A SERIES OF CLASS NOTES TO INTRODUCE LINEAR AND NONLINEAR PROBLEMS TO ENGINEERS, SCIENTISTS, AND APPLIED MATHEMATICIANS

REMEDIAL CLASS NOTES

A COLLECTION OF HANDOUTS FOR REMEDIATION

IN FUNDAMENTAL CONCEPTS FOR

PROBLEM SOLVING IN MATHEMATICS

CHAPTER 3

Mapping Concepts and Mapping Problems for Scalar Valued Functions of a Scalar Variable

Recall that the concept of a set may be taken to be a primative and then the concepts of ordered pair, Cartesian product, function, graph, and binary operation can be formally defined.. Informally, an **ordered pair** from the sets A and B is an element of the form (a,b) where $a \in A$ and $b \in B$. The ordered pair (a,b) equals the ordered pair (c,d) if (and only if) a = c and b = d. The **Cartesian product** of the sets A and B is the set of ordered pairs $A \times B = \{(x,y): x \in A \text{ and } y \in B\}$. To develop an intuitive understanding of the concept of function, we use the informal definition of a function as a **rule of correspondence** between two sets. A **function** is a rule of correspondence which assigns to each element in a first set (called the **domain** of the function) exactly one element in a second set (called the **co-domain** of the function). Often we denote the function or rule by f. If x is any element in the domain, then y = f(x) indicates the element y in the co-domain that the rule defined by f associates with the element x in the domain. Then x is the **independent variable** and y is the **dependent variable**. We use the notations $f: A \rightarrow B$ and $A \xrightarrow{f} B$ to indicate that A is the domain and B is the co-domain of the function f. We may also denote the domain of f by D_f or by D(f). Although not standard, we follow this lead and denote the codomain of f by CoD_f or CoD(f). The **graph** of $f:A \rightarrow B$ is the set $G = \{ (x,f(x)) \in AxB: x \in A \}$. It is a subset of the Cartesian product $AxB = \{(x,y): x \in A \text{ and } y \in B\}$ Since there is a natural one-to-one correspondence between functions and their graphs, we use the definition of the graph of a function as a formal definition of a function in terms of sets. Make a distinction between the graph of a function defined as a set and the geometric pictures of the graphs of real valued functions of a real variable that you have drawn on paper and the chalk board using Cartesian coordinates. When a distinction is needed, we refer to these as **picture graphs.** A **binary operation** on a set A is a mapping from $A \times A$ to A. Thus, for example, addition of real numbers associates with two given numbers, say x and y, a new number, say z. Instead of using function notation and writing +(x,y) = z, we usually write x+y = z.

We can consider functions from any set to any other set. You are familiar with real valued functions of a real variable, that is, functions that map a subset of the real numbers **R** to the set of real numbers **R**. It is important to note that to define a function, we must first define two sets, the domain and the co-domain, before giving the rule of correspondence. Thus, these two sets are part of the definition of a function. A function is not completely defined unless both of these two sets have been specified. (Often in high school algebra texts these are omitted since the codomain is almost always **R** and the "natural" domain is the subset of **R** where the formula or rule which defines the function is defined.)

Although we can consider functions from any set to any other set, we are particularly interested in functions from the real numbers **R** to the real numbers **R** which we denote by $\mathcal{F}(\mathbf{R},\mathbf{R})$. To distinguish (the picture graphs of) real functions of a real variable from (the picture graphs of) **curves** in $\mathbf{R}^2$ (which may also be rigorously defined using the Cartesian Product $\mathbf{R} \times \mathbf{R} = \mathbf{R}^2$), we say that a function is **well-defined** provided one has clearly specified exactly one element in the co-domain (y-axis) for each element of the domain (x-axis). This is often referred to as the **vertical line test** since a vertical line will intersect the picture graph of a function in at most one point. Thus the equation $x^2 + y^2 = 1$ does not define a function since a vertical line between x= −1 and x=1 crosses its graph in two places. This equation (a circle of radius 1), in fact, defines two functions. (What are they?)

