Binary Subtrees with Few Path Labels

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SIAM Discrete Math 2008
Burlington, VT
19 June 2008
History

Rod Downey  Noam Greenberg  Carl Jockusch

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The Problem

Ternary tree $T$ with depth $n = 3$.

- Let $T$ be a $\{0, 1\}$-edge-labeled perfect ternary tree of depth $n$. 
The Problem

This path has path label 100.

- Let $T$ be a $\{0, 1\}$-edge-labeled perfect ternary tree of depth $n$.
- Each path from the root to a leaf gives a path label in $\{0, 1\}^n$. 
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Let $T$ be a $\{0, 1\}$-edge-labeled perfect ternary tree of depth $n$.
Each path from the root to a leaf gives a path label in $\{0, 1\}^n$.
Let $f(T)$ be the min., over all perfect binary subtrees $S \subseteq T$ of depth $n$, of the number of path labels along paths in $S$. 
This subtree contains 3 path labels, so $f(T) \leq 3$.

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![Diagram of a ternary tree with path labels]
The Problem

In fact, \( f(T) = 2 \).

- Let \( T \) be a \( \{0, 1\} \)-edge-labeled perfect ternary tree of depth \( n \).
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- Let $f(n)$ be the max., over all $\{0, 1\}$-edge-labeled perfect ternary trees $T$ of depth $n$, of $f(T)$.
- From now on, all trees are perfect and $\{0, 1\}$-edge-labeled; all subtrees have full depth.
Main Result

Theorem

There exist positive constants $c_1$ and $c_2$ such that

$$2^{\frac{n-3}{\lg 3}} \leq f(n) \leq c_1 2^{n-c_2 \sqrt{n}}.$$
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Corollaries

- $\lim_{n \to \infty} \frac{f(n)}{2^n} = 0$

- $1.54856 \approx 2^{\frac{1}{\lg 3}} \leq \lim_{n \to \infty} (f(n))^{1/n} \leq 2$
Proposition

If $r$ and $s$ are non-negative integers, then $f(r + s) \geq f(r)f(s)$. 

Preliminaries

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Proof.

Let $R$ be a ternary tree of depth $r$ which maximizes $f(R)$.
Preliminaries

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- Let $S$ be a ternary tree of depth $s$ which maximizes $f(S)$.  

Corollary
$\lim_{n \to \infty} \left( \frac{f(n)}{n} \right) = \sup \left\{ \frac{f(n)}{n} \mid n \geq 1 \right\}$
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Proof.

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Let \( S \) be a ternary tree of depth \( s \) which maximizes \( f(S) \).

Attach a copy of \( S \) to each leaf of \( R \).
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- Attach a copy of $S$ to each leaf of $R$.
- Every binary subtree contains at least $f(r)f(s)$ path labels.
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Every binary subtree contains at least $f(r)f(s)$ path labels.

Corollary

$$\lim_{n \to \infty} (f(n))^{1/n} = \sup \left\{ (f(n))^{1/n} \mid n \geq 1 \right\}$$
Lower Bound: Overview

- To obtain a lower bound on $f(n)$, we construct a ternary tree in which every binary subtree has many path labels.
Lower Bound: Overview

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- The construction uses two different kinds of trees.
Lower Bound: Construction of $R_n$

Proposition
Let $a_0 = 1$ and $a_n = \lceil 3a_{n-1}/2 \rceil$ for $n \geq 0$. If $n \geq 0$, then there exists a ternary tree $R_n$ of depth $n$ in which each path label occurs at most $a_n$ times.

Corollary
If $n \geq 0$, then there exists a ternary tree $R_n$ of depth $n$ in which each path label occurs at most $2 \left(\frac{3}{2}\right)^n$ times.
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By induction on $n$. Extend $R_{n-1}$ to $R_n$ as follows.
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By induction on $n$. Extend $R_{n-1}$ to $R_n$ as follows.

- For each $x \in \{0, 1\}^{n-1}$, let $L_x$ be the set of leaves in $R_{n-1}$ that are endpoints of paths with path label $x$. Note that $|L_x| \leq a_{n-1}$. 

