Calculate the limit of a function: Slightly More Complicated Situations

(1) Cancelling zero factors before evaluating for the $\frac{0}{0}$ type limits When facing a $\frac{0}{0}$ type limit, one must try to get rid of the zero factor(s) before evaluating the limit. In any case, $\frac{0}{0}$ is always a wrong answer. See Examples 1-3.

(2) Use of the basic limit $\lim_{x \to 0} \frac{\sin x}{x} = 1 = \lim_{x \to 0} \frac{x}{\sin x}$ in $\frac{0}{0}$ type limits involving trigonometric functions. Apply trigonometric identities and substitution to convert the current situation into one that can apply the basic limit. See Examples 4-6.

(3) Use of Squeeze Law See Examples 7-8.

Example 1 Find $\lim_{x \to 1} \frac{x - 1}{x^2 - 1}$.

Solution: When $x \to 1$, both $x - 1$ and $x^2 - 1$ go to 0, and so this is a $\frac{0}{0}$ type limit. Factor $x^2 - 1 = (x - 1)(x + 1)$. Then cancel the zero factors before evaluating the limit:

$$\lim_{x \to 1} \frac{x - 1}{x^2 - 1} = \lim_{x \to 1} \frac{x - 1}{(x - 1)(x + 1)} = \lim_{x \to 1} \frac{1}{x + 1} = \frac{1}{2}.$$  

Example 2 Find $\lim_{x \to 0} \frac{\sqrt{x^2 + 9} - 3}{x^2}$.

Solution: When $x \to 0$, both $\sqrt{x^2 + 9} - 3$ and $x^2$ go to 0, and so this is a $\frac{0}{0}$ type limit.

In order to cancel the zero factors, we need to "free" $x^2 + 9$ out of the radical. Apply the algebraic identity $(A + B)(A - B) = A^2 - B^2$ with $A = \sqrt{x^2 + 9}$ and $B = 3$ to get

$$\sqrt{x^2 + 9} - 3 = \frac{(\sqrt{x^2 + 9} - 3)(\sqrt{x^2 + 9} + 3)}{\sqrt{x^2 + 9} + 3} = \frac{x^2 + 9 - 9}{\sqrt{x^2 + 9} + 3} = \frac{x^2}{\sqrt{x^2 + 9} + 3}.$$  

Therefore,

$$\lim_{x \to 0} \frac{\sqrt{x^2 + 9} - 3}{x^2} = \lim_{x \to 0} \frac{x^2}{x^2(\sqrt{x^2 + 9} + 3)} = \lim_{x \to 0} \frac{1}{\sqrt{x^2 + 9} + 3} = \frac{1}{6}.$$  

Example 3 Find $\lim_{x \to 1} \left[ \frac{1}{x - 1} - \frac{2}{x^2 - 1} \right]$.

Solution: When $x \to 1$, both $x - 1$ and $x^2 - 1$ go to 0, and so this is an $\infty - \infty$ type limit. One can use algebra to convert it into a $\frac{0}{0}$ type limit, by combining the two fractions, as follows.

$$\frac{1}{x - 1} - \frac{2}{x^2 - 1} = \frac{(x + 1) - 2}{x^2 - 1} = \frac{x - 1}{x^2 - 1}.$$  

1
This becomes the problem of Example 1.

**Example 4** Find \( \lim_{x \to 0} \frac{1 - \cos x}{x^2} \).

**Solution:** Using the algebraic identity \((a - b)(a + b) = a^2 - b^2\), we have \((1 - \cos x)(1 + \cos x) = 1 - \cos^2 x\); and applying a trigonometric identity \(\sin^2 x + \cos^2 x = 1\), we have \((1 - \cos x)(1 + \cos x) = 1 - \cos^2 x = \sin^2 x\). Thus

\[
\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{1 - \cos x}{x^2} \frac{1 + \cos x}{1 + \cos x} = \lim_{x \to 0} \frac{(1 - \cos x)(1 + \cos x)}{x^2} \cdot \frac{1}{1 + \cos x} = \lim_{x \to 0} \frac{\sin^2 x}{x^2} \lim_{x \to 0} \frac{1}{1 + \cos x} = \lim_{x \to 0} \frac{1}{1 + \cos 0} = \frac{1}{2}.
\]

**Example 5** Find \( \lim_{x \to 0} \frac{x}{\tan(3x)} \).

**Solution:** Use substitution \( u = 3x \). Then \( x = \frac{u}{3} \), and as \( x \to 0 \), we also have \( u \to 0 \). Note that \( \tan u = \frac{\sin u}{\cos u} \). Thus apply the Basic Limit and \(\cos 0 = 1\) to get

\[
\lim_{x \to 0} \frac{x}{\tan(3x)} = \lim_{u \to 0} \frac{1}{3} \frac{\sin u}{\sin u} = \lim_{u \to 0} \frac{1}{3} \frac{\cos u}{\sin u} = \frac{1}{3} \cdot 1 = \frac{1}{3}.
\]

**Example 6** Find \( \lim_{x \to \frac{\pi}{2}} (x - \frac{\pi}{2}) \tan(x) \).

**Solution:** Rewrite \( \tan x = \frac{\sin x}{\cos x} \). Then \(\cos \frac{\pi}{2} = 0\), and so in fraction form, this limit is a \(\frac{0}{0}\)-type.