Many, functions in $\mathcal{F}$ (**R**,**R**) are defined by **algebraic expressions** or **algebraic formulas**. Examples are **polynomials** (e.g. f(x) = mx + b and $f(x) = ax^2 + bx + c$) and **rational functions**, f(x) = p(x)/q(x) where p and q are polynomials. Familiar examples of functions not defined by algebraic formulas are the **trigonometric functions** (e.g. f(x) = sin x and f(x) = cos x) and the **exponential function** $f(x) = \exp(x) = e^x$. You should have some familiarity with these functions, particularly when the rule of correspondence is defined by an algebraic formula. In fact, consideration of familiar examples should help motivate interest in understanding <u>fundamental concepts</u> for functions.

<u>DEFINITION</u>. Let f: X → Y. Then the **range** of f is the set $R_f = \{y \in Y: \exists\, x \in X \text{ s.t. } f(x) = y\}$.

Informally, the domain of a function can be described as the set of things that get mapped and the range as the set of things that get mapped into. That is, the range is the set of things in the co-domain (target set) for which there exist an element in the domain (quiver set) that gets mapped (or shot) into those things (ie., the things that actually get hit). If A ⊆ X where f:X → Y, then by definition, the **<u>image</u>** <u>of A</u> is the set $f(A) = \{ y \in Y : \exists\, x \in A \text{ s.t. } f(x) = y \}$. Hence the range R_f is the image of the domain. On the other hand (OTOH), if B⊆Y, then the **<u>inverse image</u>** <u>of B</u> is the set $f^{1}(B) = \{x\in X: f(x) \in B\}$. That is, the inverse image of B is the subset of X that gets mapped into B. Note that all elements in B need not have an element in X that maps into that element in B. Hence the range need not be the entire codomain.

If f∈$\mathcal{F}$ (D,**R**), where D⊆**R**, then an <u>algebraic formula</u> or <u>algebraic expression</u> always defines a clear rule of correspondence between the domain D and the co-domain **R** of a **real valued function of a real variable** using the <u>binary operations</u> of addition and multiplication (and subtraction and division, but we define these as the <u>inverse operations</u> of addition and multiplication rather than as distinct operations). The algorithm for evaluation of the formula (or expression) is clearly specified using parentheses (), brackets [], and braces {} as well as standard conventions to establish the order in which the operations are to be carried out.

After specifying D, (explicitly or implicitly) we may consider subsets of $\mathcal{F}$ (D,**R**) by requiring a property that all functions in the subset must have. If p:**R** → **R** is defined by $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ... + a_n x^n$ where $a_0,...,a_n \in$ **R** and $a_n \neq 0$, then p is a **polynomial** of degree n. The domain is D(p) = **R**, but the range, R(p), depends on the constants $a_0,...,a_n$. We denote the set of all such polynomials of degree less than or equal to n by $\mathcal{P}_n$(**R**,**R**). The set of all polynomials of any degree is $\mathcal{P}$(**R**,**R**). The constant functions are $\mathcal{P}_0$(**R**,**R**).

The domain of a real valued function of a real variable is not always the entire set of real numbers **R** (e.g. $f(x) = \sqrt{x}$ or f(x) = 1/x). **Rational functions**, which we denote by $\mathcal{Q}$(D,**R**), are quotients of polynomials, f(x) = p(x) / q(x) where p and q are polynomials. The domain as well as the range of a rational function depends on the constants in the polynomials, D(f) = {x∈**R**: q(x)≠0}. We may write $f\colon \underset{\mathbf{R}}{\overset{}{D_f}} \to \mathbf{R}$ (with $D_f \subseteq \mathbf{R}$) to indicate that the domain of f is a subset of **R**. However, in an informal discussion, we may say that f is a real valued function of a real variable even when the domain of f is not the entire set of real numbers (e.g. f(x) = 1/x), but we avoid writing $f:\mathbf{R} \to \mathbf{R}$ unless the domain of f is the entire set of real numbers. Often the domain of a function is well known or the context makes it clear and it is not given explicitly. However, when writing proofs, it is essential that the domain (and co-domain) be given explicitly and correctly. The algebraic operations are addition, multiplication, raising to a power, and their inverse operations (subtraction, division, and extraction of roots). Functions that can be defined by a finite number of algebraic operations belong to the set of **algebraic functions**.