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- Of the $3|L_x|$ edges in $R_n$ below vertices in $L_x$, arbitrarily choose $\lceil 3|L_x|/2 \rceil$ to have label 0; the remaining $\lfloor 3|L_x|/2 \rfloor$ edges are labeled 1.

□
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In $R_n$, a path label $x \in \{0, 1\}^n$ occurs at most $\lceil 3a_{n-1}/2 \rceil$ times.

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**Remark**

The trees $R_n$ are best possible: in each ternary tree of depth $n$, some path label occurs at least $a_n$ times.
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**Remark**
The trees $R_n$ are best possible: in each ternary tree of depth $n$, some path label occurs at least $a_n$ times.

**Corollary**
If $n \geq 0$, then there exists a ternary tree $R_n$ of depth $n$ in which each path label occurs at most $2 \left(\frac{3}{2}\right)^n$ times.
Lower Bound: Uniform Trees

For each bitstring $y$, let $Q_y$ be the ternary tree labeled so that all paths in $Q_y$ have the same path label $y$. 
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- $Q_{01}$

- $Q_{10}$
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![Diagrams of $Q_{00}$, $Q_{01}$, $Q_{10}$, and $Q_{11}$ trees.](image-url)
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Lemma

If \( m \geq 0 \) and \( s = \lceil \log_2 \left( \frac{3}{2} \right)^m \rceil \), then \( f(m + s) \geq 2^m \).
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- For each \( u \in L_x \), arbitrarily choose a distinct bitstring \( y(u) \in \{0, 1\}^s \).
- Because \( |L_x| \leq 2 \left( \frac{3}{2} \right)^m \leq 2^s \), enough bitstrings are available.
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- Because \( |L_x| \leq 2 \left( \frac{3}{2} \right)^m \leq 2^s \), enough bitstrings are available.
- At each \( u \in L_x \), attach a copy of \( Q_{y(u)} \).
Lemma

If \( m \geq 0 \) and \( s = \lceil \lg 2 \left( \frac{3}{2} \right)^m \rceil \), then \( f(m + s) \geq 2^m \).

Proof.

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- At each \( u \in L_x \), attach a copy of \( Q_{y(u)} \).
- Repeat for each \( x \in \{0, 1\}^m \).
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Lemma
If $m \geq 0$ and $s = \lceil \log_2 \left( \frac{3}{2} \right)^m \rceil$, then $f(m + s) \geq 2^m$.

Theorem
If $n \geq 0$, then $f(n) \geq 2^{\frac{n-3}{\log_3}}$. 
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Theorem
If $n \geq 0$, then $f(n) \geq 2^{\frac{n-3}{\lg 3}}$.

Proof (sketch).
Either $n$ or $n - 1$ is of the form $m + \lceil \lg 2 \left( \frac{3}{2} \right)^m \rceil$ for some integer $m$, in which case the Lemma applies.
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Corollary
\[
\lim_{n \to \infty} (f(n))^{1/n} \geq 2^{\frac{1}{\lg 3}} \approx 1.54856
\]
Upper Bound: Overview

To obtain an upper bound on $f(n)$, we argue that every ternary tree of depth $n$ contains a binary subtree that uses few path labels.
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Upper bound uses several lemmas.
Lemma (Monochromatic Subtree Lemma; folklore?)

Let $T$ be a ternary tree in which each leaf is colored red or blue. There exists a binary subtree $S \subseteq T$ such that all leaves in $S$ share a common color.
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Proof.
Upper Bound: Orthogonal Partitions

- Let \( \Upsilon \) be a finite ground set.

Definition

A pair of partitions \( \{X, X\} \) and \( \{Y, Y\} \) of \( \Upsilon \) is \( \alpha \)-orthogonal if all four of the cross intersections \( X \cap Y, X \cap Y, X \cap Y, \) and \( X \cap Y \) have size at least \( \alpha \|\Upsilon\|/4 \).

