INTRODUCTION TO ABSTRACT LINEAR MAPPING PROBLEMS

We consider the (abstract) equation (of the first kind)

\[ T(\vec{x}) = \vec{b} \quad \text{(Nonhomogeneous)} \]  

(1)

where \( T \) is a linear operator from the vector space \( V \) to the vector space \( W \) \(( T:V \rightarrow W )\). We view (1) as a mapping problem; that is, we wish to find those \( \vec{x} \)'s that are mapped by \( T \) to \( \vec{b} \).

THEOREM #1. For the nonhomogeneous equation (1) there are three possibilities:
1) There are no solutions.
2) There is exactly one solution.
3) There are an infinite number of solutions.

THEOREM #2. For the homogeneous equation

\[ T(\vec{x}) = \vec{0} \quad \text{(Homogeneous)} \]  

(2)

there are only two possibilities:
1) There is exactly one solution, namely \( \vec{x} = \vec{0} \); that is the null space of \( T \) (i.e. the set of vectors that are mapped into the zero vector) is \( \text{N}(T) = \{ \vec{0} \} \).
2) There are an infinite number of solutions. If the null space of \( T \) is finite dimensional, say has dimension \( k \in \mathbb{N} \), then the general solution of (2) is of the form

\[ \vec{x} = c_1 \vec{x}_1 + \cdots + c_k \vec{x}_k = \sum_{i=1}^{k} c_i \vec{x}_i \]  

(3)

where \( \{ \vec{x}_1, \ldots, \vec{x}_k \} \) is a basis for \( \text{N}(T) \) and \( c_i, i=1,\ldots,k \) are arbitrary constants.

THEOREM #3. The nonhomogeneous equation (1) has at least one solution if \( \vec{b} \) is contained in the range space of \( T \), \( \text{R}(T) \), (the set of vectors \( \vec{w} \in W \) for which there exist \( \vec{v} \in V \) such that \( T[\vec{v}] = \vec{w} \)). If this is the case, then the general solution of (1) is of the form

\[ \vec{x} = \vec{x}_p + \vec{x}_h \]  

(4)

where \( \vec{x}_p \) is a particular (i.e. any specific) solution to (1) and \( \vec{x}_h \) is the general (e.g. a parametric formula for all) solution(s) of (2). If \( \text{N}(T) \) is finite dimensional then

\[ \vec{x} = \vec{x}_p + \vec{x}_h = \vec{x}_p + c_1 \vec{x}_1 + \cdots + c_k \vec{x}_k = \vec{x}_p + \sum_{i=1}^{k} c_i \vec{x}_i \]  

(5)

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where \( B = \{ \vec{x}_1, \ldots, \vec{x}_k \} \) is a basis of \( N(T) \). For the examples, we assume some previous knowledge of determinants and differential equations. Even without this knowledge, you should get a feel for the theory. And if you lack the knowledge, you may wish to reread this handout after obtaining it.

**EXAMPLE 1  OPERATORS DEFINED BY MATRIX MULTIPLICATION**

We now apply the general linear theory to operators defined by matrix multiplication. We look for the unknown column vector \( \vec{x} = [x_1, x_2, \ldots, x_n]^T \). (We use the transpose notation on a row vector to indicate a column vector to save space and trees.) We consider the operator \( T[\vec{x}] = A\vec{x} \) where \( A \) is an \( m \times n \) matrix.

**THEOREM 4.** If \( \vec{b} \) is in the column (range) space of the matrix (operator) \( A \), then the general solution to the nonhomogeneous system of algebraic equation(s)

\[
A_{mxn} \vec{x} = \vec{b}_{mx1}
\]

can be written in the form

\[
\vec{x} = \vec{x}_p + c_1 \vec{x}_1 + \cdots + c_k \vec{x}_k = \vec{x}_p + \sum_{i=1}^{k} c_i \vec{x}_i
\]

(7)

where \( \vec{x}_p \) is a particular (i.e. any) solution to (6) and

\[
\vec{x}_h = c_1 \vec{x}_1 + \cdots + c_k \vec{x}_k = \sum_{i=1}^{k} c_i \vec{x}_i
\]

(8)

is the general solution (i.e. a parametric formula for all solutions) to the complementary homogeneous equation

\[
A_{mxn} \vec{x} = \vec{0}_{mx1}
\]

(9)

Here \( B = \{ \vec{x}_1, \ldots, \vec{x}_k \} \) is a basis for the null space \( N(T) \) (also denoted by \( N(A) \)) which has dimension \( k \). All of the vectors \( \vec{x}_p, \vec{x}_1, \ldots, \vec{x}_k \) can be founded together using the computational technique of Gauss Elimination. If \( N(T) = \{ \vec{0} \} \), then the unique solution of \( A_{mxn} \vec{x} = \vec{b}_{mx1} \) is \( \vec{x}_p \)

(and the unique solution to \( A_{mxn} \vec{x} = \vec{0}_{mx1} \) is \( \vec{x}_h = \vec{0}_{nx1} \)).