DEFINITION. Let $f: D \to \mathbf{R}$. Then f is an **algebraic function** if y = f(x) satisfies an equation of the form

$$a_n(x)\, y^n + a_{n-1}(x)\, y^{n-1} + \cdots + a_1(x)\, y + a_0(x) = 0,\ x \in D$$

where the coefficient functions $a_n(x)$, $a_{n-1}(x)$, $\cdots$, $a_1(x)$, $a_0(x)$ are polynomials in x. If f is not algebraic, it is said to be **transcendental**.

The function $y = f(x) = x^{1/3}$ is an algebraic function on $D = \mathbf{R}$ since it satisfies $y^3 - x = 0$ on $\mathbf{R}$ To determine the domain of the function y = f(x) defined implicitly by the equation $y^3 + y = x$, we compute the derivative implicitly as $3y^2\,[dy/dx] + dy/dx = 1$, so that $dy/dx = 1/[1+3y^2] \geq 1 > 0$. Hence y is always increasing and hence is defined y implicitly on $\mathbf{R}$ by this equation. Even though we have not given an explicit algebraic formula for this function, it is an algebraic function. (Can you find an explicit formula for this function?) Hence we see that even though all algebraic formulas define algebraic functions, not all algebraic functions need be defined explicitly by algebraic formulas.

There are non-algebraic functions where the rule of correspondence is not simple. Recall the **trigonometric functions** (e.g., sine, cosine and tangent). Functions that are not algebraic are called **transcendental functions**. Although we can use our knowledge of trigonometry to evaluate these function exactly for some values of x (e.g. $\pi/6$, $\pi/4$,...) , more often we find approximate values of these functions using a table or a calculator. Other examples of transcendental functions are **exponentiation** and **logarithms**. We refer to trigonometric, exponential, and logarithmic functions as the **elementary transcendental functions**. Functions that can be defined by a finite number of algebraic operations involving algebraic and elementary transcendental functions will be referred to as **elementary functions**. These are the functions that you are most familiar with. We denote the elementary functions on D by E(D,**R**), where D is the domain where we wish to consider them (not necessarily the natural domain of the functions).

There are subsets of *F*(D,**R**) where D may be I = (a,b), $\bar{I}$ = [a,b], or some other set that are of interest. C(I,**R**) is the set of all functions that are continuous on I. *A*(I,**R**) is the set of all functions that are analytic on I. PC($\bar{I}$,**R**) is the set of all functions that are piecewise continuous on $\bar{I}$.

From high school algebra, trigonometry, and geometry, you should be familiar with the concepts of "**set**" "**number**", "**order**", and "**mapping**". These concepts were reviewed briefly in Chapter 1. In this chapter, we consider further the concept of a "mapping". The terms "**mapping**", "**function**", "**operator**", and "**transformation**" all have essentially the same denotation (meaning). Different words are used in different contexts because they have different connotations that indicate the nature of the structure of the **domain** and **codomain** of the mapping under consideration. We use the word mapping for any domain and codomain (e.g., when the structure of domain and codomain is not known, when they are just sets that have no structure or when no other term is appropriate). The word **function** is always used for mappings between **R** and **R** and this is the focus of this chapter. The word function is sometimes used as a synonym for mapping to relieve boredom.

To understand the theory required to solve linear and nonlinear **vector equations** we need to consider sets of **sequences** (**n-tuples**), sets of **matrices** (**arrays** of numbers), and sets of mappings (e.g., functions from **R** to **R**). We note here, but explain how later, that sequences and matrices can be considered to be mappings. Later we take the concept of mappings one step further and consider **"vector valued" functions**. Although the notation for sets of functions (**function spaces**) is not as standard as that for logic, set theory, and number systems, the need for a notation warrants its early introduction. We choose the notation $F(X,Y)$ for the set of all functions from X to Y, rather than the set theoretic X^Y, since we think of a function as a **rule of correspondence** rather than as a **set of ordered pairs**.

