Infinite Limits and Limits at Infinity

(1) **Infinity Limits** In the limiting process of a fraction, if the denominator approaches 0, while the numerator remains or approaches a positive (negative, respectively) constant, the limiting quantity is positive and can be arbitrarily large (negative with arbitrarily large absolute value, respectively), and so the limit does not exist. For convenience, we also write \( \lim_{x \to a} f(x) = \infty \) (or \(-\infty, \) respectively) to indicate the fact that the limiting quantity can be positively arbitrarily large, (negative with arbitrarily large absolute value, respectively). (Note that \( \infty \) is not a number).

(2) **Limits at Infinity** In the limiting process, if \( x \) increases unboundedly, then we write \( x \to \infty \); if \( x \) decreases unboundedly, then we write \( x \to -\infty \).

Facts about limits at infinity:

(i) If \( n \) is a positive real number, then \( \lim_{x \to \infty} x^n = 0 \).

(ii) If \( n \) is a positive real number, then \( \lim_{x \to -\infty} x^n = \infty \).

**Example 1** Compute the limits \( \lim_{x \to 1^-} \frac{1 - 2x}{x^2 - 1}, \lim_{x \to 1^+} \frac{1 - 2x}{x^2 - 1} \) and \( \lim_{x \to -\infty} \frac{1 - 2x}{x^2 - 1} \).

**Solution:** When \( x \to 1^- \), \( 1 - 2x \to 1 - 2 = -1 \not= 0 \). When \( x < 1 \) and \( x \to 1^- \), we have \( 0 < x^2 < 1 \) and so \( x^2 - 1 < 0 \). Therefore, as \( x \to 1^- \), both \( 1 - 2x < 0 \) and \( x^2 - 1 < 0 \), and so \( \lim_{x \to 1^-} \frac{1 - 2x}{x^2 - 1} = \infty \).

When \( x \to 1^+ \), \( 1 - 2x \to 1 - 2 = -1 \not= 0 \). When \( x > 1 \) and \( x \to 1^+ \), we have \( x^2 > 1 \) and so \( x^2 - 1 > 0 \).

Therefore, as \( x \to 1^+ \), \( 1 - 2x < 0 \) but \( x^2 - 1 > 0 \) and so \( \lim_{x \to 1^+} \frac{1 - 2x}{x^2 - 1} = -\infty \).

Combine these two limits to conclude that \( \lim_{x \to 1^+} \frac{1 - 2x}{x^2 - 1} \) does not exist.

**Example 2** Compute the limits \( \lim_{x \to -2^+} \frac{-x}{\sqrt{4 - x^2}} \) and \( \lim_{x \to -2^-} \frac{-x}{\sqrt{4 - x^2}} \).

**Solution:** When \( x \to -2^+ \), \( -x \to -2 + \not= 0 \). When \( x > -2 \) and \( x \to -2^+ \), we have \( \sqrt{4 - x^2} \to 0 \), and so \( \lim_{x \to -2^+} \frac{-x}{\sqrt{4 - x^2}} = \infty \).

When \( x \to -2^- \), \( -x \to -2 \not= 0 \). When \( x < -2 \) and \( x \to -2^- \), we have \( \sqrt{4 - x^2} = -\infty \), and so \( \lim_{x \to -2^-} \frac{-x}{\sqrt{4 - x^2}} = -\infty \).

**Example 3** Compute the limits \( \lim_{x \to \infty} \frac{\sin x}{x} \) and \( \lim_{x \to \infty} \frac{\cos x}{x} \).

**Solution:** Note that \( |\sin x| \leq 1 \). Therefore, \( \left| \frac{\sin x}{x} \right| \leq \frac{1}{|x|} \). As \( \lim_{x \to \infty} \frac{1}{|x|} = 0 \), we also have \( \lim_{x \to \infty} \sin x = 0 \).

One can also use Squeeze Law to compute this limit. From \(-1 \leq \sin x \leq 1 \), we have, for \( x > 0 \), \(-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x} \). As \( \lim_{x \to \infty} \frac{1}{x} = 0 \), by Squeeze Law, we have \( \lim_{x \to \infty} \sin x = 0 \). Similarly, \( \lim_{x \to \infty} \cos x = 0 \).