A family of partitions \( F \) of \( \Upsilon \) is \( \alpha \)-orthogonal if each pair of (distinct) partitions in \( \Upsilon \) is \( \alpha \)-orthogonal.
Upper Bound: Orthogonal Partitions

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- A family of partitions $\mathcal{F}$ of $\Upsilon$ is $\alpha$-orthogonal if each pair of (distinct) partitions in $\Upsilon$ is $\alpha$-orthogonal.
Lemma (Orthogonal Family Lemma)

If $|\Upsilon| = t$ and $0 \leq \alpha \leq 1$, then there exists an $\alpha$-orthogonal family of partitions $\mathcal{F}$ of $\Upsilon$ with

$$|\mathcal{F}| \geq \left\lfloor \frac{\sqrt{2}}{2} e^{\frac{(1-\alpha)^2}{16}} t \right\rfloor.$$
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Proof (sketch).

Let $r = \left\lfloor \frac{\sqrt{2}}{2} e^{\frac{(1-\alpha)^2}{16} t} \right\rfloor$. 

\begin{itemize}
  \item For each $1 \leq j \leq r$, choose a subset $X_j \subseteq \Upsilon$ uniformly and independently at random.
  \item Let $\mathcal{F} = \{\{X_j, X_j\} | 1 \leq j \leq r\}$.
  \item Chernoff bound: $\mathcal{F}$ is $\alpha$-orthogonal with positive probability.
\end{itemize}
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$$|\{x \in \Upsilon \mid x \text{ is a path label in some } S_j\}| \leq \left(1 - \frac{\alpha}{4}\right) 2^n.$$
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$$|\{x \in \Upsilon \mid x \text{ is a path label in some } S_j\}| \leq \left(1 - \frac{\alpha}{4}\right) 2^n.$$

Proof.

Consider a ptn. $\{X_1, \overline{X_1}\} \in \mathcal{F}$. 

\[\begin{array}{cccc}
\triangle & \triangle & \triangle & \triangle \\
 T_1 & T_2 & T_3 & T_4
\end{array}\]
Upper Bound: Binary Subtrees Lemma (1)

Lemma (Binary Subtrees Lemma (1))

Let $T_1, T_2, \ldots, T_k$ be ternary trees of depth $n$ and let $\Upsilon = \{0, 1\}^n$. If there exists an $\alpha$-orthogonal family of partitions $\mathcal{F}$ of $\Upsilon$ with $|\mathcal{F}| > 2^{k-1}$, then there exists binary subtrees $S_1, S_2, \ldots, S_n$ with $S_j \subseteq T_j$ such that

$$|\{x \in \Upsilon \mid x \text{ is a path label in some } S_j\}| \leq \left(1 - \frac{\alpha}{4}\right) 2^n.$$ 

Proof.

Consider a ptn. $\{X_1, \overline{X_1}\} \in \mathcal{F}$.

Color a leaf $u$ in $T_j$ red if the path label ending at $u$ is in $X_1$, and blue otherwise.
Upper Bound: Binary Subtrees Lemma (1)

Lemma (Binary Subtrees Lemma (1))

Let $T_1, T_2, \ldots, T_k$ be ternary trees of depth $n$ and let $\Upsilon = \{0, 1\}^n$. If there exists an $\alpha$-orthogonal family of partitions $\mathcal{F}$ of $\Upsilon$ with $|\mathcal{F}| > 2^{k-1}$, then there exists binary subtrees $S_1, S_2, \ldots, S_n$ with $S_j \subseteq T_j$ such that

$$|\{x \in \Upsilon \mid x \text{ is a path label in some } S_j\}| \leq \left(1 - \frac{\alpha}{4}\right) 2^n.$$

Proof.

Consider a ptn. $\{X_1, \overline{X_1}\} \in \mathcal{F}$.

Color a leaf $u$ in $T_j$ red if the path label ending at $u$ is in $X_1$, and blue otherwise.