SETS OF SEQUENCES OF FINITE LENGTH (n-TUPLES)

	Math Symbol	English Translation
1.	$\mathbf{R}$	The set of all sequences of real numbers of length one. This set is isomorphic with the set of real numbers and hence we denote it by **R**. A geometric interpretation is a **line** (the real number line).
2.	$\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$	The set of all sequences of real numbers of length two. This set is isomorphic with the Cartesian product of the real number system with itself $\{(x,y): x,y \in \mathbf{R}\}$ (i.e., the set of all **ordered pairs**). A geometric interpretation is a **plane** (Two dimensional space).
3.	$\mathbf{R}^3 = \mathbf{R}^2 \times \mathbf{R}$ $= \mathbf{R} \times \mathbf{R}^2$ $= \mathbf{R} \times \mathbf{R} \times \mathbf{R}$	The set of all sequences of real numbers of length three. This set is isomorphic with the Cartesian product of $\mathbf{R}^2$ with **R**, with the Cartesian product **R** with $\mathbf{R}^2$, and with the set of **ordered triples** $\{(x,y,z): x,y,z \in \mathbf{R}\}$. A geometric interpretation is three dimensional space.

4. $\mathbf{R}^n = \underset{\leftarrow \text{ n times } \rightarrow}{\mathbf{R}\times\cdots\times\mathbf{R}}$ The set of all sequences of real numbers of (finite) length n. This set is isomorphic with the set of **n-tuples** $\{(x_1,x_2,...,x_n): x_i\in \mathbf{R}, i=1, 2, 3, ...,n\}$and with any Cartesian products of **R** with itself n times. There is no geometrical interpretation of this set if n>3.

7. **C** The set of all sequences of complex numbers of length one. This set is isomorphic with the set of complex numbers and hence we denote it by **C**. A geometric interpretation is a plane (i.e., the **complex plane**).

8. $\mathbf{C}^2 = \mathbf{C} \times \mathbf{C}$ The set of all sequences of complex numbers of length two. This set is isomorphic with the Cartesian product of the complex number system with itself $\{(z_1,z_2): z_1,z_2 \in \mathbf{C}\}$ (i.e., the set of all ordered pairs of complex numbers). There is no geometric interpretation since four real dimensions would be required.

9. $\mathbf{C}^3 = \mathbf{C}^2 \times \mathbf{C} = \mathbf{C} \times \mathbf{C}^2 = \mathbf{C} \times \mathbf{C} \times \mathbf{C}$ The set of all sequences of complex numbers of length three. This set is isomorphic with the Cartesian product of $\mathbf{C}^2$ with **C**, with the Cartesian product **C** with $\mathbf{C}^2$, and with the set of ordered triples of complex numbers, $\{(z_1, z_2, z_3):z_1\ z_2, z_3 \in \mathbf{C}\}$. There is no geometric interpretation of this set.

10. $\mathbf{C}^n = \underset{\leftarrow \text{ n times } \rightarrow}{\mathbf{C}\times\cdots\times\mathbf{C}}$ The set of all sequences of complex numbers of (finite) length n. This set is isomorphic with the set of n-tuples of complex numbers, $\{(z_1, z_2, ...z_n): z_1, z_2,...,z_n \in \mathbf{C}\}$ and with the Cartesian product of **C** with itself n times. There is no geometrical interpretation of this set if n>1.

SETS OF MATRICES

1. $\mathbf{R}^{m\times n}$ The set of all m by n matrices with entries from **R**. The set of all real matrices of a given size. All **arrays** of real numbers.

2. $\mathbf{C}^{m\times n}$ The set of all m by n matrices with entries from **C**. The set of all complex matrices of a given size. All **arrays** of complex numbers.

