PART 1. This portion of the exam is to test basic knowledge and calculation skills to a correct solution. The questions are multiple choice with “None of these” as a possible valid choice. No partial credit is given in this section, so work very carefully. Value: 4 points each

1. The expression \( \frac{4}{\sqrt[3]{x^2}} \), when written in exponential form, is

   (A) \( \frac{4x^2}{3} \)  
   (B) \( 4x^{-\frac{3}{2}} \)  
   (C) \( -4x^{\frac{2}{3}} \)  
   (D) \( 4x^{-\frac{2}{3}} \)  
   (E) \( 4^{-1}x^{-\frac{3}{2}} \)  
   (AB) None of these.

Solution: With fractional exponents, \( \sqrt[3]{x^2} = x^{\frac{2}{3}} \). Since this is in the denominator, we need to use negative exponents and so the answer is \( 4x^{-\frac{2}{3}} \).

2. The radian measure \( \frac{2\pi}{3} \), when converted to degree measure, is equal to

   (A) \( \left( \frac{2\pi}{3} \right)^\circ \)  
   (B) \( \left( \frac{3}{8\pi} \right)^\circ \)  
   (C) \( 30^\circ \)  
   (D) \( 120^\circ \)  
   (E) \( 135^\circ \)  
   (AB) None of These.

Solution: \( \frac{2\pi}{3} = \frac{2\pi}{3} \cdot \frac{180^\circ}{\pi} = 120^\circ \).

3. The limit \( \lim_{x\to0} \frac{4x}{\sin(x)} \) equals:

   (A) \( \frac{1}{4} \)  
   (B) \( 4 \)  
   (C) DNE  
   (D) \( 0 \)  
   (E) \( 1 \)  
   (AB) None of these.

Solution: Apply Properties of limits to get
\[
\lim_{x\to0} \frac{4x}{\sin(x)} = \lim_{x\to0} \frac{4}{\sin(x)} = \frac{4}{\lim_{x\to0} \frac{\sin(x)}{x}} = \frac{4}{1} = 4.
\]
4. The solutions of the inequality \(x^2 - 3x - 10 < 0\) are

(A) \(\{x \in \mathbb{R} | -5 < x < 2\}\)  
(B) \(\{x \in \mathbb{R} | x < -10\}\)  
(C) \(\{x \in \mathbb{R} | x > -10\}\)  
(D) \(\{x \in \mathbb{R} | -2 < x < 5\}\)  
(E) \(\{x \in \mathbb{R} | x > 5\}\)  
(AB) \(\{x \in \mathbb{R} | x < -2\}\)  
(AC) \(\{x \in \mathbb{R} | x < -2\ or\ x > 5\}\)  
(AD) None of these.

**Solution:** Factor \(x^2 - 3x - 10 = (x+2)(x-5)\) to see that the parabola \(y = x^2 - 3x - 10\) has an upward opening which crosses the \(x\)-axis at \(x = -2\) and \(x = 5\). Therefore, the \(y\)-coordinates of the parabola \(y = x^2 - 3x - 10\) is less than zero (below the \(x\)-axis) if and only if \(-2 < x < 5\), and so the solutions of \(x^2 - 3x - 10 < 0\) are the points in \(\{x \in \mathbb{R} | -2 < x < 5\}\).

5. The limit \(\lim_{x \to 2} \frac{2(x^2 - 2x - 3)}{x^2 - 6}\) equals:

(A) 2  
(B) -1  
(C) DNE  
(D) \(\frac{2}{3}\)  
(E) \(\frac{-2}{3}\)  
(AB) \(\frac{-3}{2}\)  
(AC) 3  
(AD) None of these.