Apply Monochromatic Subtree Lemma.
Upper Bound: Binary Subtrees Lemma (1)

Lemma (Binary Subtrees Lemma (1))
Let $T_1, T_2, \ldots, T_k$ be ternary trees of depth $n$ and let $\Upsilon = \{0, 1\}^n$. If there exists an $\alpha$-orthogonal family of partitions $\mathcal{F}$ of $\Upsilon$ with $|\mathcal{F}| > 2^{k-1}$, then there exists binary subtrees $S_1, S_2, \ldots, S_n$ with $S_j \subseteq T_j$ such that

$$|\{x \in \Upsilon \mid x \text{ is a path label in some } S_j\}| \leq \left(1 - \frac{\alpha}{4}\right)2^n.$$

Proof.

Repeat for each ptn. in $\mathcal{F}$. 
Lemma (Binary Subtrees Lemma (1))

Let $T_1, T_2, \ldots, T_k$ be ternary trees of depth $n$ and let $\Upsilon = \{0, 1\}^n$. If there exists an $\alpha$-orthogonal family of partitions $\mathcal{F}$ of $\Upsilon$ with $|\mathcal{F}| > 2^{k-1}$, then there exists binary subtrees $S_1, S_2, \ldots, S_n$ with $S_j \subseteq T_j$ such that

$$|\{x \in \Upsilon \mid x \text{ is a path label in some } S_j\}| \leq \left(1 - \frac{\alpha}{4}\right) 2^n.$$ 

Proof.

repeat for each ptn. in $\mathcal{F}$. 

\{$x_1, \overline{x_1}\}$ 

\{$x_2, \overline{x_2}\}$
Upper Bound: Binary Subtrees Lemma (1)

Lemma (Binary Subtrees Lemma (1))

Let $T_1, T_2, \ldots, T_k$ be ternary trees of depth $n$ and let $\Upsilon = \{0, 1\}^n$. If there exists an $\alpha$-orthogonal family of partitions $\mathcal{F}$ of $\Upsilon$ with $|\mathcal{F}| > 2^{k-1}$, then there exists binary subtrees $S_1, S_2, \ldots, S_n$ with $S_j \subseteq T_j$ such that

$$|\{x \in \Upsilon \mid x \text{ is a path label in some } S_j\}| \leq \left(1 - \frac{\alpha}{4}\right)2^n.$$ 

Proof.

Repeat for each ptn. in $\mathcal{F}$. 

\[ \begin{align*}
T_1 & \quad T_2 & \quad T_3 & \quad T_4 & \quad \{x_1, \overline{x}_1\} \\
T_1 & \quad T_2 & \quad T_3 & \quad T_4 & \quad \{x_2, \overline{x}_2\} \\
T_1 & \quad T_2 & \quad T_3 & \quad T_4 & \quad \{x_3, \overline{x}_3\}
\end{align*} \]
Upper Bound: Binary Subtrees Lemma (1)

**Lemma (Binary Subtrees Lemma (1))**

Let $T_1, T_2, \ldots, T_k$ be ternary trees of depth $n$ and let $\Upsilon = \{0, 1\}^n$. If there exists an $\alpha$-orthogonal family of partitions $\mathcal{F}$ of $\Upsilon$ with $|\mathcal{F}| > 2^{k-1}$, then there exists binary subtrees $S_1, S_2, \ldots, S_n$ with $S_j \subseteq T_j$ such that

$$|\{x \in \Upsilon \mid x \text{ is a path label in some } S_j\}| \leq \left(1 - \frac{\alpha}{4}\right)2^n.$$

**Proof.**

- Repeat for each ptn. in $\mathcal{F}$. 
Upper Bound: Binary Subtrees Lemma (1)

Lemma (Binary Subtrees Lemma (1))

Let $T_1, T_2, \ldots, T_k$ be ternary trees of depth $n$ and let $\Upsilon = \{0, 1\}^n$. If there exists an $\alpha$-orthogonal family of partitions $\mathcal{F}$ of $\Upsilon$ with $|\mathcal{F}| > 2^{k-1}$, then there exists binary subtrees $S_1, S_2, \ldots, S_n$ with $S_j \subseteq T_j$ such that

$$|\{x \in \Upsilon \mid x \text{ is a path label in some } S_j\}| \leq \left(1 - \frac{\alpha}{4}\right) 2^n.$$

Proof.

