Topics to be covered on a PhD entrance exam in topology, Spring 2000

- Examples of topological spaces.
- Separation axioms (T_0-, T_1-, Hausdorff, regular, and normal spaces).
- Metric space topology (completeness, equivalent forms of compactness).
- Continuity.
- Connected spaces.
- Compactness.

Suggested reference books.

- Dugundji, *Topology*, Allyn & Bacon. (Chapters I-IX and XI.)
- Kelly, *General Topology*, D. van Nostrand. (Chapters: all except II, VI, and Appendix.)
- Gemignani, *Elementary Topology*, Addison-Wesley. (Chapters: all except XI.)

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Topology Ph.D. Entrance Exam, August 2000

In the exercises that follow $\overline{A}$ stands for the closure of A, and $A \setminus B$ for the set difference: $A \setminus B = \{x \in A : x \notin B\}$.

Ex. 1. (a) Define a T_0 topological space.

(b) Show that a topological space X is a T_0-space if and only if $\overline{\{x\}} \neq \overline{\{y\}}$ for every distinct $x, y \in X$.

Ex. 2. A topological space X is said to be *completely regular* provided that for each $p \in X$ and closed set A in X such that $p \notin A$, there is a continuous function $f: X \to [0,1]$ such that $f(p) = 0$ and $f[A] = \{1\}$.

Prove that any subspace of a completely regular space is completely regular.

Ex. 3. Let X be a topological space and let A and B be non-empty proper closed subsets of X such that $X = A \cup B$. Show that $X \setminus (A \cap B)$ is not connected.

Ex. 4. (a) Give an example of sets A_i ($i = 1, 2, 3, \ldots$) in a topological space for which

$$\overline{\bigcup_{i=1}^{\infty} A_i} \neq \bigcup_{i=1}^{\infty} \overline{A_i}.$$

(b) Show that for any family $\{A_i : i = 1, 2, 3, \ldots\}$ of subsets of a topological space X the following formula holds:

$$\overline{\bigcup_{i=1}^{\infty} A_i} = \bigcup_{i=1}^{\infty} \overline{A_i} \cup \bigcap_{k=1}^{\infty} \overline{\bigcup_{i=k}^{\infty} A_i}.$$

Ex. 5. Let $S = \langle \mathbb{R}, \tau_S \rangle$ be a Sorgenfrey line, $D(\mathbb{N}) = \langle \mathbb{N}, \tau_D \rangle$ be a discrete topology on $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $D(\mathbb{R}) = \langle \mathbb{R}, \tau_D \rangle$ be a discrete topology on $\mathbb{R}$.

Show that there is a continuous mapping from S *onto* $D(\mathbb{N})$ but that there is no continuous mapping from S *onto* $D(\mathbb{R})$.

Ex. 6. Let X be a normal space and let U_1 and U_2 be open subsets of X such that $X = U_1 \cup U_2$. Show that there are open sets V_1 and V_2 such that $\overline{V_1} \subset U_1$, $\overline{V_2} \subset U_2$, and $X = V_1 \cup V_2$.

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Topology Ph.D. Entrance Exam, April 2001

Ex. 1. Let $\langle X_0, \tau_0\rangle$ and $\langle X_1, \tau_1\rangle$ be connected topological spaces. Show that $X_0 \times X_1$ with the product topology is connected.

Ex. 2. Consider the real line $\mathbb{R}$ with the topology τ generated by the family of intervals:

$$\mathcal{F} = \{[a, b) \colon a \in \mathbb{Q} \;\&\; b \in \mathbb{R} \;\&\; a < b\},$$

where $\mathbb{Q}$ stands for the set of rational numbers. Let X be the product of $\langle \mathbb{R}, \tau\rangle$ with itself (with the product topology). Prove or disprove that X is normal.

Ex. 3. Prove or find a counterexample for the statement:

> A compact subset of a topological space $\langle X, \tau\rangle$ is closed in X.

Ex. 4. Let τ be the usual topology on the real line $\mathbb{R}$. Answer one of the following two questions.

(a) Does there exists a topology $\tau_0 \subset \tau$ such that $\langle \mathbb{R}, \tau_0\rangle$ is homeomorphic to figure eight (i.e., two circles tangent at a point)?

(b) Does there exists a topology $\tau_0 \subset \tau$ such that $\langle \mathbb{R}, \tau_0\rangle$ is homeomorphic to the unit circle $S^1 = \{\langle x, y\rangle \in \mathbb{R}^2 \colon x^2 + y^2 = 1\}$?

Ex. 5. Let $\langle X, \tau\rangle$ and $\langle Y, \tau'\rangle$ be the topological spaces and let $f \colon X \to Y$ be a function. Consider the graph $G(f) = \{\langle x, f(x)\rangle \colon x \in X\}$ of f as a subspace of the cartesian product $X \times Y$ (with the product topology). Prove or disprove each the following.

(a) If f is continuous, then $G(f)$ is homeomorphic to X.

(b) If $G(f)$ is homeomorphic to X, then f is continuous.

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Topology Ph.D. Entrance Exam, August 2001

Ex. 1. Let $\mathbb{R}^2$ be the euclidean plane (i.e., with natural topology). Let

$$X = \{\langle x, y\rangle \in \mathbb{R}^2 \colon x^2 + y^2 = 1\} \cup \{\langle x, 0\rangle \in \mathbb{R}^2 \colon -1 \le x \le 1\},$$

$$Y = \{\langle x, y\rangle \in \mathbb{R}^2 \colon x^2 + y^2 = 1\} \cup \{\langle x, 0\rangle \in \mathbb{R}^2 \colon -1 \le x \le 2\}.$$

Are X and Y homeomorphic? Give reasons for your answer.

Ex. 2. Prove that every compact metric space has a countable base for its topology.

Ex. 3. Let $\langle X, d\rangle$ be a compact metric space, and let $f \colon X \to X$ satisfy

$$d(f(x_1), f(x_2)) < d(x_1, x_2) \text{ for all distinct } x_1, x_2 \in X.$$

Show that there is a point $p \in X$ such that $f(p) = p$.

Ex. 4. A topological space X is said to have *countable pseudo character* provided every singleton in X is a G_δ-set (i.e., it is a countable intersection of open sets). Show that every compact Hausdorff space with countable pseudo character is first countable, that is, it has a countable local base at every point $x \in X$.

Ex. 5. Let $\mathcal{F}$ be the family of all *non-zero* polynomials of the form

$$w(x, y) = a_0x^2 + a_1y^2 + a_2xy + a_3x + a_4y + a_5$$

with rational coefficients and for every $w \in \mathcal{F}$. Let

$$E_w = \{\langle x, y\rangle \in \mathbb{R}^2 \colon w(x, y) = 0\}.$$

Show that the plane $\mathbb{R}^2$ is not covered by the sets E_w with $w \in \mathcal{F}$, that is, that $\mathbb{R}^2 \ne \bigcup_{w\in\mathcal{F}} E_w$.