**Example 3** Compute the limits \( \lim_{x \to \infty} (\sqrt{x^2 + 3x} - x) \).

**Solution:** Use the identity \((A - B)(A + B) = A^2 - B^2\) to remove the square root before computing the limit:

\[
\lim_{x \to \infty} (\sqrt{x^2 + 3x} - x) = \lim_{x \to \infty} \frac{(\sqrt{x^2 + 3x} - x)(\sqrt{x^2 + 3x} + x)}{\sqrt{x^2 + 3x} + x} = \lim_{x \to \infty} \frac{(x^2 + 3x) - x^2}{\sqrt{x^2 + 3x} + x} = \lim_{x \to \infty} \frac{3x}{\sqrt{x^2 + 3x} + x} \lim_{x \to \infty} \frac{3}{\sqrt{1 + \left(\frac{3}{x}\right)^2 + 1}} = \frac{3}{\sqrt{1 + 0}} = \frac{3}{2}.
\]
Asymptotes

(1) A line \( y = L \) is a horizontal asymptote of the graph of \( y = f(x) \) if

\[
\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L.
\]

(2) A line \( x = x_0 \) is a vertical asymptote of the graph of \( y = f(x) \) if \( x_0 \) is not in the domain of \( f(x) \) and if

\[
\lim_{x \to x_0^+} f(x) = \infty \quad \text{or} \quad \lim_{x \to x_0^-} f(x) = -\infty.
\]

(3) A non vertical line \( y = ax + b \) is a slant asymptote of the graph of \( y = f(x) \) if

\[
\lim_{x \to \pm\infty} \left( f(x) - (ax + b) \right) = 0.
\]

Example 4 Determine all horizontal, slant, and vertical asymptotes of \( y = \frac{1}{x^2 + 1} \).

Solution: First consider the domain of \( f(x) \). Since \( 4 + x^2 > 0 \) for all reals, the domain of \( f(x) \) is \(( -\infty, \infty ) \). Therefore, \( y = f(x) \) has no vertical asymptotes.

To determine if \( y = f(x) \) has a horizontal asymptote, compute

\[
\lim_{x \to \pm\infty} \frac{x}{\sqrt{4 + x^2}} = \lim_{x \to \pm\infty} \frac{1}{\sqrt{\frac{4}{x^2} + 1}} = \frac{1}{\lim_{x \to \pm\infty} \sqrt{\frac{4}{x^2} + 1}} = \frac{1}{1} = 1.
\]

Therefore, \( y = 1 \) is a horizontal asymptote and \( y = f(x) \) does not have a slant asymptote.

Example 5 Determine all horizontal, slant, and vertical asymptotes of \( f(x) = \frac{x^2}{4 - x^2} \).

Solution: First consider the domain of \( f(x) \). Since \( 4 - x^2 = 0 \) if and only if \( x = 2 \) and \( x = -2 \), the domain of \( f(x) \) is \(( -\infty, -2) \cup (-2, 2) \cup (2, \infty) \). Therefore, \( y = f(x) \) has possible vertical asymptotes at \( x = -2 \) and \( x = 2 \).

As \( \lim_{x \to -2^-} \frac{x^2}{4 - x^2} = -\infty \) and \( \lim_{x \to -2^+} \frac{x^2}{4 - x^2} = \infty \), \( x = -2 \) is a vertical asymptote. As \( \lim_{x \to 2^-} \frac{x^2}{4 - x^2} = \infty \) and \( \lim_{x \to 2^+} \frac{x^2}{4 - x^2} = -\infty \), \( x = 2 \) is also a vertical asymptote.

To determine if \( y = f(x) \) has a horizontal asymptote, compute

\[
\lim_{x \to \pm\infty} \frac{x^2}{4 - x^2} = \lim_{x \to \pm\infty} \frac{1}{\frac{4}{x^2} - 1} = \frac{1}{\lim_{x \to \pm\infty} \frac{4}{x^2} - 1} = \frac{1}{-1} = -1.
\]

Therefore, \( y = -1 \) is a horizontal asymptote and \( y = f(x) \) does not have a slant asymptote.

Example 6 Determine all horizontal, slant, and vertical asymptotes of \( f(x) = \frac{x^3}{4 - x^2} \).

Solution: First consider the domain of \( f(x) \). Since \( 4 - x^2 = 0 \) if and only if \( x = 2 \) and \( x = -2 \), the domain of \( f(x) \) is \(( -\infty, -2) \cup (-2, 2) \cup (2, \infty) \). Therefore, \( y = f(x) \) has possible vertical asymptotes at \( x = -2 \) and \( x = 2 \).

As in the previous example, both \( x = -2 \) and \( x = 2 \) are vertical asymptotes.

To determine if \( y = f(x) \) has a horizontal asymptote, compute

\[
\lim_{x \to \pm\infty} \frac{x^3}{4 - x^2} = \lim_{x \to \pm\infty} \frac{x}{\frac{4}{x^2} - 1} = -\infty.
\]

Therefore, \( y = f(x) \) does not have a horizontal asymptote.

To determine if \( y = f(x) \) has a slant asymptote, apply division to get

\[
\frac{x^3}{4 - x^2} = -x + \frac{4}{4 - x^2}, \quad \text{and} \quad \lim_{x \to \pm\infty} \frac{4}{4 - x^2} = 0.
\]

Therefore, \( y = x \) is a slant asymptote of \( y = f(x) \).