- Repeat for each ptn. in $\mathcal{F}$.
- $\mathcal{F}$ is large, so some pair $\{X, \overline{X}\}$ and $\{Y, \overline{Y}\}$ give the same red/blue ptn. of $\{T_1, \ldots, T_k\}$. 

\[ \begin{align*} 
&\{x_1, \overline{x_1}\} \\
&\{x_2, \overline{x_2}\} \\
&\{x_3, \overline{x_3}\} \\
&\{x_4, \overline{x_4}\} \\
&\{x_5, \overline{x_5}\} 
\end{align*} \]
Lemma (Binary Subtrees Lemma (1))

Let $T_1, T_2, \ldots, T_k$ be ternary trees of depth $n$ and let $\Upsilon = \{0, 1\}^n$. If there exists an $\alpha$-orthogonal family of partitions $\mathcal{F}$ of $\Upsilon$ with $|\mathcal{F}| > 2^{k-1}$, then there exists binary subtrees $S_1, S_2, \ldots, S_n$ with $S_j \subseteq T_j$ such that

$$|\{x \in \Upsilon \mid x \text{ is a path label in some } S_j\}| \leq \left(1 - \frac{\alpha}{4}\right)2^n.$$  

Proof.

- Repeat for each ptn. in $\mathcal{F}$.
- $\mathcal{F}$ is large, so some pair $\{X, \overline{X}\}$ and $\{Y, \overline{Y}\}$ give the same red/blue ptn. of $\{T_1, \ldots, T_k\}$. 

\begin{itemize}
  \item $T_1$
  \item $T_2$
  \item $T_3$
  \item $T_4$
  \item $\{x_3, \overline{x_3}\}$
\end{itemize}

\begin{itemize}
  \item $T_1$
  \item $T_2$
  \item $T_3$
  \item $T_4$
  \item $\{x_5, \overline{x_5}\}$
\end{itemize}
Lemma (Binary Subtrees Lemma (1))

Let $T_1, T_2, \ldots, T_k$ be ternary trees of depth $n$ and let $\Upsilon = \{0, 1\}^n$. If there exists an $\alpha$-orthogonal family of partitions $\mathcal{F}$ of $\Upsilon$ with $|\mathcal{F}| > 2^{k-1}$, then there exists binary subtrees $S_1, S_2, \ldots, S_n$ with $S_j \subseteq T_j$ such that

$$\left| \{ x \in \Upsilon \mid x \text{ is a path label in some } S_j \} \right| \leq \left( 1 - \frac{\alpha}{4} \right) 2^n.$$

Proof.

$\mathcal{T}_1 \quad \mathcal{T}_2 \quad \mathcal{T}_3 \quad \mathcal{T}_4 \quad \{X, \overline{X}\}$

$\mathcal{T}_1 \quad \mathcal{T}_2 \quad \mathcal{T}_3 \quad \mathcal{T}_4 \quad \{Y, \overline{Y}\}$

- Repeat for each ptn. in $\mathcal{F}$.
- $\mathcal{F}$ is large, so some pair $\{X, \overline{X}\}$ and $\{Y, \overline{Y}\}$ give the same red/blue ptn. of $\{T_1, \ldots, T_k\}$. 
Upper Bound: Binary Subtrees Lemma (1)

Lemma (Binary Subtrees Lemma (1))

Let $T_1, T_2, \ldots, T_k$ be ternary trees of depth $n$ and let $\Upsilon = \{0, 1\}^n$. If there exists an $\alpha$-orthogonal family of partitions $\mathcal{F}$ of $\Upsilon$ with $|\mathcal{F}| > 2^{k-1}$, then there exists binary subtrees $S_1, S_2, \ldots, S_n$ with $S_j \subseteq T_j$ such that

$$|\{x \in \Upsilon \mid x \text{ is a path label in some } S_j\}| \leq \left(1 - \frac{\alpha}{4}\right)2^n.$$

Proof.

