Important: Your should write your solutions in the space given. If you have to write your solution outside the space provided, you must indicated clearly where your solutions are. Failing to do so may result in losing credit points as grader will ignore solutions that are not written in the given space.

PART 1. This portion of the exam is to test basic knowledge and calculation skills to a correct solution. The questions are multiple choice with “None of these” as a possible valid choice. No partial credit is given in this section, so work very carefully. Value: 4 points each

1. The limit \( \lim_{x \to 0} \frac{1 - e^x}{\sin(x)} \) equals:

(A) \(-1\)  (B) 1  (C) 0  (D) \(\frac{1}{2}\)  (E) \(-\frac{1}{2}\)  (F) None of these.

Solution: As the limit is a \(\frac{0}{0}\)-Type, we apply L'Hôpital's Rule twice to get

\[
\lim_{x \to 0} \frac{1 - e^x}{\sin(x)} = \lim_{x \to 0} \frac{-e^x}{\cos(x)} = -\frac{1}{1} = -1.
\]

2. The limit \( \lim_{x \to 0} \frac{x^2}{x - \ln(1 + x)} \) equals:

(A) \(-1\)  (B) 1  (C) 0  (D) 2  (E) \(-2\)  (F) None of these.

Solution 1: As the limit is a \(\frac{0}{0}\)-Type, we apply L'Hôpital's Rule twice (check the limit is a \(\frac{0}{0}\)-Type before applying the rule) to get

\[
\lim_{x \to 0} \frac{x^2}{x - \ln(1 + x)} = \lim_{x \to 0} \frac{2x}{1 - \frac{1}{1+x}} = \lim_{x \to 0} \frac{2}{\frac{1}{1+x}} = \lim_{x \to 0} \frac{2}{1} = 2.
\]

Solution 2: As the limit is a \(\frac{0}{0}\)-Type, we apply L'Hôpital's Rule and algebra to get

\[
\lim_{x \to 0} \frac{x^2}{x - \ln(1 + x)} = \lim_{x \to 0} \frac{2x}{1 - \frac{1}{1+x}} = \lim_{x \to 0} \frac{2x}{\frac{1+x-1}{1+x}} = \lim_{x \to 0} \frac{2x(1 + x)}{x} = \lim_{x \to 0} \frac{2(1 + x)}{1} = 2.
\]

3. For the function \( f(x) = x^3 - 3x \) on \([0, 2]\), which of the following statement is a correct statement?
(A) $f(x)$ has absolute maximum at $x = 0$ and absolute minimum at $x = 2$.
(B) $f(x)$ has absolute maximum at $x = 0$ and absolute minimum at $x = 1$.
(C) $f(x)$ has absolute maximum at $x = 2$ and absolute minimum at $x = 0$.
(D) $f(x)$ has absolute maximum at $x = 1$ and absolute minimum at $x = 2$.
(E) $f(x)$ has absolute maximum at $x = 2$ and absolute minimum at $x = 1$.
(F) $f(x)$ has absolute maximum at $x = 1$ and absolute minimum at $x = 0$.
(G) None of these.

Solution: $f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$. There is only one critical number $x = 1$ inside the interval $[0, 2]$. Compute

$$f(0) = 0, f(1) = 1 - 3 = -2, \text{ and } f(2) = 8 - 6 = 2,$$

$f(x)$ has absolute maximum at $x = 2$ and absolute minimum at $x = 1$.

4. For the function $f(x) = x\sqrt{x^2 + 1}$, which of the following statement is a correct statement?

(A) $f(x)$ is decreasing on $(-\infty, \infty)$.
(B) $f(x)$ is increasing on $(-\infty, \infty)$.
(C) $f(x)$ is decreasing on $(-\infty, 1)$ and increasing on $(1, \infty)$.
(D) $f(x)$ is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$.
(E) $f(x)$ is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.
(F) $f(x)$ is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.
(G) None of these.

Solution: The domain of $f(x)$ is $(-\infty, \infty)$. Compute $f'(x) = \sqrt{x^2 + 1} + x \cdot \frac{2x}{2\sqrt{x^2 + 1}} = \frac{2x^2 + 1}{\sqrt{x^2 + 1}} > 0$. Thus $f(x)$ is increasing on $(-\infty, \infty)$.