**THEOREM 5.** If \( n = m \), then we consider two cases (instead of three) for equation (6):

1) \( \det A \neq 0 \) so that \( A \) is nonsingular; then the matrix \( A \) has a unique inverse, \( A^{-1} \) (which is almost never computed), and for any \( \vec{b} \in \mathbb{R}^n \), \( A_{nxn} \vec{x} = \vec{b}_{nx1} \) always has the unique solution

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Thus the operator $\mathbf{T}(\mathbf{x}) = A\mathbf{x}$ is one-to-one and onto so that any vector $\mathbf{b}$ is always in the range space $\text{R}(A)$ and only the vector $\mathbf{x} = A^{-1}\mathbf{b}$ maps to it. Again, the matrix $A$ defines an operator that is a one-to-one and onto mapping from $\mathbf{R}^n$ to $\mathbf{R}^n$ (or $\mathbf{C}^n$ to $\mathbf{C}^n$).

2) $\text{det } A = 0$ so that $A$ is singular; then either there is no solution or if there is a solution, then there are an infinite number of solutions. Whether there is no solution or an infinite numbers of solutions depends on $\mathbf{b}$, specifically, on whether $\mathbf{b} \in \text{R}(A)$ or not. The operator defined by the matrix $A$ is not one-to-one or onto and the dimension of $\text{N}(A)$ is greater than or equal to one.

**EXAMPLE 2 LINEAR DIFFERENTIAL EQUATIONS**

To avoid using $x$ as either the independent or dependent variable, we look for the unknown function $u$ (dependent variable) as a function of $t$ (independent variable). We let the domain of $u$ be $I = (a,b)$ and think of the function $u$ as a vector in an (infinite dimensional) vector (function) space.

**THEOREM 6.** If $g$ is in the range space $\text{R}(L)$ of the linear differential operator $L$ (i.e. $g \in \text{R}(L)$) then the **general solution to the nonhomogeneous equation**

$$L[ u(t) ] = g(t) \quad \forall t \in I$$

(10)

can be written in the form

$$u(t) = u_p(t) + u_h(t) \quad \text{(11)}$$

where $u_p$ is a **particular solution** to (10) and $u_h$ is the **general solution to the homogeneous equation**

$$L[ u(t) ] = 0 \quad \forall t \in I$$

(12)

**Special cases:**

1) $L[ u(t) ] = u'' + p(t)u' + q(t)u$. Second Order Scalar Equation.

   For this case, we let $I = (a,b)$ and $L: \mathbf{A}(I,\mathbf{R}) \to \mathbf{A}(I,\mathbf{R})$. It is known that the dimension of the null space is two so that

   $$u_h(t) = c_1 u_1(t) + c_2 u_2(t).$$

2) $L[ u(t) ] = p_o(t) \frac{d^n u}{dt^n} + \cdots + p_n(t) u(t)$. $n^{th}$ Order Scalar Equation.

   Again we let $I = (a,b)$ and $L: \mathbf{A}(I,\mathbf{R}) \to \mathbf{A}(I,\mathbf{R})$. For this case, the dimension of the null space is $n$ so that

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\[ u_h(t) = c_1 u_1(t) + \cdots + c_n u_n(t) = \sum_{i=1}^{n} c_i u_i(t). \]

3) \[ L[ \ddot{u}(t) ] = \frac{d\ddot{u}}{dt} - P(t)\dddot{u}(t) \quad \text{First Order System ("Vector" Equation)} \]

Again we let \( I = (a,b) \), but now \( L: A(I, \mathbb{R}^n) \to A(I, \mathbb{R}^n) \) where \( A(I, \mathbb{R}^n) = \{ \ddot{u}(t) : I \to \mathbb{R}^n \} \);

that is the set of all time varying "vectors". Here the word "vector" means an \( n \)-tuple of functions. We replace (10) with

\[ L[ \ddot{u}(t) ] = \ddot{g}(t) \]

and (12) with

\[ L[ \ddot{u}(t) ] = \ddot{0}. \]

Then

\[ \ddot{u}(t) = \ddot{u}_p(t) + \ddot{u}_h(t) \]

where

\[ \ddot{u}_h(t) = c_1 \ddot{u}_1(t) + \cdots + c_n \ddot{u}_n(t) \quad (\text{i.e. the null space is } n \text{ dimensional}). \]