If $T_j$ is red under $\{X, \overline{X}\}$, then $T_j$ has a binary subtree $S_j$ in which every path label is in $X$. If $T_j$ is blue under $\{Y, \overline{Y}\}$, then $T_j$ has a binary subtree $S_j$ in which every path label is in $Y$. \qed
Upper Bound: Binary Subtrees Lemma (1)

Lemma (Binary Subtrees Lemma (1))

Let \( T_1, T_2, \ldots, T_k \) be ternary trees of depth \( n \) and let \( \Upsilon = \{0, 1\}^n \). If there exists an \( \alpha \)-orthogonal family of partitions \( \mathcal{F} \) of \( \Upsilon \) with \( |\mathcal{F}| > 2^{k-1} \), then there exists binary subtrees \( S_1, S_2, \ldots, S_n \) with \( S_j \subseteq T_j \) such that

\[
|\{x \in \Upsilon \mid x \text{ is a path label in some } S_j\}| \leq \left(1 - \frac{\alpha}{4}\right)2^n.
\]

Proof.

If \( T_j \) is red under \( \{X, \overline{X}\} \), then \( T_j \) has a binary subtree \( S_j \) in which every path label is in \( X \).

If \( T_j \) is blue under \( \{Y, \overline{Y}\} \), then \( T_j \) has a binary subtree \( S_j \) in which every path label is in \( \overline{Y} \).
Upper Bound: Binary Subtrees Lemma (1)

Lemma (Binary Subtrees Lemma (1))

Let $T_1, T_2, \ldots, T_k$ be ternary trees of depth $n$ and let $\Upsilon = \{0, 1\}^n$. If there exists an $\alpha$-orthogonal family of partitions $\mathcal{F}$ of $\Upsilon$ with $|\mathcal{F}| > 2^{k-1}$, then there exists binary subtrees $S_1, S_2, \ldots, S_n$ with $S_j \subseteq T_j$ such that

$$|\{x \in \Upsilon \mid x \text{ is a path label in some } S_j\}| \leq \left(1 - \frac{\alpha}{4}\right)2^n.$$ 

Proof.

Every path label in each $S_j$ is in $X \cup \overline{Y}$. 

\begin{tikzpicture}
    \draw (0,0) -- (1,1) -- (2,0) -- (0,0);
    \draw (3,0) -- (4,1) -- (5,0) -- (3,0);
    \draw (0,-1) -- (1,-2) -- (2,-1) -- (0,-1);
    \draw (3,-1) -- (4,-2) -- (5,-1) -- (3,-1);
    \node at (1,1) {$T_1$}; \node at (4,1) {$T_2$}; \node at (7,1) {$T_3$}; \node at (10,1) {$T_4$}; \node at (1,0) {$\Upsilon$}; \node at (4,0) {$\{X, X\}$}; \node at (7,0) {$\{Y, Y\}$}; \node at (10,0) {$\overline{\Upsilon}$}; \node at (12,0) {\text{Every path label in each } S_j \text{ is in } X \cup \overline{Y}$};
\end{tikzpicture}
Upper Bound: Binary Subtrees Lemma (1)

Lemma (Binary Subtrees Lemma (1))

Let $T_1, T_2, \ldots, T_k$ be ternary trees of depth $n$ and let $\Upsilon = \{0, 1\}^n$. If there exists an $\alpha$-orthogonal family of partitions $\mathcal{F}$ of $\Upsilon$ with $|\mathcal{F}| > 2^{k-1}$, then there exists binary subtrees $S_1, S_2, \ldots, S_n$ with $S_j \subseteq T_j$ such that

$$|\{x \in \Upsilon \mid x \text{ is a path label in some } S_j\}| \leq \left(1 - \frac{\alpha}{4}\right) 2^n.$$

Proof.