5. The limit $\lim_{x \to 0} \frac{x^2}{\sin(x)}$ equals:

(A) 2 (B) 1 (C) 0 (D) $\frac{1}{2}$ (E) $-\frac{1}{2}$ (F) None of these.

Solution: As the limit is a $\frac{0}{0}$-Type, we apply L'Hôpital's Rule to get

$$\lim_{x \to 0} \frac{x^2}{\sin(x)} = \lim_{x \to 0} \frac{2x}{\cos(x)} = \frac{2 \cdot 0}{\cos(0)} = \frac{0}{1} = 0.$$  

6. For a function $f(x) = x^3$ on the interval $[0, 1]$, a point $c$ inside the interval $(0, 1)$ that satisfies the conclusion of the Mean Value Theorem is
(A) \(0\) (B) \(1\) (C) \(\sqrt{3}\) (D) \(-\sqrt{3}\) (E) \(\frac{1}{\sqrt{3}}\) (F) \(\frac{1}{3}\) (G) \(\frac{1}{\sqrt{2}}\) (H) None of these.

**Solution:** Compute \(f'(x) = 3x^2\), \(f(1) = 1\) and \(f(0) = 0\). We want a number \(c\) such that
\[
3c^2 = f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{1 - 0}{1 - 0} = 1,
\]
and so \(c = \pm \frac{1}{\sqrt{3}}\). Inside the interval \([0, 1]\), we must have \(c = \frac{1}{\sqrt{3}}\).

7. The linear approximation of \(\sqrt{15}\) is equal to

(A) \(4\) (B) \(\frac{33}{16}\) (C) \(\frac{61}{32}\) (D) \(\frac{1}{32}\) (E) \(\frac{63}{32}\) (F) None of These.

**Solution:** Choose \(f(x) = \sqrt{x}\) and \(x_0 = 16\) (as \(\sqrt{16} = 2\)). Compute \(f'(x) = \frac{1}{4\sqrt{x}}\).
The linear approximation of \(f(x)\) near \(x_0\) is
\[
L(x) = f(16) + f'(16)(x - 16).
\]
Thus the linear approximation of \(\sqrt{15}\) is
\[
L(15) = \sqrt{16} + \frac{1}{4\sqrt{16^3}}(15 - 16) = 2 - \frac{1}{4}\left(\sqrt{16}\right)^3 = 2 - \frac{1}{32} = \frac{63}{32}.
\]

8. For a function \(f(x) = x^3 - 12x - 100\), which of the following is a correct statement?

(A) \(f(x)\) has a local maximum at \(x = 2\) and does not have a local minimum.
(B) \(f(x)\) has a local minimum at \(x = 2\) and does not have a local maximum.
(C) \(f(x)\) has a local maximum at \(x = 2\) and a local minimum at \(x = -2\).
(D) \(f(x)\) has a local maximum at \(x = -2\) and a local minimum at \(x = 2\).
(E) \(f(x)\) has a local maximum at \(x = 2\) and a local minimum at \(x = 0\).
(F) \(f(x)\) has a local maximum at \(x = 0\) and a local minimum at \(x = 2\).
(G) None of these.

**Solution:** The domain of \(f(x)\) is \((-\infty, \infty)\). Compute \(f'(x) = 3x^2 - 12 = 3(x-2)(x+2)\) and so \(x = \pm 2\) are the critical numbers of \(f(x)\). As \(f'(x) > 0\) in \((-\infty, -2)\) and \((2, \infty)\), and as \(f'(x) < 0\) in \((-2, 2)\), \(f(x)\) has a local maximum at \(x = -2\) and a local minimum at \(x = 2\).

9. For a function \(f(x) = x^3 - 12x^2\), which of the following is a correct statement?
11. (10 points) For a function \( f(x) \) without supporting work shown on the paper will receive NO credit.

**PART 2:** This portion of the exam will be graded on a partial credit basis. **Answers without supporting work shown on the paper will receive NO credit.**

10. For a function \( f(x) = 3x^4 + 4x^3 - 12x^2 \), which of the following is a correct statement?

(A) \( f(x) \) has two local maxima and one local minimum.
(B) \( f(x) \) has one local maximum and two local minima.
(C) \( f(x) \) has two local maxima and no local minimum.
(D) \( f(x) \) has no local maximum and one local minimum.
(E) \( f(x) \) has one local maximum and no local minimum.
(F) \( f(x) \) has two local maxima and two local minima.
(G) None of these.

