is one method for computing \( \det(A) \).

_____ 5. If \( A \in \mathbb{R}^{n \times n} \) (or \( \mathbb{C}^{n \times n} \)), use of Gauss Elimination is one method for computing \( \det(A) \).
6. If $A \in \mathbb{R}^{n \times n}$ (or $\mathbb{C}^{n \times n}$) and $B$ is obtained from $A$ by exchanging two rows, then $\det(B) = -\det(A)$.

7. If $A \in \mathbb{R}^{n \times n}$ (or $\mathbb{C}^{n \times n}$) and $B$ is obtained from $A$ by multiplying a row of $A$ by a $\neq 0$, then $\det(B) = a \det(A)$.

8. If $A \in \mathbb{R}^{n \times n}$ (or $\mathbb{C}^{n \times n}$) and $B$ is obtained from $A$ by replacing a row of $A$ by itself plus a scalar multiple of another row, then $\det(B) = \det(A)$.

9. If $A \in \mathbb{R}^{n \times n}$ (or $\mathbb{C}^{n \times n}$) and $U$ is the upper triangular matrix obtained from $A$ by Gauss elimination (forward sweep) using only ERO's of type 3, then $\det(A) = \det(U)$.

10. If $A \in \mathbb{R}^{n \times n}$ (or $\mathbb{C}^{n \times n}$) and $U \in \mathbb{R}^{n \times n}$ (or $\mathbb{C}^{n \times n}$) is upper triangular, then $\det(U)$ is equal to the product of the diagonal elements.

11. If $A \in \mathbb{R}^{n \times n}$ (or $\mathbb{C}^{n \times n}$) and $A$ has a row of zeros, then $\det(A) = 0$.

12. If $A \in \mathbb{R}^{n \times n}$ (or $\mathbb{C}^{n \times n}$) and $A$ has a column of zeros, then $\det(A) = 0$.

13. If $A \in \mathbb{R}^{n \times n}$ (or $\mathbb{C}^{n \times n}$) and $A$ has two rows that are equal, then $\det(A) = 0$.

14. If $A \in \mathbb{R}^{n \times n}$ (or $\mathbb{C}^{n \times n}$) and $A$ has two columns that are equal, then $\det(A) = 0$.

15. If $A \in \mathbb{R}^{n \times n}$ (or $\mathbb{C}^{n \times n}$) and $A$ has one row that is a scalar multiple of another row, then $\det(A) = 0$.

16. If $A \in \mathbb{R}^{n \times n}$ (or $\mathbb{C}^{n \times n}$) and $A$ has one column that is a scalar multiple of another column, then $\det(A) = 0$.

17. If $A, B \in \mathbb{R}^{n \times n}$ (or $\mathbb{C}^{n \times n}$) then $\det(AB) = \det(A) \det(B)$.

18. If $A \in \mathbb{R}^{n \times n}$ (or $\mathbb{C}^{n \times n}$) and $\det(A) \neq 0$, then $\det(A^{-1}) = 1/\det(A)$.

**EXERCISE #2.** Compute $\det A$ where $A = \begin{bmatrix} 0 & 5 \\ 0 & -1 \end{bmatrix}$.

**EXERCISE #3.** Compute $\det A$ where $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$.

**EXERCISE #4.** Compute $\det A$ where $A = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$.

**EXERCISE #5.** Compute $\det A$ where $A = \begin{bmatrix} 1 & i \\ i & 0 \end{bmatrix}$.

**EXERCISE #6.** Compute $A^{-1}$ if $A = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}$.

**EXERCISE #7.** Compute $A^{-1}$ if $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$.

**EXERCISE #8.** Compute $A^{-1}$ if $A = \begin{bmatrix} 1 & i \\ 1 & 2 \end{bmatrix}$.

**EXERCISE #9.** Compute $A^{-1}$ if $A = \begin{bmatrix} 1 & i \\ -i & 0 \end{bmatrix}$.

**EXERCISE #10.** Compute $A^{-1}$ if $A = \begin{bmatrix} 0 & 5 \\ 0 & -1 \end{bmatrix}$.

**EXERCISE #11.** Compute $A^{-1}$ if $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$.

**EXERCISE #12.** Compute $A^{-1}$ if $A = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$.

Ch. 6 Pg. 3
We give an example of how to compute a determinant using Laplace expansion.

**EXAMPLE.** Compute $\det(A)$ where $A=$

\[
\begin{bmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 2
\end{bmatrix}
\]

using Laplace expansion.