Recall that we chose the notation *F*(X,Y) for the set of all functions from X to Y. If the codomain has been clearly established, we may shorten this notation to *F*(X). If both the domain and codomain have both been clearly established, we may shorten further to *F*. Typically, the sets X and Y have structure. This gives the set *F*(X,Y) structure. Hence we may specify subsets of *F*(X,Y) based on this structure. Instead of just giving names for specific subsets, we develop a method for naming such subsets. We denote subsets by a letter (or letters) followed by (X,Y) (or (X) or nothing) where the letter (or letters) denotes a property that the functions must satisfy. For example, if we let I = (a,b), then C(I,**R**) denotes the set of all functions that are continuous on the open interval I = (a,b). If in a given setting our functions all have domain I and codomain **R**, we might shorten this to C(I) or just C. The letter may have subscripts or superscripts. For example, $C^1(I,\mathbf{R}) = \{f:I\to R \mid f'(x)$ exists and is continuous on I$\}$. The letter *A* denotes analytic and *H* denotes holomorphic. The letter *P* denotes a set of polynomials. The letter *Q* denotes a set of rational functions. *A* denote algebraic functions. *E* denote elementary functions. PC denotes piecewise continuous functions. *M* denotes measurable functions. E denotes the set f elementary functions. $\widetilde{PC}$ denotes the set of equivalence classes of piecewise continuous functions. $M^{\sim}$ denotes set of equivalence classes of measurable functions. *E* denotes even functions and *O* denotes odd functions (see later handout). We give some examples.

SETS OF FUNCTIONS (FUNCTION SPACES)

1. *F*(D,CoD) $\{f:D\to CoD)$ = the set of all functions where the common domain is specified as D (e.g., D=**R**) and the common codomain CoD (e.g., CoD = **R**) is also specified (e.g., we are discussing real valued functions of a real variable).
2. *F*(I,**R**) $\{f:I\to \mathbf{R})$ where the common domain is D=I=(a,b) where I is the open interval between a and b and the common codomain is **R**. We are discussing the real valued functions on the interval I.
3. *F*(I) $\{f:I\to CoD)$ when the common codomain, (e.g., CoD = **R**, has been previously specified and is understood i.e., we are discussing real valued functions), but the common domain is given as I where I is an open interval I=(a,b). This abbreviated notation is used in a discussion when the domain might change, but the common codomain does not.
4. $P_n(D,\mathbf{R})$ $\{f:D\to R: f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ where $a_i \in\mathbf{R}$ and $x\in D\}$ = the set of all polynomials of degree less than or equal to n on the common domain D.
5. $P_n(I)$ $\{f:I\to CoD: f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ where $a_i \in\mathbf{R}$ and $x\in I\}$ = the set of all polynomials of degree less than or equal to n on the common domain I=(a,b) where the common codomain has been specified (say R) and is understood.
6. *A*(D,**R**) $\{f:D\to \mathbf{R}$: f is analytic on D$\}$ = the set of all functions on the common domain D which are analytic at all points in D.
7. *A*(I) $\{f:I\to R$: f is analytic on the interval I$\}$ = the set of all functions on the interval I (say I=(a,b)) which are analytic on I when the common codomain has been specified (say **R**) and is understood.
8. C(D,**R**) $\{f:D\to\mathbf{R}$: f is continuous on D$\}$ = the set of all continuous functions on the common domain D which is specified and common codomain, the set **R** of real numbers.
9. C(I) $\{f:I\to R$: f is continuous on the interval I$\}$ = the set of all continuous functions on the interval I when the interval I (say I=(a,b)) as well as the common codomain has been previously specified and are understood. This abbreviated notation is used in a discussion when the domain might change, but the common codomain **R** does not.

10. $C^1(D,\mathbf{R})$ — $\{f:D \to CoD: f'(x)$ exists and is continuous on $D\}$ = the set of all functions on the common domain D which have a continuous derivative at all points in D.