**Solution:** The domain of \( f(x) \) is \((-\infty, \infty)\). Compute \( f'(x) = 12x^3 + 12x^2 - 24x = 12x(x^2 + x - 2) = 12x(x - 1)(x + 2) \). Critical numbers are \( x = 0, -2, 1 \). As \( f'(x) < 0 \) in \((-\infty, -2) \) and \((0, 1)\), \( f'(x) > 0 \) in \((-2, 0) \) and \((1, \infty)\), \( f(x) \) has one local maximum (at \( x = 0 \)) and two local minima (at \( x = -2 \) and \( x = 1 \)).

11. (10 points) Find the absolute extrema of \( f(x) = \frac{4}{3}x^3 - x \) on the interval \([0, 2]\).

**Solution:** Compute \( f'(x) = 4x^2 - 1 = (2x - 1)(2x + 1) \). The critical point inside \([0, 2]\) is \( x = \frac{1}{2} \). Compute
\[
  f(0) = 0, \quad f\left(\frac{1}{2}\right) = \frac{4}{3} \cdot \frac{1}{2^2} - \frac{1}{2} = \frac{1}{6} - \frac{1}{2} = -\frac{1}{3}, \quad \text{and} \quad f(2) = \frac{4}{3} \cdot 2^3 - 2 = \frac{26}{3}.
\]
Therefore, the absolute maximum is \( f(2) = \frac{26}{3} \) and the absolute minimum is \( f\left(\frac{1}{2}\right) = -\frac{1}{3} \).

12. (14 points) Given a function \( f(x) = xe^{-\frac{x}{4}} \), do each of the following.

(a) Determine the intervals where \( f(x) \) is increasing and the intervals where \( f(x) \) is decreasing.

**Solution:** The domain of \( f(x) \) is \((-\infty, \infty)\). Compute \( f'(x) = e^{-\frac{x}{4}} - \frac{x}{4}e^{-\frac{x}{4}} = e^{-\frac{x}{4}}\left(1 - \frac{x}{4}\right) \). Therefore, the critical number is \( x = 4 \).
This critical number $x = 4$ partitions the domain $(-\infty, \infty)$ into two intervals $(-\infty, 4)$ and $(4, \infty)$. Determine the sign of $f''(x)$ in these intervals: $f''(x) < 0$ (and so $f(x)$ is decreasing) in $(4, \infty)$, and $f''(x) > 0$ (and so $f(x)$ is increasing) in $(-\infty, 4)$.

(b) Determine the intervals where $f(x)$ is concave up and the intervals where $f(x)$ is concave down.

**Solution:** Compute $f''(x) = -\frac{1}{4}e^{-\frac{x}{2}} \left( 1 - \frac{x}{4} \right) - \frac{1}{4}e^{-\frac{x}{4}} = -\frac{1}{4}e^{-\frac{x}{4}} \left( 2 - \frac{x}{4} \right)$. Therefore, the only candidate of an inflection point has $x$-coordinate $x = 8$.

This point $x = 8$ partitions the domain $(-\infty, \infty)$ into two intervals $(-\infty, 8)$ and $(8, \infty)$. Determine the sign of $f''(x)$ in these intervals: $f''(x) > 0$ (and so $f(x)$ is concave up) in $(8, \infty)$, and $f''(x) < 0$ (and so $f(x)$ is concave down) in $(-\infty, 8)$.

(The inflection point has $x$-coordinate $x = 8$.)

13. (7 points) Compute the limit $\lim_{x \to 1} \frac{e^{x-1} - x}{x^3 - 2x^2 + x}$.

**Solution:** As $e^{1-1} - 1 = 0$ and $1^3 - 2 \cdot 1^2 + 1 = 0$, this is a $0^0$-type of limit. We apply L'Hôpital's Rule to get

$$
\lim_{x \to 1} \frac{e^{x-1} - x}{x^3 - 2x^2 + x} = \lim_{x \to 1} \frac{e^{x-1} - 1}{3x^2 - 4x + 1}.
$$

The resulting limit is again a $0^0$-type of limit. We continue applying L'Hôpital's Rule to get

$$
\lim_{x \to 1} \frac{e^{x-1} - 1}{3x^2 - 4x + 1} = \lim_{x \to 1} \frac{e^{x-1}}{6x - 4} = \frac{e^{1-1}}{6 - 4} = \frac{1}{2}.
$$

14. (7 points) Sketch the graph $y = f(x)$ with the following properties: $f(x)$ is defined on $(-\infty, \infty)$; $f(-3) = 2, f(-2) = 4, f(0) = 0, f(3) = -3$ and $f(4) = -2$; $f'(x) > 0$ for $-\infty < x < -2$ and $x > 3$, $f'(x) < 0$ for $-2 < x < 3$; $f''(x) > 0$ for $x < -3$ and $0 < x < 4, f''(x) < 0$ for $-3 < x < 0$ and $x > 4$.

