Math 156 - Volumes

$f(x) \geq g(x)$ on $[a, b]$.

Both $y = f(x)$, and $y = g(x)$ are one-to-one on $[a, b]$.

Inverse functions (defined on $[c, d]$) can be represented as $x = f^{-1}(y)$ and $x = g^{-1}(y)$.

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<th>rotation about $x$-axis, or the line parallel to it</th>
<th>rotation about $y$-axis, or the line parallel to it</th>
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<td>cuts / washers</td>
<td>integration with respect to $x$ functions used: $f(x)$ and $g(x)$ limits of integration: $a$ and $b$</td>
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<td>cylindrical shells</td>
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The functions in the graph above are:
\[ y = f(x) = 2x, \text{ and } y = g(x) = (x - 1)^2 + 2 = x^2 - 2x + 2. \]
Their inverses are:
\[ x = f^{-1}(y) = \frac{y}{2}, \text{ and } x = g^{-1}(y) = \sqrt{y - 2} + 1. \]

The **volume of a solid obtained by rotating the area described above**

1. Rotation about \(x\)-axis
   
   (a) cuts/washers
   \[ \int_1^3 \pi [(2x)^2 - (x^2 - 2x + 3)^2] \, dx \]
   (b) cylindrical shells
   \[ \int_2^6 2\pi y [\sqrt{y - 2} + 1 - \frac{y}{2}] \, dy \]

2. Rotation about \(y = -1\)
   
   (a) cuts/washers
   \[ \int_1^3 \pi [(2x + 1)^2 - (x^2 - 2x + 3 + 1)^2] \, dx \]
   (b) cylindrical shells
   \[ \int_2^6 2\pi (y + 1) [\sqrt{y - 2} + 1 - \frac{y}{2}] \, dy \]
3. Rotation about $y = 7$

(a) cuts/washers
\[ \int_{1}^{3} \pi [(7 - (x^2 - 2x + 3))^2 - (7 - 2x)^2] \, dx \]

(b) cylindrical shells
\[ \int_{2}^{6} 2\pi (7 - y)[\sqrt{y - 2} + 1 - \frac{y}{2}] \, dy \]

4. Rotation about $y$-axis

(a) cuts/washers
\[ \int_{2}^{6} \pi \left[ (\sqrt{y - 2} + 1)^2 - \left( \frac{y}{2} \right)^2 \right] \, dy \]

(b) cylindrical shells
\[ \int_{1}^{3} 2\pi x[(2x) - (x^2 - 2x + 3)] \, dx \]

5. Rotation about $x = -1$

(a) cuts/washers
\[ \int_{2}^{6} \pi \left[ (\sqrt{y - 2} + 1 + 1)^2 - \left( \frac{y}{2} + 1 \right)^2 \right] \, dy \]

(b) cylindrical shells
\[ \int_{1}^{3} 2\pi (x + 1)[(2x) - (x^2 - 2x + 3)] \, dx \]

6. Rotation about $x = 5$

(a) cuts/washers
\[ \int_{2}^{6} \pi \left[ (5 - \frac{y}{2})^2 - (5 - (\sqrt{y - 2} + 1))^2 \right] \, dy \]

(b) cylindrical shells
\[ \int_{1}^{3} 2\pi (5 - x)[(2x) - (x^2 - 2x + 3)] \, dx \]