To apply the Basic Limit, we use substitution \( u = x - \frac{\pi}{2} \). As \( x \to \frac{\pi}{2} \), we have \( u \to 0 \). With this substitution, we have

\[
\tan(x) = \tan\left(x - \frac{\pi}{2} + \frac{\pi}{2}\right) = \tan\left(u + \frac{\pi}{2}\right) = \frac{\sin(u + \frac{\pi}{2})}{\cos(u + \frac{\pi}{2})}.
\]

Apply trigonometric identities

\[
\sin\left(u + \frac{\pi}{2}\right) = \cos u, \text{ and } \cos\left(u + \frac{\pi}{2}\right) = -\sin u
\]

\(}\]

2
to get the answer:
\[
\lim_{x \to \frac{\pi}{2}} (x - \frac{\pi}{2}) \tan(x) = \lim_{u \to 0} u \frac{\sin(u + \frac{\pi}{2})}{\cos(u + \frac{\pi}{2})} \quad \text{apply the trig. identities above}
\]
\[
= \lim_{u \to 0} u \frac{-\sin u}{\cos u} \quad \text{rewrite the expression to get}\ \frac{u}{\sin u}
\]
\[
= \lim_{u \to 0} -\cos u \quad \text{apply the Basic Limit and } \cos 0 = 1
\]
\[
= (-1) \cdot 1 = -1.
\]

The **Squeeze Theorem** states that if

\[ f(x) \leq g(x) \leq h(x) \]

hold for all \( x \) in some interval \((c, d)\), except possibly at the point \( a \in (c, d)\), and if for some finite number \( L \),

\[ \lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L, \]

then

\[ \lim_{x \to a} g(x) = L. \]

**Example 7** Find \( \lim_{x \to 0} x^2 \sin \left( \frac{1}{x} \right) \).

**Solution** To apply Squeeze Theorem, here \( g(x) = x^2 \sin \left( \frac{1}{x} \right) \). Note that \(-1 \leq \sin(u(x)) \leq 1\) for any function \( u(x) \) (here \( u(x) = \frac{1}{x} \)). From

\[ -1 \leq \sin \left( \frac{1}{x} \right) \leq 1, \]

we can get, by multiplying a non negative quantity \( x^2 \) everywhere in the inequalities above,

\[ -x^2 \leq x^2 \sin \left( \frac{1}{x} \right) \leq x^2. \]

Hence we can choose \( f(x) = -x^2 \) and \( h(x) = x^2 \). Then for the interval \((-1, 1)\) excluding \( x = 0 \), we have \( f(x) \leq g(x) \leq h(x) \). Moreover, for \( L = 0 \),

\[ \lim_{x \to a} f(x) = \lim_{x \to a} -x^2 = 0 \quad \text{and} \quad \lim_{x \to a} h(x) = \lim_{x \to a} x^2 = 0. \]

It follows by the Squeeze Theorem that

\[ \lim_{x \to 0} x^2 \sin \left( \frac{1}{x} \right) = 0. \]
Example 8 Find \( \lim_{x \to 0^+} \sqrt{x} \cos^2 \left( \frac{1}{x} \right) \).

**Solution:** To apply Squeeze Theorem, here \( g(x) = \sqrt{x} \cos^2 \left( \frac{1}{x} \right) \). Note that \(-1 \leq \cos(u(x)) \leq 1\) for any function \( u(x) \) (here \( u(x) = \frac{1}{x} \)). From 

\[
0 \leq \cos^2 \left( \frac{1}{x} \right) \leq 1,
\]

we can get, by multiplying a non negative quantity \( \sqrt{x} \) everywhere in the inequalities above,

\[
0 \leq \sqrt{x} \cos^2 \left( \frac{1}{x} \right) \leq \sqrt{x}.
\]

Hence we can choose \( f(x) = 0 \) and \( h(x) = \sqrt{x} \). Then for the interval \((0,1)\), we have \( f(x) \leq g(x) \leq h(x) \). Moreover, for \( L = 0 \),

\[
\lim_{x \to a} f(x) = \lim_{x \to a} 0 = 0 \quad \text{and} \quad \lim_{x \to a} h(x) = \lim_{x \to a} \sqrt{x} = 0.
\]

It follows by the Squeeze Theorem that

\[
\lim_{x \to 0^+} \sqrt{x} \cos^2 \left( \frac{1}{x} \right) = 0.
\]

Example 9 Suppose that \( F(x) \) is bounded: that is, there exist constants \( m \) and \( n \) such that \( m \leq F(x) \leq n \) for all \( x \). Use the Squeeze Theorem to prove that \( \lim_{x \to 0} x^2 F(x) = 0 \).

**Solution:** To apply Squeeze Theorem, here \( g(x) = x^2 F(x) \). From

\[
m \leq F(x) \leq n,
\]

we can get, by multiplying a non negative quantity \( x^2 \) everywhere in the inequalities above,

\[
-mx^2 \leq x^2 F(x) \leq nx^2.
\]

Hence we can choose \( f(x) = -mx^2 \) and \( h(x) = nx^2 \). Then for all \( x \), we have \( f(x) \leq g(x) \leq h(x) \). Moreover, for \( L = 0 \),

\[
\lim_{x \to a} f(x) = \lim_{x \to a} -mx^2 = 0 \quad \text{and} \quad \lim_{x \to a} h(x) = \lim_{x \to a} nx^2 = 0.
\]

It follows by the Squeeze Theorem that

\[
\lim_{x \to 0} x^2 F(x) = 0.
\]