11. $C^2(D,\mathbf{R})$ — $\{f:D \to \mathbf{R}: f''(x)$ exists and is continuous on $D\}$ = the set of all functions on the common domain D which have a continuous second derivative at all points in D.

12. $C^2(I)$ — $\{f:I \to R: f''(x)$ exists and is continuous on the interval $I\}$ = the set of all functions on the interval I (say I=(a,b)) which have a continuous second derivative on I when the common codomain **R** is understood. This abbreviated notation is used in a discussion when the domain might change, but the common codomain **R** does not.

13. $C^n(D,\mathbf{R})$ — $\{f:D \to \mathbf{R}: f^{(n)}(x)$ exists and is continuous on $D\}$ = the set of all functions on the common domain D which have a continuous n^{th} derivative at all points in D.

14. $C^\infty(I)$ — $\{f:D \to CoD: f^{(n)}(x)$ exists and is continuous on the interval I for n = 0,1,2,...$\}$ = the set of all functions on the common domain D which have a continuous n^{th} derivative for all $n \in \mathbf{N}$ at all points in I when the common codomain **R** is understood.

15. $\mathcal{A}(D,\mathbf{R})$ — $\{f:D \to \mathbf{R}$: f is analytic on $D\}$ = the set of all algebraic functions on the common domain D.

16. $\mathcal{A}(I)$ — $\{f:I \to R$: f is analytic on the interval $I\}$ = the set of all algebraic functions on the interval I (say I=(a,b)) which are analytic on I when the common codomain **R** is understood.

17. $E(D,\mathbf{R})$ — $\{f:D \to \mathbf{R}$: f is an elementary function on $D\}$ = the set of all elementary functions on the common domain D.

18. $E(I)$ — $\{f:I \to R$: f is an elementary function the interval $I\}$ = the set of all elementary functions on the interval I (say I=(a,b)) which are analytic on I when the common codomain **R** is understood.

19. $\mathscr{F}(\Omega,\mathbf{C})$ — $\{f:\Omega \to \mathbf{C})$ where the common domain is $D=\Omega$ where Ω is a region (open connected subset of **C**) and the common codomain is **C**. We are discussing the complex valued functions on Ω.

20. $H(\Omega, \mathbf{C})$ — $\{f:\Omega \to \mathbf{C}$: f is analytic on $\Omega\}$ = the set of all functions on the common domain Ω which are analytic (holomorphic) at all points in Ω.

21. $E(\mathbf{R}, \mathbf{R})$ — $\{f:\mathbf{R} \to \mathbf{R}: \forall x \in R, f(-x) = f(x)\}$ = the set of all even functions.

22. $\mathcal{O}(\mathbf{R}, \mathbf{R})$ — $\{f:\mathbf{R} \to \mathbf{R}: \forall x \in R, f(-x) = -f(x)\}$ = the set of all odd functions.

Handout #3 ODD, EVEN, ONE-TO-ONE, AND ONTO REAL VALUED FUNCTIONS OF A REAL VARIABLE Professor Moseley

Once a function has been defined (including the <u>domain</u> and <u>co-domain</u> as well as specifying the <u>rule of correspondence</u>), we can consider properties of functions. For f:**R**→**R** you are familiar with the concepts of **continuity** and **differentiability** at a point. A function is said to have these properties if they hold for all x in the <u>domain</u> D. We have given names to these subsets of *F*(D,**R**) that have these properties. Here we discuss the easy properties of <u>odd</u> and <u>even</u> and then consider the more complicated properties of <u>one-to-one</u> and <u>onto</u> where the domain and co-domain need not be the real numbers.

<u>DEFINITION #1</u>. A function f:**R**→**R** is **even** if ∀ x ∈ **R**, f(−x) = f(x). f is **odd** if ∀ x ∈ **R**, f(−x) = −f(x).