(Graph is omitted)

15. (22 points) Given a function $f(x) = \frac{2x^2}{x^2 - 1}$, and

$$
f'(x) = \frac{-4x}{(x^2 - 1)^2}, \quad \text{and} \quad f''(x) = \frac{12x^2 + 4}{(x^2 - 1)^3},
$$

sketch the graph $y = f(x)$ by completing each of the following.

(a) Determine the domain of $f(x)$.

**Solution:** As $x^2 = 1$ has solution $x = \pm 1$, the domain is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$. 

5
(b) Determine the vertical asymptote(s) of \( y = f(x) \).

**Solution:** As \( x = \pm 1 \) are discontinuities of \( f(x) \), we compute these limits

\[
\lim_{x \to -1^-} \frac{2x^2}{x^2 - 1} = +\infty, \quad \lim_{x \to -1^+} \frac{2x^2}{x^2 - 1} = -\infty, \quad \lim_{x \to 1^-} \frac{2x^2}{x^2 - 1} = -\infty, \quad \lim_{x \to 1^+} \frac{2x^2}{x^2 - 1} = +\infty.
\]

Therefore, \( x = 1 \) and \( x = -1 \) are vertical asymptotes of \( y = f(x) \).

(c) Determine the horizontal asymptote(s) of \( y = f(x) \).

**Solution:** Compute the limit

\[
\lim_{x \to \pm\infty} \frac{2x^2}{x^2 - 1} = \lim_{x \to \pm\infty} \frac{2}{1 - \frac{1}{x^2}} = \frac{2}{1} - 0 = 2,
\]

and so \( y = 2 \) is a horizontal asymptote of \( y = f(x) \).

(d) Determine the interval(s) where \( f(x) \) is increasing, and the interval(s) where \( f(x) \) is decreasing.

**Solution:** From \( f'(x) = \frac{-4x}{(x^2 - 1)^2} \), we determine the critical number is \( x = 0 \). This number \( x = 0 \) partitions the domain of \( f(x) \) into four intervals: \(( -\infty, -1) \), \((-1, 0) \), \((0, 1) \) and \((1, \infty) \). Determine the sign of \( f'(x) \) in these intervals: \( f'(x) > 0 \) (and so \( f(x) \) is increasing) in \((-\infty, -1) \) and \((-1, 0) \), and \( f'(x) < 0 \) (and so \( f(x) \) is decreasing) in \((0, 1) \) and \((1, \infty) \).

(e) For each of the critical numbers, determine whether it represents a local maximum, a local minimum, or neither.

**Solution:** As \( f'(x) > 0 \) in \((-1, 0) \) and \( f'(x) < 0 \) in \((0, 1) \), \( f(0) = 0 \) is a local maximum value (or we can also say that \( y = f(x) \) has a local maximum at \( x = 0 \)).

(f) Determine the interval(s) where \( f(x) \) is concave up, and the interval(s) where \( f(x) \) is concave down.

**Solution:** From \( f''(x) = \frac{12x^2 + 4}{(x^2 - 1)^3} \), there is no point in the domain of \( f(x) \) at which \( f''(x) = 0 \) or at which \( f''(x) \) does not exist. We conclude that \( y = f(x) \) does not have an inflection point. Determine the sign of \( f'(x) \) in each interval of the domain: \( f''(x) > 0 \) (and so \( f(x) \) is concave up) in \((-\infty, -1) \) and \((1, \infty) \), and \( f'(x) < 0 \) (and so \( f(x) \) is concave down) in \((-1, 1) \).

(g) Identify any inflection point(s).

**Solution:** As shown above, \( y = f(x) \) does not have an inflection point.

(h) (4 points) Sketch the graph \( y = f(x) \) reflecting all the features above.

(Graph is omitted)