**Solution:** Apply Properties of limits to get

\[
\lim_{x \to 2} \frac{2(x^2 - 2x - 3)}{x^2 - 6} = \frac{\lim_{x \to 2} 2(x^2 - 2x - 3)}{\lim_{x \to 2} x^2 - 6} = \frac{2(2^2 - 2 \cdot 2 - 3)}{2^2 - 6} = \frac{2 \cdot (-3)}{-2} = 3.
\]

6. The domain of the function \(f(x) = \frac{1}{\sqrt{4 - x^2}}\) is

(A) \(\{x \in \mathbb{R} | x > 0\}\)  
(B) \(\{x \in \mathbb{R} | x > 4\}\)  
(C) \(\{x \in \mathbb{R} | -4 < x < 4\}\)  
(D) \(\{x \in \mathbb{R} | -2 < x < 2\}\)  
(E) \(\{x \in \mathbb{R} | -4 \leq x \leq 4\}\)  
(AB) \(\{x \in \mathbb{R} | -2 \leq x \leq 2\}\)  
(AC) \(\{x \in \mathbb{R} | x < -2\ or\ x > 2\}\)  
(AD) None of these.

**Solution:** As whatever inside the square root cannot be negative, we must have \(4 - x^2 \geq 0\); as whatever in the denominator cannot be zero, we must have \(4 - x^2 \neq 0\). Combine such considerations to conclude that the domain of the given function must be the points satisfying \(4 - x^2 > 0\).

Note that the parabola \(y = 4 - x^2\) has a downward opening which crosses the \(x\)-axis at \(x = -2\) and \(x = 2\). Therefore, the \(y\)-coordinates of the parabola \(y = 4 - x^2\) is bigger than zero (above the \(x\)-axis) if and only if \(-2 < x < 2\), and so the domain of given function are the points in \(\{x \in \mathbb{R} | -2 < x < 2\}\).
7. The limit \( \lim_{x \to 2} \frac{4(x^2 - x - 2)}{4 - x^2} \) equals:

- (A) \(-\frac{3}{4}\)
- (B) \(-3\)
- (C) DNE
- (D) \(\frac{3}{4}\)
- (E) 2

Solution: As this is a \(0/0\)-type of limit, we must cancel zero factors first. Use \(2 - x = -(x - 2)\) to get

\[
\lim_{x \to 2} \frac{4(x^2 - x - 2)}{4 - x^2} = \lim_{x \to 2} \frac{4(x - 2)(x + 1)}{(2 - x)(2 + x)} = \lim_{x \to 2} \frac{4(x + 1)}{-4} = -3.
\]

8. A line passing through the point (2, 3) and perpendicular to the line \(2y - x = 10\) has equation

- (A) \(y - 3 = 2(x - 2)\)
- (B) \(y - 3 = \frac{1}{2}(x - 2)\)
- (C) \(y - 3 = -(x - 2)\)
- (D) \(2y - 3 = 2(x - 2)\)
- (E) \(y - 3 = 2(x - 2) + 10\) (AB) None of These.

Solution: Rewrite \(2y - x = 10\) to \(y = \frac{1}{2}x + 5\) to see that the slope of this line is \(\frac{1}{2}\). Any line perpendicular to this line must have slope \(-2\). As none of the lines in (A)-(E) has \(-2\) as a slope, the answer must be (AB).

9. The limit \( \lim_{x \to 4^-} \frac{|x - 4|}{x - 4} \) equals:

- (A) \(\frac{1}{4}\)
- (B) 4
- (C) DNE
- (D) 0
- (E) \(-1\)

Solution: When \(x \to 4^-\), we have \(x < 4\) or \(x - 4 < 0\), and so \(|x - 4| = -(x - 4)\). Thus

\[
\lim_{x \to 4^-} \frac{|x - 4|}{x - 4} = \lim_{x \to 4^-} \frac{-(x - 4)}{x - 4} = \lim_{x \to 4^-} -1 = -1.
\]

10. Given the function

\[
f(x) = \begin{cases} 
3x - 1 & \text{if } x < 1 \\
x^2 + 1 & \text{if } x > 1 
\end{cases}
\]

which of the following statements is correct?

- (A) \(f(x)\) is continuous at \(x = 1\).
- (B) The discontinuity of \(f(x)\) at \(x = 1\) can be removed by defining \(f(1) = 2\).
- (C) \(f(x)\) has a non removable discontinuity at \(x = 1\).
- (D) The discontinuity of \(f(x)\) at \(x = 1\) can be removed by defining \(f(1) = 3\).
- (E) None of above is correct.
**Solution:** When \( x \neq 1 \), \( f(x) \) is a polynomial in either interval and so \( x = 1 \) is the only discontinuity of the function. Since
\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 3x - 1 = 2 \quad \text{and} \quad \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^2 + 1 = 2,
\]
\( \lim_{x \to 1^-} f(x) = 2 \) exists. Thus the discontinuity of \( f(x) \) at \( x = 1 \) can be removed by defining \( f(1) = 2 \).