Solution: Expanding in terms of the first row we have

\[
det(A) = 2 \begin{vmatrix} -1 & 0 \\ -1 & 2 \\ 0 & 2 \end{vmatrix} - \begin{vmatrix} -1 & 0 \\ -1 & 2 \\ 0 & -1 \end{vmatrix} + \begin{vmatrix} 0 & 2 \\ 0 & -1 \end{vmatrix}
\]

so the last two 3x3's are zero. Hence expanding the first remaining 3x3 in terms of the first row and the second terms of the first column we have

\[
det(A) = 2[4 - 1] + [4 - 1] = 2 + 3 = 5
\]
**EXERCISES** on Computation Using Laplace Expansion

**EXERCISE #1.** Using Laplace expansion, compute det A where $A = \begin{bmatrix} 3 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$

**EXERCISE #2.** Using Laplace expansion, compute det A where $A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 3 \end{bmatrix}$

**EXERCISE #3.** Using Laplace expansion, compute det A where $A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 2 \end{bmatrix}$

**EXERCISE #4.** Using Laplace expansion, compute det A where $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 1 & 4 \end{bmatrix}$

**EXERCISE #5.** Using Laplace expansion, compute det A where $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 2 & 1 & 4 \end{bmatrix}$

**EXERCISE #6.** Using Laplace expansion, compute det A where $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$
We give an example of how to compute a determinant using Gauss elimination.

**EXAMPLE.** Compute det(A) where $A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$ using Gauss elimination.

Recall

**THEOREM.** Let $A \in \mathbb{R}^{n \times n}$ (or $\mathbb{C}^{n \times n}$). Then

3. (ERO's of type 3) If $B$ is obtained from $A$ by replacing a row of $A$ by itself plus a scalar multiple of another row, then $\det(B) = \det(A)$.

4. If $U$ is the upper triangular matrix obtained from $A$ by Gauss elimination (forward sweep) using only ERO's of type 3, then $\det(A) = \det(U)$.

**EXERCISES on Computation Using Gauss Elimination**

**EXERCISE #1.** Using Gauss elimination, compute $\det A$ where $A = \begin{bmatrix} 3 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$

**EXERCISE #2.** Using Gauss elimination, compute $\det A$ where $A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 3 \end{bmatrix}$

**EXERCISE #3.** Using Gauss elimination, compute $\det A$ where $A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 2 \end{bmatrix}$

Ch. 6 Pg. 6
Cramer’s rule is a method of solving \( A\mathbf{x} = \mathbf{b} \) when \( A \) is square and the determinant of \( A \) which we denote by \( D \neq 0 \). The good news is that we have a formula. The bad news is that, computationally, it is not efficient for large matrices and hence is never used when \( n > 3 \). Let \( A \) be an \( nxn \) matrix and \( A = [a_{ij}] \). Let \( \mathbf{x} = [x_1, x_2, \ldots, x_n]^T \) and \( \mathbf{b} \) be \( nx1 \) column vectors and \( \mathbf{x} = [x_i] \) and \( \mathbf{b} = [b_i] \). Now let \( A_i \) be the matrix obtained from \( A \) by replacing the \( i \)th column of \( a \) by the column vector \( \mathbf{b} \). Denote the determinant of \( A_i \) by \( D_i \). Then \( x_i = \frac{D_i}{D} \) so that \( \mathbf{x} = [x_1, x_2, \ldots, x_n]^T = [D_i / D]^T \).

**EXERCISES** on Introduction to Cramer’s Rule

**EXERCISE #1.** Use Cramer’s rule to solve \( A\mathbf{x} = \mathbf{b} \) where \( A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \) and \( \mathbf{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \)

**EXERCISE #2.** Use Cramer’s rule to solve \( A\mathbf{x} = \mathbf{b} \) where \( A = \begin{bmatrix} 2i & -3 \\ 3 & 5i \end{bmatrix} \) and \( \mathbf{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \)

**EXERCISE #3.** Use Cramer’s rule to solve \( A\mathbf{x} = \mathbf{b} \) where \( A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -3 \\ 2 & 1 & -1 \end{bmatrix} \) and \( \mathbf{b} = \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix} \)

**EXERCISE #4.** Use Cramer’s rule to solve
\[
\begin{align*}
x_1 + x_2 + x_3 + x_4 &= 1 \\
x_1 + 2x_2 + x_3 &= 0 \\
x_3 + x_4 &= 1 \\
x_2 + 2x_3 + x_4 &= 1
\end{align*}
\]