\begin{itemize}
  \item Every path label in each $S_j$ is in $X \cup \overline{Y}$.
  \item Set of path labels in $\{S_1, \ldots, S_j\}$ and $\overline{X} \cap Y$ are disjoint.
\end{itemize}
Lemma (Binary Subtrees Lemma (1))

Let $T_1, T_2, \ldots, T_k$ be ternary trees of depth $n$ and let $\Upsilon = \{0, 1\}^n$. If there exists an $\alpha$-orthogonal family of partitions $\mathcal{F}$ of $\Upsilon$ with $|\mathcal{F}| > 2^{k-1}$, then there exists binary subtrees $S_1, S_2, \ldots, S_n$ with $S_j \subseteq T_j$ such that

$$|\{x \in \Upsilon \mid x \text{ is a path label in some } S_j\}| \leq \left(1 - \frac{\alpha}{4}\right) 2^n.$$  

Proof.

\begin{itemize}
\item Every path label in each $S_j$ is in $X \cup \overline{Y}$.
\item Set of path labels in $\{S_1, \ldots, S_j\}$ and $\overline{X} \cap Y$ are disjoint.
\item $\mathcal{F}$ is $\alpha$-orthogonal: $|\overline{X} \cap Y| \geq \frac{\alpha}{4} 2^n$.
\end{itemize}
Upper Bound: Binary Subtrees Lemma (2)

Setting $\alpha = 1/2$ in the Orthogonal Family Lemma and applying the Binary Subtrees Lemma (1) yields:

Lemma (Binary Subtrees Lemma (2))

Let $T_1, \ldots, T_k$ be ternary trees of depth $n \geq 6 + \lg k$, and let $\Upsilon = \{0, 1\}^n$. There exist binary subtrees $S_1, \ldots, S_k$ with $S_j \subseteq T_j$ such that

$$|\{x \in \Upsilon | x \text{ is a path label in some } S_j\}| \leq \left(\frac{7}{8}\right)^2 n.$$
Upper Bound: Binary Subtrees Lemma (2)

Setting $\alpha = 1/2$ in the Orthogonal Family Lemma and applying the Binary Subtrees Lemma (1) yields:

Lemma (Binary Subtrees Lemma (2))

Let $T_1, \ldots, T_k$ be ternary trees of depth $n \geq 6 + \lg k$, and let $\Upsilon = \{0, 1\}^n$. There exist binary subtrees $S_1, \ldots, S_k$ with $S_j \subseteq T_j$ such that

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Setting $\alpha = 1/2$ in the Orthogonal Family Lemma and applying the Binary Subtrees Lemma (1) yields:

**Lemma (Binary Subtrees Lemma (2))**

Let $T_1, \ldots, T_k$ be ternary trees of depth $n \geq 6 + \lg k$, and let $\Upsilon = \{0, 1\}^n$. There exist binary subtrees $S_1, \ldots, S_k$ with $S_j \subseteq T_j$ such that

$$|\{x \in \Upsilon \mid x \text{ is a path label in some } S_j\}| \leq \left(\frac{7}{8}\right) 2^n.$$

The assumption $n \geq 6 + \lg k$ is tight up to an additive constant.

Indeed, if $k = 2^n$:

![Diagrams](image-url)
Upper Bound

Theorem

Let $c_1 = \sqrt{\log(16/15)} \approx 0.3051$ and $c_2 = 2^{c_1 \sqrt{540} - 1} \approx 68.156$. If $n \geq 0$, then $f(n) \leq c_2 2^{n - c_1 \sqrt{n}}$. 
Upper Bound

Theorem
Let $c_1 = \sqrt{\log(16/15)} \approx 0.3051$ and $c_2 = 2^{c_1 \sqrt{540}} - 1 \approx 68.156$. If $n \geq 0$, then $f(n) \leq c_2 2^{n - c_1 \sqrt{n}}$.

Proof (sketch).