**PART 2:** This portion of the exam will be graded on a partial credit basis. **Answers without supporting work shown on the paper will receive NO credit.**

11. (8 points each) Compute each of the limit below.

(a) \( \lim_{x \to 2^-} \frac{|x - 2|}{x^2 - 4} \).

**Solution:** When \( x \to 2^- \), we have \( x < 2 \) or \( x - 2 < 0 \), and so \( |x - 2| = -(x - 2) \). It follows
\[
\lim_{x \to 2^-} \frac{|x - 2|}{x^2 - 4} = \lim_{x \to 2^-} \frac{-(x - 2)}{(x - 2)(x + 2)} = \lim_{x \to 2^-} \frac{-1}{x + 2} = -\frac{1}{4}.
\]

(b) \( \lim_{x \to \infty} \sqrt{x^2 + x} - x \).

**Solution:** Use \((a - b)(a + b) = a^2 - b^2 \) (view \( a = \sqrt{x^2 + x} \) and \( b = x \)) to get
\[
\lim_{x \to \infty} \sqrt{x^2 + x} - x = \lim_{x \to \infty} \frac{(\sqrt{x^2 + x} - x)(\sqrt{x^2 + x} + x)}{\sqrt{x^2 + x} + x} = \lim_{x \to \infty} \frac{x^2 + x - x^2}{\sqrt{x^2 + x} + x} = \lim_{x \to \infty} \frac{1}{\sqrt{1 + \frac{1}{x} + 1}} = \frac{1}{2}.
\]

(c) \( \lim_{x \to 0} x \cot(x) \).

**Solution:** Note that \( \cot(x) = \frac{\cos(x)}{\sin(x)} \). Apply the limit \( \lim_{x \to 0} \frac{\sin(x)}{x} = 1 \) and Limit Properties to get
\[
\lim_{x \to 0} x \cot(x) = \lim_{x \to 0} x \frac{\cos(x)}{\sin(x)} = \lim_{x \to 0} \frac{\cos(x)}{\sin(x)} = \lim_{x \to 0} \frac{\cos(x)}{\frac{\sin(x)}{x}} = \frac{\cos(0)}{1} = 1.
\]
12. *(6 points)* Given an example of a function \( f(x) \) such that \( \lim_{x \to 0} f(x) \) exists but \( f(0) \) is not defined and such that \( f(1) \) is defined but \( \lim_{x \to 1} f(x) \) does not exist. (You may sketch the graph of your example with such properties).

**Solution:** Two examples of correct solutions are given below. (You can either give the expressions of these functions or the graphs of such functions).

\[
f(x) = \begin{cases} 
  x & \text{if } x < 0 \\
  x^2 & \text{if } 0 < x < 1 \\
  1 - x & \text{if } x \geq 1 
\end{cases}
\]

or even simpler,

\[
f(x) = \begin{cases} 
  -1 & \text{if } x < 0 \\
  -1 & \text{if } 0 < x \leq 1 \\
  1 & \text{if } x > 1 
\end{cases}
\]

13. *(12 points)* Given a function

\[
f(x) = \frac{x^2 - 2x - 3}{x^2 - 9},
\]

do each of the following:

(a) Determine the domain and all discontinuities of \( f(x) \).

**Solution:** The domain: \( \{ x \in \mathbb{R} \mid x \neq \pm 3 \} \) or \( (-\infty, -3) \cup (-3, 3) \cup (3, \infty) \). As fractional functions are continuous in their domains, the discontinuities of the function are \( x = -3 \) and \( x = 3 \).

(b) Find all horizontal and vertical asymptotes of the graph \( y = f(x) \).