E(**R**, **R**) = {f:**R** → **R**: ∀x∈R, f (−x) = f(x)} is the set of all even functions. *O*(**R**, **R**) = f:**R** → **R**: ∀x∈R,
f (−x) = −f(x)} = the set of all odd functions. First note that most functions are neither odd nor even (e.g., $f(x) = e^x$ and $f(x) = x+x^2$). One way to determine if a function is odd or even (or neither) directly using the definition (DUD) is by computing f(−x). If it happens to be f(x), then the function is even. If it happens to be −f(x), then the function is odd. Since there is no particular reason why either of these should happen, most functions are neither odd nor even. Another way to determine if a function is odd or even (or neither) is to look at the picture graph of the function y = f(x). If ∀ x ∈ R, we have that the point (x, y) being on the picture graph implies that the point (−x,y) is on the picture graph, then f is even. Similarly, if ∀ x ∈ R, we have that the point (x, y) being on the picture graph implies that the point (−x,−y) is on the picture graph, then f is odd.

<u>THEOREM</u>. If f∈$\mathscr{F}$(**R**,**R**) is both odd and even, then f(x) = 0 ∀x∈**R** (i.e., f is the zero function). Proof. Let f∈$\mathscr{F}$(**R**,**R**) be both odd and even. Then by the definitions of odd and even, we have for all x∈**R** that f(−x) = f(x) = −f(x). Hence 2f(x) = 0 so that f(x) = 0; that is, f is the zero function.

A function is **one-to-one** (1-1) if every element in the domain gets mapped to a different element in the co-domain. A function is **onto** if every element in the co-domain has an element in the domain which maps into it. The formal definition of one-to-one is the **contrapositive** of the above informal definition. (The contrapositive of a <u>statement</u> is equivalent to the statement, but is stated with <u>negations</u>. Since our informal definition involves negation, the contrapositive has positives. This is why we use it instead of the informal definition.) The formal definition of onto is stated in terms of the range.

<u>DEFINITION #2</u>. Let f: X → Y. The function f is said to be **one-to-one** (**injective**) if $f(x_1) = f(x_2)$ implies $x_1 = x_2$. (This is the contra positive of the <u>statement</u>, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$. That is, distinct values of x in X get mapped to distinct values of y in Y. Recall that the negation of a negative is the positive.) The function f is said to be **onto** (**surjective**) if

R(f) = Y. (That is, f is onto if the range is the entire co-domain.) If f is both 1-1 and onto, then it is said to be **bijective** or to form a **one-to-one correspondence** between the domain and the co-domain. (If f is 1-1, then it always forms a 1-1 correspondence between its domain and its range.) The **identity function** (denoted by I or i_X) from a set X to itself is the function i_X:X → X defined by $i_X(x) = x \ \forall x \in X$.

THEOREM. If f∈$\mathscr{F}$(D, CoD) is one-to-one and onto (i.e, it is a bijection), then its inverse function f^{-1} from CoD to D, defined by ∀y∈CoD, $f^{-1}(y) = x$ if (and only if) f(x) = y exists. (For example, $i_X(x)$is its own inverse function.)

Proof. Let f∈$\mathscr{F}$(**R**,**R**) be one-to-one and onto and y∈CoD. Then since f is onto, there exists x∈D such that f(x) = y. Since f is one-to-one, there is at most one such x. (Suppose there were a second x_0 such that $f(x_0) = y$. Then since f is one-to-one, $f(x_0) = y = f(x)$ implies $x_0 = x$. Hence no different x_0 exists.) Hence f^{-1} defined by $f^{-1}(y) = x$ is a well defined function..

Q.E.D.

Thus if f∈$\mathscr{F}$(D, CoD) is a bijection, then its inverse exists. We say f is invertible.

Let E_{inv}(D,**R**) = {f:D → **R**: f is analytic on D and is invertible.} be the set of all elementary functions on the common domain D that are invertible. For some algebraic functions, the inverse function can be obtained by algebraic operations.

As exercises consider these concepts using elementary functions. First use your intuition to determine whether a given functions is one-to-one and onto. Then try to develop precise arguments (i.e. proofs) to validate your intuition.