Let $T$ be a ternary tree with depth $n$. ▶
Upper Bound

Theorem

Let $c_1 = \sqrt{\lg(16/15)} \approx 0.3051$ and $c_2 = 2^{c_1 \sqrt{540} - 1} \approx 68.156$. If $n \geq 0$, then $f(n) \leq c_2 2^{n - c_1 \sqrt{n}}$.

Proof (sketch).

- Let $T$ be a ternary tree with depth $n$.
- Let $T'$ be the ternary subtree of $T$ up to depth $m \approx n - c_1 \sqrt{n}$.
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Let $c_1 = \sqrt{\log(16/15)} \approx 0.3051$ and $c_2 = 2^{c_1\sqrt{540}-1} \approx 68.156$. If $n \geq 0$, then $f(n) \leq c_2 2^{n-c_1\sqrt{n}}$.

Proof (sketch).

- Let $T$ be a ternary tree with depth $n$.
- Let $T'$ be the ternary subtree of $T$ up to depth $m \approx n - c_1\sqrt{n}$.
- Obtain a binary subtree $S' \subseteq T'$ that uses few path labels.
Upper Bound

Theorem
Let $c_1 = \sqrt{\log(16/15)} \approx 0.3051$ and $c_2 = 2^{c_1 \sqrt{540}-1} \approx 68.156$. If $n \geq 0$, then $f(n) \leq c_2 2^{n-c_1 \sqrt{n}}$.

Proof (sketch).

- Fix $x \in \{0,1\}^m$ and let $L_x$ be the set of leaves in $S'$ that are endpoints of a path with path label $x$. 
Upper Bound

Theorem

Let $c_1 = \sqrt{\log(16/15)} \approx 0.3051$ and $c_2 = 2^{c_1 \sqrt{540-1}} \approx 68.156$. If $n \geq 0$, then $f(n) \leq c_2 2^{n-c_1 \sqrt{n}}$.

Proof (sketch).

- Fix $x \in \{0, 1\}^m$ and let $L_x$ be the set of leaves in $S'$ that are endpoints of a path with path label $x$.
- Two cases: if $L_x$ is large, then extend $S'$ at vertices in $L_x$ arbitrarily.
Upper Bound

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Let \( c_1 = \sqrt{\lg(16/15)} \approx 0.3051 \) and \( c_2 = 2^{c_1\sqrt{540}-1} \approx 68.156 \). If \( n \geq 0 \), then \( f(n) \leq c_2 2^{n-c_1\sqrt{n}} \).

Proof (sketch).

Fix \( x \in \{0, 1\}^m \) and let \( L_x \) be the set of leaves in \( S' \) that are endpoints of a path with path label \( x \).

Two cases: if \( L_x \) is large, then extend \( S' \) at vertices in \( L_x \) arbitrarily.

If \( L_x \) is small, apply Binary Subtrees Lemma (2) to extend \( S' \) at vertices in \( L_x \).
Summary & Open Problems

Theorem

There exist positive constants $c_1$ and $c_2$ such that

$$2^{\frac{n-3}{\lg 3}} \leq f(n) \leq c_1 2^{n-c_2 \sqrt{n}}.$$

Corollary

$$1.54856 \approx 2^{\frac{1}{\lg 3}} \leq \lim_{n \to \infty} (f(n))^{1/n} \leq 2.$$
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Open Problems

▶ Improve the bounds on $f(n)$ and $\lim_{n \to \infty} (f(n))^{1/n}$. 

▶ Is it true that $\lim_{n \to \infty} (f(n))^{1/n} < 2$?

▶ For each $p < q$, consider the analogous problem on $\{0, 1, \ldots, p-1\}$-edge-labeled perfect $q$-ary trees. Nothing is known except our results for $(p, q) = (2, 3)$. 

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- Improve the bounds on $f(n)$ and $\lim_{n \to \infty} (f(n))^{1/n}$.
- Is it true that $\lim_{n \to \infty} (f(n))^{1/n} < 2$?
- For each $p < q$, consider the analogous problem on $\{0, 1, \ldots, p - 1\}$-edge-labeled perfect $q$-ary trees. Nothing is known except our results for $(p, q) = (2, 3)$. 