**Solution:** First, we note that when \( x \neq \pm 3 \),

\[
f(x) = \frac{x^2 - 2x - 3}{x^2 - 9} = \frac{(x - 3)(x + 1)}{(x - 3)(x + 3)} = \frac{x + 1}{x + 3}.
\]

At \( x = -3 \), as

\[
\lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} \frac{x + 1}{x + 3} = +\infty, \\
\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} \frac{x + 1}{x + 3} = -\infty,
\]

\( x = -3 \) is a vertical asymptote of \( y = f(x) \).

At \( x = 3 \), as

\[
\lim_{x \to 3^-} f(x) = \lim_{x \to 2} \frac{x + 1}{x + 3} = \frac{4}{6} = \frac{2}{3},
\]

\( x = 3 \) is not a vertical asymptote, and so \( x = -3 \) is the only vertical asymptote of \( y = f(x) \).

For horizontal asymptote, we compute

\[
\lim_{x \to \pm \infty} f(x) = \lim_{x \to \pm \infty} \frac{x^2 - 2x - 3}{x^2 - 9} = \lim_{x \to \pm \infty} \frac{1 - \frac{2}{x} - \frac{3}{x^2}}{1 - \frac{9}{x^2}} = \frac{1 - 0 - 0}{1 - 0} = 1.
\]

Thus \( y = 1 \) is the only horizontal asymptote of \( y = f(x) \).
14. (6 points each)

(a) (For Exam 1B) Find all real solutions of \(2 \sin^2 x + \sin x - 1 = 0\).

**Solution:** As \(2 \sin^2 x + \sin x - 1 = (2 \sin x - 1)(\sin x + 1) = 0\), either \(2 \sin x - 1 = 0\) or \(\sin x + 1 = 0\). (One can also use quadratic formula to get solutions \(\sin x = \frac{1}{2}\)). If \(2 \sin x - 1 = 0\), then \(\sin x = \frac{1}{2}\), which has only one solution \(x = \frac{\pi}{4}\) in \([0, 2\pi]\); If \(\sin x + 1 = 0\), then \(\sin x = -1\), which has only one solution \(x = \pi\) in \([0, 2\pi]\). Therefore, all real solutions of the equation are \(\frac{\pi}{4} + 2k\pi, \, \pi + 2k\pi\), where \(k = 0, \pm 1, \pm 2, \pm 3, \ldots\).

(For Exam 1A) Find all real solutions of \(2 \sin^2 x + 3 \sin x - 2 = 0\).

**Solution:** As \(2 \sin^2 x + 3 \sin x - 2 = (2 \sin x - 1)(\sin x + 2) = 0\), either \(2 \sin x - 1 = 0\) or \(\sin x + 2 = 0\). (One can also use quadratic formula to get solutions \(\sin x = -2\) and \(\sin x = \frac{1}{2}\)). If \(2 \sin x - 1 = 0\), then \(\sin x = \frac{1}{2}\), which has only one solution \(x = \frac{\pi}{4}\) in \([0, 2\pi]\); If \(\sin x + 2 = 0\), then \(\sin x = -2\), which does not have any real solution. Therefore, all real solutions of the equation are \(\frac{\pi}{4} + 2k\pi\), where \(k = 0, \pm 1, \pm 2, \pm 3, \ldots\).

(b) Solve the equation \(\ln(x^4) = 8\).

**Solution:** Apply properties of logarithm to get \(x^4 = e^{\ln x^4} = e^8\). Thus \(x^2 = e^4\) and so \(x = \pm e^2\).

15. (6 points) Use the Intermediate Value Theorem to verify that \(f(x) = x^5 - 3x + 1\) must have at least one zero in the interval \([0, 1]\).

**Solution:** The polynomial \(x^5 - 3x + 1\) is continuous on all the real numbers and so \(f(x)\) is continuous on \([0, 1]\). As 
\[f(0) = 0^5 - 3 \cdot 0 + 1 = 1 > 0 \text{ and } f(1) = 1^5 - 3 \cdot 1 + 1 = -2 < 0,\]
the Intermediate Value Theorem of continuous functions assures that for some point \(c\) inside the interval \((0, 1)\), we have \(f(c) = 0\), and so \(f(x) = x^5 - 3x + 1\) must have at least one zero in the interval \([0, 1]\).